Numerical semigroups associated to algebraic curves

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Iberian Meeting on Numerical Semigroups, Porto 2008

2 Problems - 3 Objects

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Classification of plane curves

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- Γ - The semigroup of a plane curve
- Δ - The Delorme Module of a plane curve

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- Γ_L - The semigroup of a Legendrian curve

Classification of (irreducible germs of) plane curves

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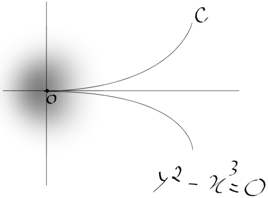

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$C = f^{-1}(0)$

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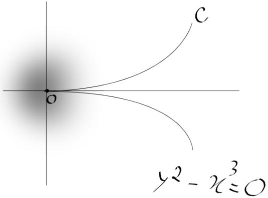

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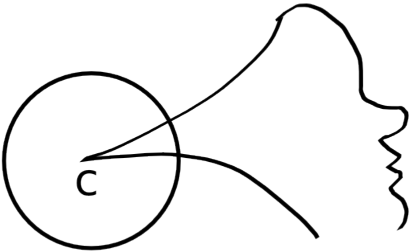
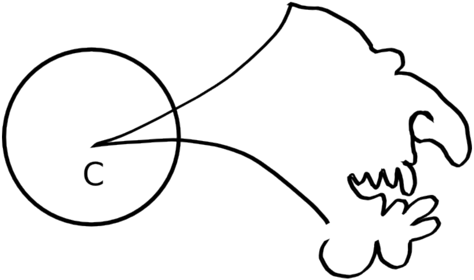
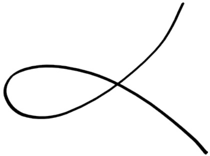
But we only care about irreducible **germs** around the origin.

Germ around the origin





Let $f(x, y) = y^2 - x^2(x + 1)$
Is f irreducible?



(1)

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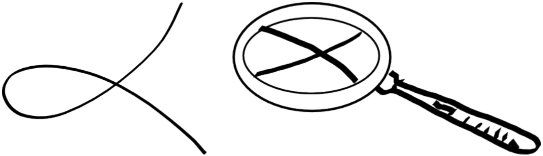
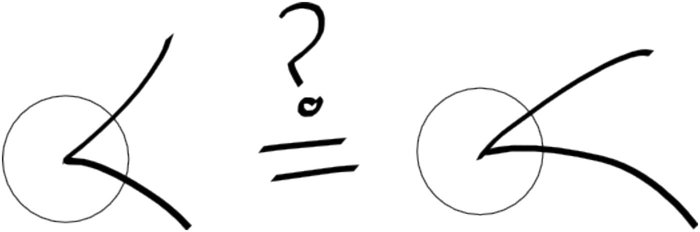


(1)

Yes, in $\mathbb{C}[x, y]$, but...
The ring of power series is a magnifying glass
$f = (y - x\sqrt{x+1})(y + x\sqrt{x+1})$
So, for us, "plane curve" means $f^{-1}(0) \in \mathbb{C}\{x, y\}$

When are they the same?

We'd like to know:
Given two (irreducible germs of) curves, C_1 and C_2 when is there an analytic isomorphism F of $\mathbb{C}^2$ such that $F(C_1) = C_2$ (as germs)?



Topological type of a curve

But first we ask:
Given to (irreducible germs of) curves, C_1 and C_2 when is there an homeomorphism F of $\mathbb{C}^2$ such that $F(C_1) = C_2$?

Puiseux Expansion

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It follows that we can always find a parametrization of C

$$t \mapsto (t^n, t^m + \sum_{i>m} a_i t^i), a_i \in \mathbb{C}$$

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example:
If $C = \{(x, y) : f(x, y) = y^2 - x^3 = 0\}$ then $Y(x) = x^{3/2}$ is the rational power series expansion of the curve and

$$C = \begin{cases} x(t) = t^2 \\ y(t) = t^3 \end{cases}$$

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in general:

$$Y(x) = \sum_{k\geq 1} a_{1,k} x^{(\frac{m_1}{n_1})k} + \sum_{k\geq 1} a_{2,k} x^{(\frac{m_2}{n_1 n_2})k} + \ldots + \sum_{k\geq 1} a_{r,k} x^{(\frac{m_r}{n_1 n_2 \cdots n_r})k}$$

The special exponents $\frac{n_i}{m_i}$ are topological invariants.
The $(n_1, m_1), \ldots, (n_r, m_r)$ are called the **Puiseux pairs** of the curve.

