Some contributions to 3D Frobenius' Problem

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INTRODUCTION

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Given a 3D *Frobenius' Set* $A = \{a, b, N\} \subset \mathbb{N}$ with

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We have $|\overline{\mathcal{R}}(A)| < \infty$ and we denote

$$\mathfrak{f}(A) = \max \overline{\mathcal{R}}(A), \ \textit{Frobenius' Number}.$$

Almost Closed Formula

An $O(h(N))$ *Almost Closed Formula* (ACF) for a value $v(a_1, ..., a_k)$, with fixed k and $a_1, ..., a_k \in O(N)$, is a closed formula $F(a_1, ..., a_k, b_1, ..., b_l) = v(a_1, ..., a_k)$ with a fixed number l of computable additional parameters $b_1, ..., b_l$ with an $O(h(N))$-time algorithm.

Some things to be computed

The *Frobenius' Problem* $FP(A)$, $A = \{a, b, N\}$, is related to several questions:

- $\mathfrak{f}(A)$?
- $|\overline{\mathcal{R}}(A)|$?
- $\overline{\mathcal{R}}(A)$?
- Given $m \in \mathbb{N}$, is $m \in \mathcal{R}(A)$ or $m \in \overline{\mathcal{R}}(A)$?
- If $m \in \mathcal{R}(A)$, $d(m; A) = |\{(x, y, z) \in \mathbb{N}^3 : \ xa + yb + zN = m\}|$?
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- Frobenius ?

$$\mathfrak{f}(a,b) = ab - a - b$$

- Sylvester 1884

$$|\overline{\mathcal{R}}(a,b)| = \frac{(a-1)(b-1)}{2}$$

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Denoting by

$$F(N) = \max_{\substack{0<a<b<c\leq N \\ \gcd(a,b,c)=1}} \mathfrak{f}(a,b,c)$$

Lewin '73 gave

$$F(N) = \begin{cases} \frac{N(N-1)}{2} - N + 1, & N \equiv 0 \pmod 2 \\ \frac{(N-1)^2}{2} - N, & N \equiv 1 \pmod 2 \end{cases}$$

and the critical sets are $\{\frac{N}{2}, N-1, N\}, \{N-2, N-1, N\}$ for even N and $\{\frac{N-1}{2}, N-1, N\}$ for odd N.

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ACF of $\mathfrak{f}(a, b, c)$ and $|\overline{\mathcal{R}}(a, b, c)|$ were given by several authors.

WEIGHTED DOUBLE-LOOP DIGRAPHS

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A *weighted double-loop* digraph $G(N; a, b; \mathfrak{a}, \mathfrak{b})$ has the set of vertices

$$V(G) = \mathbb{Z}_N = \{0, 1, ..., N-1\}$$

and the set of arcs

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Cheng & Hwang '88 pointed to the identity

$$D(N; a, b; a, b) - N = \mathfrak{f}(a, b, N).$$

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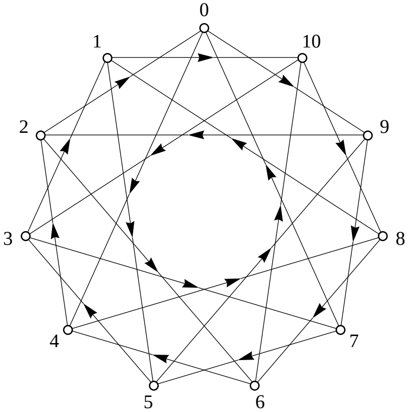

$G(11;4,9;4,9)$

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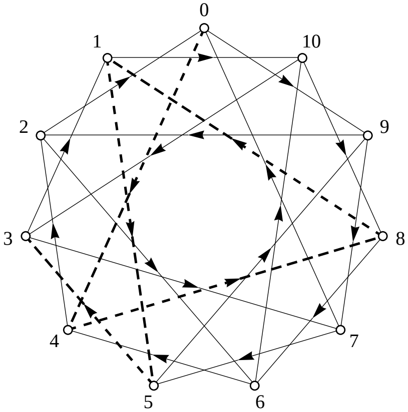
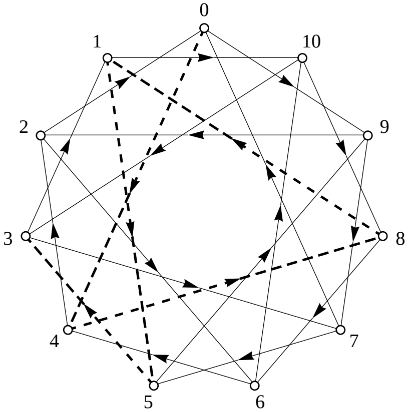


$$D(11; 4, 9; 4, 9) = d(0, 3) = 4 \times 4 + 1 \times 9 = 25$$

Links with FP

$\mathfrak{f}(4, 9, 11) = D(11; 4, 9; 4, 9) - 11 = 14$



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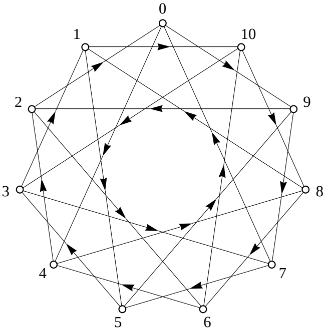

