ALMOST SYMMETRIC NUMERICAL SEMIGROUPS

Valentina Barucci
Dipartimento di Matematica
Sapienza, Università di Roma

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In this talk are mentioned notions and results due to Herzog, Kunz, Jäger, Heinzer, Huckaba, Papick, Fröberg, B., ...

We fix for all the section the following notation. S is a *numerical semigroup*, i.e. a subsemigroup of $\mathbb{N}$, with zero and with finite complement in $\mathbb{N}$. $M = S \setminus \{0\}$ is the *maximal ideal* of S, e is the *multiplicity* of S, that is the smallest positive integer of S, g is the *Frobenius number* of S, that is the greatest integer which does not belong to S. Moreover we set $n = \text{Card}(\{s \in S \mid s < g\})$.

A *relative ideal* of S is a nonempty subset I of $\mathbb{Z}$ (which is the quotient group of S) such that $I + S \subseteq I$ and $I + s \subseteq S$, for some $s \in S$. A relative ideal which is contained in S is an *integral ideal* of S.

If I, J are relative ideals of S, then the following are relative ideals too:

$$I + J = \{i + j;\ i \in I, j \in J\}$$

$$I - J = \{z \in \mathbb{Z} \mid z + J \subseteq I\}$$

$$I \cap J$$

$$I \cup J$$

Moreover the ideal generated by $z_1, \ldots, z_h$ is

$$(z_1 + S) \cup \cdots \cup (z_h + S)$$

We have also

$$I \subseteq S-(S-I) = \bigcap_{I \subseteq z+S} (z+S)$$

and if the equality holds I is *bidual*. More generally

$$I \subseteq J-(J-I) = \bigcap_{I \subseteq z+J} (z+J)$$

If I is a relative ideal of S, then $\mathrm{Ap}_e(I) = \mathrm{Ap}(I) = I \setminus (e+I)$ is the set of the e smallest elements in I in the e congruence classes mod e and is called the *Apery set* of I (with respect to e). In particular $\mathrm{Ap}(S)$ is the Apery set of S. Since g is the greatest gap of S, $g+e$ is the largest element in $\mathrm{Ap}(S)$.

The following Lemma, corresponding to Nakayama's Lemma for local rings, is very easy to prove for numerical semigroups:

Lemma 1 *If I is a relative ideal of S, then the unique minimal set of generators of I is $I \setminus (M + I)$.*

Since $e + I \subseteq M + I$, then $I \setminus (M + I) \subseteq I \setminus (e + I) = \mathrm{Ap}_e(I)$ and by Lemma 1 each relative ideal I of S needs at most e generators.

In particular $M \setminus 2M$ is the minimal set of generators of M and its cardinality is the *embedding dimension* ν of S. We have $\nu \leq e$ and if equality holds the semigroup S is called of *maximal embedding dimension*.

Problem The following problem was asked by Wilf and is still unsolved

$$g + 1 \le n\nu \text{ ?}$$

A particular relative ideal of S plays a special role. It is the *canonical ideal* $\Omega = \{g - x \mid x \in \mathbb{Z} \setminus S\}$. So

$$x \notin S \Rightarrow g - x \in \Omega$$

$$x \in S \Rightarrow g - x \notin \Omega$$

Of course it is $S \subseteq \Omega \subseteq \mathbb{N}$ ($s \in S$ then $x = g - s \notin S$, so $s = g - (g - s) = g - x \in \Omega$).

Proposition 2 (I. Serban) *If* $Ap_e(S) = \{p_0, \ldots, p_{e-1}\}$, *then* $Ap_e(\Omega) = \{p_{e-1} - p_{e-1}, p_{e-1} - p_{e-2} \cdots, p_{e-1} - p_0\}$.

Proof. Recall that $g + e = p_{e-1}$.

We have $p_i \in \mathrm{Ap}_e(S) = S \setminus (e + S)$. Thus

$$p_i \in S \;\Rightarrow\; g - p_i \notin \Omega \;\Rightarrow\; g + e - p_i = p_{e-1} - p_i \notin \Omega + e$$

On the other hand

$$p_i \notin S + e \;\Rightarrow\; p_i - e \notin S \;\Rightarrow\; g - (p_i + e) = g + e - p_i \in \Omega$$

Proposition 3 *1)* $\Omega - (\Omega - I) = I$*, for each ideal* I *of* S*.*

In particular $\Omega - (\Omega - S) = \Omega - \Omega = S$

2) $\Omega - (I \cap J) = (\Omega - I) \cup (\Omega - J)$

3) Ω *is an irreducible relative ideal, i.e.* Ω *is not the intersection of any set of relative ideals properly containing* Ω*.*

Proof. 2) Since

$$(\Omega - I) \;\subseteq\; (\Omega - I) \cup (\Omega - J) = H$$

then

$$(\Omega - H) \;\subseteq\; \Omega - (\Omega - I) = I$$

In the same way

$$(\Omega - H) \subseteq J$$

Thus

$$\Omega - H \subseteq I \cap J$$

hence

$$\Omega - (I \cap J) \subseteq \Omega - (\Omega - H) = H$$

The other inclusion is trivial.