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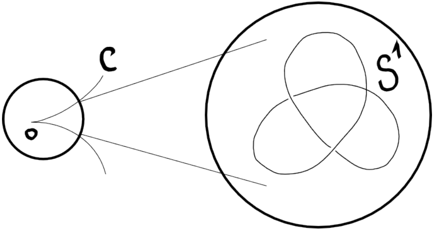
Example:
$C = y^2 - x^3 = 0$ defines $Y(x) = x^{3/2}$ has only one Puiseux pair which is $(2, 3)$.

Intersecting a curve with a small sphere gives a knot.
example:
The intersection of $y^2 - x^3 = 0$ with a small sphere gives the treefoil knot.



The Puiseux pairs determine the knot. So the Puiseux pairs define the curve germ up to homeomorphism.

Some Algebraic Tools

Let $C = f^{-1}(0)$. Let $(x(t) = t^n, y(t) = t^m + O(t^{m+1}))$ be a parametrization of C.
Let $\mathcal{O}(C) = \mathbb{C}\{x, y\}/(f)$. $\mathcal{O}(C)$ is called the ring of the curve. It can be shown that $\mathcal{O}(C) = \mathbb{C}\{t\}/(x(t), y(t))$.
There is a natural valuation on $\mathbb{C}\{t\}$, given by

$$\begin{array}{rcl} \mathbb{C}\{t\} & \rightarrow & \mathbb{N} \\ \sum_{i\geq 0} a_i t^i & \mapsto & inf\{i : a_i \neq 0\} \end{array}$$

which induces a valuation on the ring of the curve

$$\begin{array}{rcccl} \mathcal{O}(C) & \hookrightarrow & \mathbb{C}\{t\} & \rightarrow & \mathbb{N} \\ g(x, y) & \mapsto & g(x(t), y(t)) & \mapsto & v(g(x(t), y(t))) \end{array}$$

The Semigroup of a plane curve

$v(\mathcal{O}(C))$ is a subsemigroup of $\mathbb{N}$. We call it the semigroup of the plane curve C.

The semigroup of a curve is a fundamental topological invariant.

It can be shown that the semigroup determines the knot of the curve, and therefore, the topological type.

But that is not all.

Analytical Classification

From now on assume there is only one Puiseux pair, (n, m).
($m > n$, gcd(n,m)=1)
Suppose we are given two curves with parametrizations of the type

$$C = \begin{cases} x(t) &= t^n \\ y(t) &= t^m + \sum_{i>m} a_i t^i \end{cases}$$

differing only on the values of the a_i. How do we know if they differ by an analytic isomorphism?

Zariski(1970)

If we act on $\mathbb{C}^2$ with an isomorphism of the type

$$\left\{ \begin{array}{rcl} x' & = & x \\ y' & = & y - \theta x^i y^j \end{array} \right.$$

we obtain new parametrization of C with $x(t)$ unchanged and

$$y'(t) = y(t) - \theta t^{v(x^i y^j)} + O(v(x^i y^j + 1))$$

and therefore by choosing θ adequately we can cut the term of order $v(x^i y^j)$.

By iterating we can cut from the parametrization any term t^i with $i \in \Gamma(C)$.

We say that a curve is in the plane short form if

$$C = \left\{ \begin{array}{rcl} x(t) & = & t^n \\ y(t) & = & t^m + \sum_{i>m} a_i t^i \end{array} \right.$$

is such that $a_i = 0$ for all $i \in \Gamma(C)$

The short form is finite.
There is a positive integer **c**, the conductor of the curve, such that for any $i > c$, i is the valuation of some $g \in \mathcal{O}(C)$.
For a curve with a single Puiseux pair (n, m), we have
$c = (n-1)(m-1)$
So the moduli space of curves can be seen as a $\mathbb{C}^{c-m-1}_{(a_{m+1},\dots,a_{c-1})}$ modulo a certain equivalence relation.