4	8	1	5	9	2	6	10	3	7
6	10	3	7	0	4	8	1	5	9
8	1	5	9	2	6	10	3	7	0
10	3	7	0	4	8	1	5	9	2
1	5	9	2	6	10	3	7	0	4
3	7	0	4	8	1	5	9	2	6
5	9	2	6	10	3	7	0	4	8
7	0	4	8	1	5	9	2	6	10
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$G(11; 4, 9; 1, 1)$

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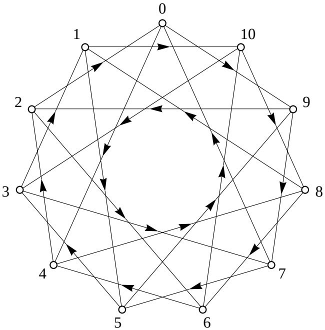

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1	5	9	2	6	10	3	7	0	4
3	7	0	4	8	1	5	9	2	6
5	9	2	6	10	3	7	0	4	8
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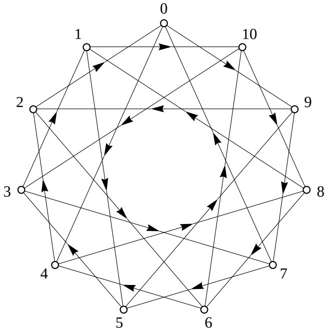

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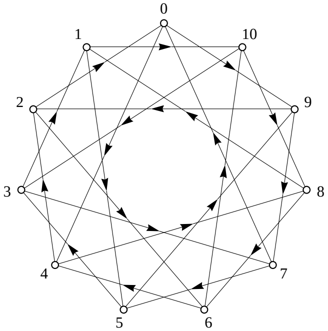

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36	40	44	48	52	56	60	64	68	72
27	31	35	39	43	47	51	55	59	63
18	22	26	30	34	38	42	46	50	54
9	13	17	21	25	29	33	37	41	45
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$$\mathcal{B}(4,9,11) = \{0,4,8,9,12,13,16,17,18,21,25\}$$

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$$\mathfrak{f}(4,9,11) = 25 - 11 = 14$$

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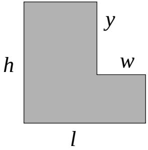
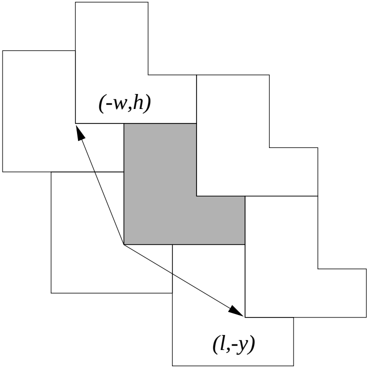
L-shapes will be denoted by their lengths $L(l, h, w, y)$





ALMOST CLOSED FORMULAE

ACF

A., Miralles & Zaragozá '06:
$\mathcal{H} = L(l, h, w, y)$ encodes $\mathcal{B}(a, b, N)$ iff $la \leq yb$ and $wa \leq hb$.
$\mathcal{H}$ can be computed in $O(\log N)$, in the worst case.

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$$\begin{aligned}
\mathfrak{f}(A) &= \max\{(l-1)a + (h-y-1)b, (l-w-1)a + (h-1)b\} - N, \\
|\overline{\mathcal{R}}(A)| &= \frac{a}{2N}[l(l-1)(h-y) + (l-w-1)(l-w)y] + \\
&+ \frac{b}{2N}[l(h-y-1)(h-y) + y(l-w)(2h-y-1)] - \frac{N-1}{2},
\end{aligned}$$

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$\mathcal{H}$ can be computed in $O(\log N)$, in the worst case.

$$\begin{aligned} \mathfrak{f}(A) &= \max\{(l-1)a + (h-y-1)b, (l-w-1)a + (h-1)b\} - N, \\ |\overline{\mathcal{R}}(A)| &= \frac{a}{2N}[l(l-1)(h-y) + (l-w-1)(l-w)y] + \\ &+ \frac{b}{2N}[l(h-y-1)(h-y) + y(l-w)(2h-y-1)] - \frac{N-1}{2}, \\ \overline{\mathcal{R}}(A) &= \bigcup_{i=0}^{l-1} \bigcup_{j=0}^{h-y-1} \mathcal{C}_{i,j} \cup \bigcup_{i=0}^{l-w-1} \bigcup_{j=h-y}^{h-1} \mathcal{C}_{i,j}, \end{aligned}$$

where $\mathcal{C}_{i,j} = \bigcup_{\alpha=1}^{\lfloor (ia+jb-1)/N \rfloor} \{ia + jb - \alpha N\}$.

SYMBOLICAL APPLICATIONS

Solving a sequence of FS

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\end{aligned}
$$

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&\cup && \bigcup_{i=1}^{3t+4} \bigcup_{\alpha=1}^{\lfloor \frac{n_{i,1,t,0}-1}{N_t} \rfloor} \{n_{i,1,t,\alpha}\} \cup \bigcup_{i=t+2}^{3t+4} \bigcup_{\alpha=1}^{\lfloor \frac{n_{i,0,t,0}-1}{N_t} \rfloor} \{n_{i,0,t,\alpha}\},\ t \geq 1,\\
\mathfrak{f}(B_t) &= && 3t^3+15t^2+21t+9,\ t \geq 0.
\end{aligned}$$

where $n_{i,j,t,\alpha} = ia_t + jb_t - \alpha N_t$.