3) If $\Omega = I \cap J$, then

$$S = \Omega - \Omega = \Omega - (I \cap J) = (\Omega - I) \cup (\Omega - J)$$

Thus

$$\Omega - I = S \quad \text{or} \quad \Omega - J = S$$

and so

$$I = \Omega - (\Omega - I) = \Omega - S = \Omega \ \text{ or } \ J = \Omega - (\Omega - J) = \Omega - S = \Omega$$

The properties 1) and 3) characterize Ω in the following sense:

Proposition 4 *Let Ω' be a relative ideal of S. Then*

$$\Omega' - (\Omega' - I) = I$$

for each ideal I of S if and only if $\Omega' = z + \Omega$, for some $z \in \mathbb{Z}$.

Proof. Since

$$\Omega' - (\Omega' - \Omega) = \Omega = \bigcap_{\Omega \subseteq z + \Omega'} (z + \Omega')$$

and Ω is irreducible, we have $\Omega = z + \Omega'$. The converse is trivial.

Proposition 5 *Let* Ω' *be a relative ideal of* S*. Then* Ω' *is irreducible if and only if* $\Omega' = z + \Omega$*, for some* $z \in \mathbb{Z}$*.*

Proof. Since

$$\Omega' = \Omega - (\Omega - \Omega') = \bigcap_{\Omega' \subseteq z + \Omega} (z + \Omega)$$

and Ω' is irreducible, we have $\Omega' = z + \Omega$. The converse is trivial.

The *type* t of a numerical semigroup S is the minimal number of generators of the canonical ideal, that is $t = \mathrm{Card}(\Omega \setminus (\Omega + M))$.

Proposition 6 *For the type t of S the following holds:*

1. $t = \mathrm{Card}((S - M) \setminus S)$

2. $S = (h_1 + \Omega) \cap \cdots \cap (h_t + \Omega)$, *for some* $h_i \in \mathbb{Z}$, *where the intersection is irredundant.*

3. *Each relative principal ideal of S is an irredundant intersection of t irreducible relative ideals.*

4. *Each relative ideal is a finite intersection of irreducible relative ideals.*

5. $1 \leq t \leq e-1$.

Proof. 1. $t = \mathrm{Card}(\Omega \setminus (\Omega + M))$

$$= \mathrm{Card}(\Omega - (\Omega + M) \setminus (\Omega - \Omega))$$

$$= \mathrm{Card}((\Omega - \Omega) - M \setminus S) = \mathrm{Card}((S - M) \setminus S)$$

2. Suppose that Ω is minimally generated by $z_1, \ldots, z_t$. Then

$$\Omega - S = \Omega = \bigcup_{z_i=1}^{t} (z_i + S) = \bigcup_{z_i=1}^{t} (z_i + (\Omega - \Omega))$$

$$= \bigcup_{z_i=1}^{t} (\Omega - (\Omega - z_i)) = \Omega - \bigcap_{z_i=1}^{t} (\Omega - z_i)$$

So that, dualizing again

$$S = (\Omega - (\Omega - S)) = \bigcap_{z_i=1}^{t} (\Omega - z_i)$$

Moreover the intersection is irredundant.

3. Easy by 2.

Fact The type t of a semigroup S is at the same time the number of components of an irredundant intersection of a principal ideal in irreducible relative ideals and in irreducible integral ideals.

The two notions are really different, in fact:

I irreducible as relative id. $\Rightarrow$ I irreducible as integral id.

but the converse does not hold. There are in general many irreducible integral ideals . On the contrary, as we saw, there is essentially only one irreducible relative ideal, the canonical ideal Ω and all those of the form $x+\Omega$, $x \in \mathbb{Z}$.

Example Let $S=<3,8,10>=\{0,3,6,8,\rightarrow\}$. The canonical ideal is

$$\Omega=\{0,2,3,5,6,8,\rightarrow\}$$

So each irreducible relative ideal of S is of the form $x+\Omega$. On the other hand, consider the partial order on S given by

$$s_1 \leq s_2 \ \Leftrightarrow s_1+s_3=s_2, \quad \text{for some} \quad s_3 \in S$$

and , if $s \in S$, set $B(s)=\{z \in S \mid z \leq s\}$. It turns out (cf. R., G.S., G.G.) that $I=S \setminus B(s)$ is an irreducible integral semigroup ideal of S (in fact each integral semigroup ideal containing I contains s) and is not necessarily of the form $x+\Omega$. For example, for $s=12$, we get

$$I=\{8,10,11,13,\rightarrow\}$$

If $t = 1$ or equivalently $\Omega = S$, then the numerical semigroup S is classically called *symmetric*.