Delorme (1978)

The action of

$$D(\alpha,\beta) = \begin{cases} x' &= x + \alpha(x,y) \qquad v(\alpha) > v(x) \\ y' &= y + \beta(x,y) \qquad v(\beta) > v(y) \end{cases}$$

over $\mathbb{C}\{t\}$ is

$$y'(t) = y(t) + \beta(t) - p\alpha(t) + O(t^{2v(\alpha)-2n+m}).$$

where $p = (dy/dx) = \frac{dy}{dt}/\frac{dx}{dt} = t^{m-n} + \cdots$.
Consider the $\mathcal{O}(C)$-module $\mathcal{D} = \mathcal{O}(C) + p\mathcal{O}(C)$. We call it Delorme's module. We consider the set of valuations $v(\mathcal{D})$. It is not a semigroup like Γ, it is just a finitely generated Γ-set. The action of $D(\alpha,\beta)$ above shows that we can cut all powers of t with valuations in $v(\mathcal{D})$ as long as we can control the error $O(t^{2v(\alpha)-2n+m})$. Let $l(i)$ be the valuation of the maximum valuation α in $\mathcal{O}(C)$ such that $v(\beta - p\alpha) = i$ for some β. If $i < 2l(i) - 2n + m$ then we can cut i.

generic stratum

Unlike the semigroup, $v(\mathcal{D})$ depends on the coefficients of the parametrization. Example: Take

$$C = \begin{cases} x(t) & = & t^5 \\ y(t) & = & t^{11} + \sum_{i>11} a_i t^i \end{cases}$$

We have $p = (11/5)t^6 + (12/5)a_{12}t^7 + \dots$ So
$y - (5/11)px = a_{12}t^{12} + \cdots$ has valuation 12 iff $a_{12} \neq 0$.
$12 \in v((D))$ if and only if this is so.
In general, $v(\mathcal{D})$ depends on a number of such equations on the coefficients a_i but is constant on a Zariski open set of the space of coeficients (set all such equations to be different from zero), and, for that generic value of $\mathcal{D}$, the error can be controlled everytime. Therefore, in the generic stratum, we can cut every t^i with $i \in v(\mathcal{D})$ (except for $inf\{i : i \in v(\mathcal{D})\backslash\Gamma\}$, for special reasons that I don't have time to go into - just take it on faith).

example

$$C = \begin{cases} x(t) & = & t^5 \\ y(t) & = & t^{11} + \sum_{i>11} a_i t^i \end{cases}$$

	0	1	2	3	4
5	x				
10	.	y			
15	.	.			
20	.	.	y^2		
25	.	.	.		
30	.	.	.	y^3	
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$$C = \begin{cases} x(t) & = & t^5 \\ y(t) & = & t^{11} + a'_{12}t^{12} + a'_{13}t^{13} + a'_{14}t^{14} + a'_{19}t^{19} \end{cases}$$

what more is there?

Definition: A homothety is an action of $\mathbb{C}^*$ over $\mathbb{C}^2$ defined by

$$(x, y) \mapsto (\theta^n x, \theta^m y) \text{ for a } \theta \in \mathbb{C}^*$$

Theorem: Two generic curves are equivalent iff their Delorme ultrashort forms differ by a homothety.

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So take our example once more:

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Supposing $a'_{12} \neq 0$. Applying a homothety has the action of (anisotropically) projectivizing the space of the a_i. We obtain $a'_{m+i} = a_{m+i}/\theta^i$ and the moduli space of the curves forms a weighted projective space of dimension 3.

Contact Geometry

M complex variety of dimension $2n+1$. $U \subset M$ open set, $\omega \in \Omega^1(U)$ is a contact form if $\omega \wedge (d\omega)^n \neq 0 \, \forall p \in U$ $Ker(\omega_p) \subset T_pM$ is called a contact hyperplane. A contact structure $\mathcal{L}$ is M with a maximal cover by equivalent contact forms. A contact morphism is a morphism between two contact varieties that preserves the contact structures, i.e., $\phi : (M_1, \omega_1) \to (M_2, \omega_2)$ is such that $\phi^* \omega_2 = \omega_1$ A Legendrian curve is a curve in a contact variety (M, ω) such that $\omega(T_pC) = 0 \, \forall p \in C_{\text{reg}}$

Take $n = 1$. Standard example of a dimension 3 contact variety is $\mathbb{P}^*\mathbb{C}^2$, the projectivization of the cotangent bundle.

$$\mathbb{C}^2_{x,y}, \mathbb{T}^*\mathbb{C}^2 \cong \mathbb{C}_{x,y,\xi,\eta}$$

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$$\mathbb{C}^2_{x,y}, \mathbb{T}^*\mathbb{C}^2 \cong \mathbb{C}_{x,y,\xi,\eta}$$

$$\mathbb{P}^*\mathcal{C}^2 = ((\mathbb{T}^*\mathbb{C}^2 - \mathbb{C}^2)/\mathbb{C}^*, \langle \theta = \xi dx + \eta dy \rangle)$$

Locally ($p = \xi/\eta, \eta \neq 0$):

$$(M, \mathcal{L}) = ((\mathbb{C}^3_{(x,y,p)}), \omega = dy - pdx) \tag{2}$$

Darboux: It is always like this

Problem: Classify Legendrian curves modulo Contact isomorphisms

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Why, Oh, why?

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1-It's a natural problem (meaning, why not?)

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- Better than a real job

HOW?

Surprise! Legendrian curves are plane curves (they just don't know it)

Conormal of a plane curve:

Take $f \in \mathbb{C}^2\{x, y\}$, $C = \{f(x, y) = 0\}$, parametrized by

$$t \mapsto (x = t^n, y = t^m + \sum_i a_i t^i)$$

Apply the Gauss map:

$$\begin{array}{rcl} C_{\mathrm{reg}} & \rightarrow & \mathbb{P}^1 \\ p & \mapsto & \langle df(p) \rangle \end{array}$$

The image is the conormal of C, $\mathbb{P}^*_C\mathbb{C}^2 \hookrightarrow \mathbb{P}^*\mathbb{C}^2$, and is a Legendrian curve. Moreover it has a parametrization given by

$$(x(t) = t^n, y(t) = t^m + \sum_i a_i t^i, p(t) = dy/dx = t^{m-n} + \sum_i (i/n) a_i t^i)$$

All Legendrian curves are conormals of plane curves (up to isomorphism)

Theorem: Modulo an adequate contact isomorphism, a legendrian curve in $(\mathbb{C}_{x,y,p}, dy - pdx)$ has a projection onto $\mathbb{C}_{x,y}$ which is a plane curve. Moreover the legendrian curve is the conormal of that plane curve projection.

Definition: We say that the equissingularity class of a Legendrian curve is the topological type of its (generic) plane curve projection.

Problem:

Fix a topological type (n, m). Let $\mathcal{G}$ be the group of contact isomorphisms. Let $L(n, m)$ be the set of legendrian curves with topological type (n, m). Classify $L(n, m)$ modulo $\mathcal{G}$.

Important: For legendrian curves we can always assume $m \geq 2n + 1$ (modulo a contact transformation).

Lifts of plane isomorphisms

Plane isomorphisms are Legendrian! (just they didn't know it - but you guessed it by now)
For every plane isomorphism

$$P(\alpha,\beta) = \begin{cases} x' &= x+\alpha(x,y) \qquad v(\alpha) > v(x) \\ y' &= y+\beta(x,y) \qquad v(\beta) > v(y) \end{cases}$$

there is a lift to a contact isomorphism, that acts in the same way on the coordinates (x,y). We can find a $\gamma \in \mathbb{C}\{x,y,p\}$ with $v(\gamma) > v(p)$ and such that

$$\text{lift}(P)(\alpha,\beta) = \begin{cases} x' &= x+\alpha(x,y) \qquad v(\alpha) > v(x) \\ y' &= y+\beta(x,y) \qquad v(\beta) > v(y) \\ p' &= p+\gamma(x,y,p) \qquad v(\gamma) > v(p) \end{cases}$$