S is *almost symmetric* if $\Omega \subseteq S - M$, equivalently if $t = g + 2 - 2n$.

Example (see board)

$$S = \langle 5, 8, 9, 12 \rangle = \{0, 5, 8, 9, 10, 12, \rightarrow\}$$

$$\Omega = \{0, 4, 5, 7, 8, 9, 10, 12, \rightarrow\} = \langle 0, 4, 7 \rangle$$

$$S - M = \{0, 4, 5, 7, \rightarrow\}$$

Theorem 7 (B.) *The following conditions are equivalent for a numerical semigroup S of maximal ideal M:*

1. *S is almost symmetric.*

2. *Each ideal of $M - M$ is bidual as ideal of S.*

3. *$M - e$ is the canonical ideal of $M - M$*

Proof. If $S = \mathbb{N}$, the three conditions trivially hold. Suppose $S \subsetneq \mathbb{N}$, so that $M - M = S - M$ is a semigroup which properly contains S.

2. $\Rightarrow$ 3. We have to prove that $M-(M-I) \subseteq I$, for each ideal I of $M-M$. First notice that

$$M - I = S - I$$

If $M-I \subsetneq S-I$, then $z+I \subseteq S$ and $z+I \not\subseteq M$ for some $z \in \mathbb{Z}$ thus $z+I = S$ and S is an ideal of $M-M$, a contradiction. Thus

$$M-(M-I) \subseteq S-(M-I) = S-(S-I) = I$$

3. $\Rightarrow$ 2. Each relative ideal I of $M-M$ is a finite intersection of irreducible relative ideals, i.e. of ideals of the form $x+M$, because $M-e$ is the canonical ideal of $M-M$. Since M is bidual in S (in fact $S-(S-M) \subsetneq S$ is equal to M) also I, as intersection of bidual ideals, is bidual.

Theorem 8 (M. La Valle) *If S is an almost symmetric semigroup, then*

$$g + 1 \leq n\nu$$

Proof. Since S is almost symmetric, $t = g + 2 - 2n$, thus $n = \frac{g-t}{2} + 1$. Thus the Wilf inequality becomes

$$g + 1 \leq (\frac{g - t}{2} + 1)\nu$$

or equivalently

$$2 + \nu t \leq (\nu - 2)g + 2\nu$$

It is enough to prove

$$\nu t \leq (\nu - 2)g$$

equivalently

$$\nu(g-t) \geq 2g$$

Since $t < e$, it is enough to prove

$$\nu(g-e) \geq 2g \quad \text{that is} \quad \nu \geq \frac{2g}{g-e}$$

Case 1. If $e < g/2$, then $g-e > g/2$ and

$$\frac{2g}{g-e} < \frac{2g}{g/2} = 4$$

Since, for $\nu \leq 3$, Wilf's inequality is true, in this case the theorem is proved.

Case 2. if $e = g+1$, it is easy: $\nu = e$ and $n = 1$ thus $g+1 = e \leq 1e$

Case 3. If $g/2 < e < g$, we have to compute the number ν of (necessary) generators of S.

We have at least:

$n - 1$ generators smaller than g

$2e - (g + 1)$ generators between $g + 1$ and $2e$

$(e + s) - 2e - 1$ generators between $2e$ and $e + s$, where $S = \{s_0 = 0, s_1 = e, s_2 = s, \dots (s_i < s_{i+1})\}$. Therefore:

$\nu \geq (n - 1) + (2e - g - 1) + (s - e - 1) = n + e + s - g - 3$

and

$$\nu n \geq n^2 + ne + ns - ng - 3n = f(n)$$

We claim that

$$f(n) \geq g+1$$

By induction on $n \geq 3$. For $n = 3$, it is

$$f(n) = 3(e+s-g)$$

But for $n = 3$ an almost symmetric semigroup is of one of the following forms:

$$S = \{0, g-3, g-1, g+1, \rightarrow\}$$

$$S = \{0, g-2, g-1, g+1, \rightarrow\}$$

In the first case

$$f(3) = 3g - 12 \geq g+1 \Leftrightarrow g \geq 7$$

In the second case

$$f(3) = 3g - 9 \geq g + 1 \Leftrightarrow g \geq 5$$

Thus, taking care of a finite small number of easy cases ($g < 7$ in the first case and $g < 5$ in the second case) the inductive hypothesis is verified.

Now the inductive step:

$$f(n+1) - f(n) = (2n-2) + (s + e - g) > 0$$

So if $f(n) > g + 1$, then also $f(n+1) > g + 1$.