preserves the contact 1-form, $\omega = dy - pdx$

short form

now, a generic legendrian transformation looks like this

$$\Phi(\alpha, \beta, \gamma) = \begin{cases} x' = x + \alpha(x, y, p) & v(\alpha) > v(x) \\ y' = y + \beta(x, y, p) & v(\beta) > v(y) \\ p' = p + \gamma(x, y, p) & v(\gamma) > v(p) \end{cases}$$

and its action on the parametrization is

$$y'(t) = y(t) + \beta(t) - p\alpha(t) + O(t^{2v(\alpha)-2n+m}).$$

Which looks like the action on the plane except that now α and β have a third coordinate p and belong to the ring $\mathbb{C}\{x(t, y(t), p(t))\}$.

For lifts of plane morphisms the action is exactly the same! So we can cut all elements in $\Gamma(n, m)$ and put the curve into the plane short form. So, again, the equivalence classes of legendrian curves have finite dimension and are a quotient of a $\mathbb{C}^{m-n-1}_{(a_{m+1},\dots,a_{c-1})}$

Some definitions:

given a curve $\mathbb{P}_C\mathbb{C}^2$ in $L(n, m)$ with parametrization $(x(y), y(t), p(t))$ we define

Ring of the legendrian curve:

$\mathcal{O}(\mathbb{P}_C\mathbb{C}^2) = \mathbb{C}\{x(t), y(t), p(t)\}$

Semigroup of the legendrian curve:

$\Gamma_L(\mathbb{P}_C\mathbb{C}^2) = v(\mathcal{O}(\mathbb{P}_C\mathbb{C}^2))$

Now, here's a thought:

In the plane, we could cut the semigroup, that is, $v(\mathcal{O}(C))$.
But the big star of the show was $v(\mathcal{O}(C) + p\mathcal{O}(C))$.
In the legendrian case, p is identified with a coordinate, and the ring of the curve is

$$\mathcal{O}(\mathbb{P}_C\mathbb{C}^2) = \mathcal{O}(C) + p\mathcal{O}(C) + p^2\mathcal{O}(C) + p^3\mathcal{O}(C) + \dots$$

We suspect that we can now cut all these valuations and furthermore that the legendrian case is neater: we conjecture that the Legendrian Semigroup classifies Legendrian curves completely.

can we do it?

It depends on what morphisms we have available. It turns out that we have enough.
Theorem: We can choose $\alpha(x, y, p)$ and $\beta(x, y, p)$ freely (minus some technicalities) and we can always find an adequate $\gamma(x, y, p)$ that makes

$$\Phi(\alpha, \beta, \gamma) = \begin{cases} x' & = & x + \alpha(x, y, p) & v(\alpha) > v(x) \\ y' & = & y + \beta(x, y, p) & v(\beta) > v(y) \\ p' & = & p + \gamma(x, y, p) & v(\gamma) > v(p) \end{cases}$$

a contact isomorphism.
Proof: Substitute Φ into $dy - pdx$, solve a bunch of PDE's, prove existence and uniqueness by Cauchy-Kowalevski. (just trust me, it works)

Still one problem

So we can put into α and β whatever is available in the ring of our curve. That means we could use the action

$$y'(t) = y(t) + \beta(t) - p\alpha(t) + O(t^{2v(\alpha)-2n+m}).$$

to cut every valuation in the legendrian semigroup. Except that we have to control the error term. Can we do it? Depends on what semigroup we have.

Unlike the plane semigroup $\Gamma(n, m)$, the legendrian semigroup depends upon the values of the coefficients a_i of the parametrization (just like Delorme's module).
The value of the semigroup depends upon a set of polynomial equations $g_i(a)$ obtained by Gauss elimination on the ring of the curve, seen as an infinite matrix with coefficients in $\mathbb{C}[a_{m+1}, \ldots, a_{c-1}]$. We can set $\prod g_i \neq 0$ and we get a Zariski open set of $\mathbb{C}^{c-m-1}_{(a_{m+1},\ldots,a_{c-1})}$ where the semigroup is a constant $\Gamma_L(n, m)$. We call $\Gamma_L(n, m)$ the **generic semigroup** of equissingularity (n, m).

generic semigroup

Theorem: The generic semigroup of type (n, m) can be obtained numerically by the following iterative construction:

Let $\sharp_0(r) = \sharp\{(i, j, k) : ni + mj + k(m - n) = r\}$.

Suppose $\sharp_q(.)$ is known. Let $s = sup\{i : \sharp_q(i) > 1\}$. Set

$$\begin{aligned} \sharp_{q+1}(s) &= \sharp_q(s) - 1 \\ \sharp_{q+1}(s+1) &= \sharp_q(s+1) + 1 \\ \sharp_{q+1}(p) &= \sharp_q(p) \text{ for all } p \neq s, s+1 \end{aligned}$$

Repeat until you reach a Q such that all $\sharp_Q(i) \leq 1$ for all $i < c$. Set

$$\Gamma_L(n, m) = \{i : \sharp_Q(i) \neq 0\}$$

What it means

Take the case $(5, 11)$ again.

$$\mathbb{P}^*_C \mathbb{C}^2 = (x(t) = t^5, y(t) = t^{11} + \sum_{i>11} a_i t^i, p(t) = (11/5)t^6 + \dots)$$

	0	1	2	3	4
5	x	p			
10	.	y	p^2		
15	.	.	yp	p^3	
20	.	.	y^2	yp^2	p^4
25	.	.	.	y^2p	p^3y

....and count monomials for each i (that's the meaning of $\sharp_0(i)$)

						*					*	*		
*	*				*	*	*			*	*	*	*	
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

What it means

						*					*	*		
*	*				*	*	*			*	*	*	*	
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

What it means

						*					*		*	
*	*				*	*	*			*	*	*	*	
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

What it means

						*					*			
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What it means

						*						*		
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5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

What it means

						*							*	
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What it means

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What it means

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What it means

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What it means

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5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

What it means

*	*				*	*	*	*		*	*	*	*	*
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

What it means

*	*				*	*	*	*		*	*	*	*	*
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

By the way, what's with the colours?

What it means

*	*				*	*	*	*		*	*	*	*	*
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

By the way, what's with the colours?

We have implicitly defined (maximal trajectories):

$\{16, ..., \text{inf}\}$, $\{11, 12, 13\}$
The theorem that states this form for the generic semigroup is proved by showing that the matrices corresponding to maximal trajectories have well-behaved minors - that means the trajectories have no jumps.

What it means

*	*				*	*	*	*		*	*	*	*	*
5	6	7	8	9	10	11	12	13	14	15	16	17	18	19

Trajectories are crucial for cutting valuations on the parametrization.

In order to cut a valuation from the parametrization, we may be forced to use monomials in α, β that have valuation in the trajectory that contains the valuation we want to cut.

The larger the trajectory the greater the possibility that α has a low valuation and the action $\beta - p\alpha$ has a valuation lower than the error $2v(\alpha) - 2n + m$

In fact you could say this theorem's importance is that it states that in the generic semigroup trajectories are as small as you could hope for. But are they small enough?

Main Theorem: Hell, yeah!

Main Theorem: Hell, yeah!

In a generic semigroup trajectories are small enough that you can control the error.
Therefore you can cut the whole legendrian semigroup (except for $inf(\Gamma_L \backslash \Gamma)$.

Proof?

Two main reasons:

1) If a (contiguous) trajectory gets too big it will end the semigroup (and then you can cut its elements by simple monomial transformations)

2) In legendrian curves, $m \geq 2n + 1$.

Classification

Definition: A legendrian curve C of type (n, m) with generic semigroup and parametrization

$$\mathbb{P}^*_C\mathbb{C}^2 = (x(t) = t^n, y(t) = t^m + \sum_{i>m} a_i t^i, p(t) = dy/dx)$$

is said to be in legendrian short form if $a_i = 0$ for all $i \in \Gamma_L(n, m)$

Theorem: Two generic legendrian curves of type (n, m) are equivalent iff their legendrian short forms differ by a homothety.

A final look at our example:

	0	1	2	3	4
5	x	p			
10	.	y	p^2	*	
15	.	.	yp	p^3	*
20	.	.	y^2	yp^2	p^4
25	.	.	.	y^2p	p^3y