Proportionally Modular Diophantine Inequalities and its Multiplicity

J.C. Rosales, M.B. Branco and P. Vasco

(Universidad de Granada, E-18071 Granada, Spain, Universidade de Évora, 7000 Évora and Universidade de Trás-os-Montes e Alto Douro, 5001-801 Vila Real, Portugal)

Main characters

A numerical semigroup S is a submonoid of $\mathbb{N}$ such that $g.c.d.(S) = 1$

• $m(S)$ the smallest element in S is the multiplicity of S.

• $H(S) = \mathbb{N} \setminus S$ is finite its elements are the gaps of S and its cardinality is the singularity degree of S.

$g(S) = \max(\mathbb{N} \setminus S)$,
the Frobenius number of S

The aim

- Study of the C-semigroups.
- Characterize the intervals of positive rational numbers I, subject to the condition that $S(I)$ has the multiplicity m.

• A proportional modular Diophantine inequality is an expression

$ax \bmod b \leq cx$ with $a, b, c \in \mathbb{Z}^+$

We prove that

$S(a, b, c) = \{x \in \mathbb{N} \ : \ ax \bmod b \leq cx\}$ is a numerical semigroup.

This semigroups are called proportionally modular numerical semigroups

If we consider

$S(I) = T \cap \mathbb{N}$ where T is the additive submonoid of $\mathbb{Q}_0^+$ generated by the interval I.

We obtain that
$S(I)$ is a numerical semigroup.

We have the following results

- Let a, b, c be a positive integers such that $c < a < b$. Then

  $\{x \in N \ : \ ax \text{ mod } b \leq cx\} = T \cap \mathbb{N}$

  where $T =< [\frac{b}{a}, \frac{b}{a-c}] >$.

  (i. e. $S(a, b, c) = S[\frac{b}{a}, \frac{b}{a-c}]$).

- Conversely let $\frac{b_1}{a_1} < \frac{b_2}{a_2}$ with a_1, a_2, b_1, b_2 a positive integers and $T =< [\frac{b_1}{a_1}, \frac{b_2}{a_2}] >$. Then

  $T \cap \mathbb{N} = \{x \in N \ : \ a_2 b_1 x \text{ mod } a_1 a_2 \leq (a_2 b_1 - a_1 b_2) x\}$.

  (i.e. $S[\frac{b_1}{a_1}, \frac{b_2}{a_2}] = S(a_1 b_2, b_1 b_2, a_1 b_2 - a_2 b_1)$).

- If I is an interval of positive rational numbers (closed or open interval) $\Rightarrow$ S(I) is a proportionally modular semigroup.

In next result establish that, numerical semigroup $< b_1, b_2 >$ is a modular numerical semigroup.

• Let b_1, b_2, a_1, a_2 positive integers such that $a_1b_2 - a_2b_1 = 1$. Then

$$< b_1, b_2 >= \{x \in \mathbb{N} \ : \ a_1b_2x \bmod b_1b_2 \leq x \ \}.$$

in view of the results above, with $a_1b_2 - a_2b_1 = 1$, we have that

$$S[\frac{b_1}{a_1}, \frac{b_2}{a_2}] =< b_1, b_2 >.$$

• We define C-semigroup as a numerical semigroup such that $S = S]\frac{b_1}{a_1}, \frac{b_2}{a_2}[$ where $a_1 < b_1$ $a_1b_2 - a_2b_1 = 1$.

In next we have the relation between a C-semigroup generated by the open interval and the semigroup generated by two elements.

• Let b_1, b_2, a_1, a_2 positive integers such that $a_1 < b_1$ and $a_1b_2 - a_2b_1 = 1$. Then

$S]\frac{b_1}{a_1}, \frac{b_2}{a_2}[=< b_1, b_2 > \backslash\{\lambda b_1 \ : \ 1 \leq \lambda \leq b_2\} \cup \{\mu b_2 \ : \ 1 \leq \mu \leq b_1\}$.

We explicit the elements of a C-semigroup

- Let b_1, b_2, a_1, a_2 positive integers such that $a_1 < b_1$ and $a_1 b_2 - a_2 b_1 = 1$.

Then $S]\frac{b_1}{a_1}, \frac{b_2}{a_2}[= \{\lambda b_1 + \mu b_2 \ : \ \lambda, \mu \in \mathbb{N} \setminus \{0\}\} \cup \{0\}$.

Note: Let $S =< n_1, n_2 >$ with n_1, n_2 positive integers. Then

- $g(S) = n_1 n_2 - n_1 - n_2$

- $\#H(S) = \frac{(n_1-1)(n_2-1)}{2}$

From the previous results we have formulas for the multiplicity, the Frobenius numbers and the singularity degree of a C-semigroup.

• Let b_1, b_2, a_1, a_2 positive integers such that $a_1 < b_1$ and $a_1 b_2 - a_2 b_1 = 1$ and $S = S]\frac{b_1}{a_1}, \frac{b_2}{a_2}[$. Then

- $m(S) = b_1 + b_2$
- $g(S) = b_1 b_2$
- $\# H(S) = \frac{b_1 b_2 + b_1 + b_2 - 1}{2}$

The set of numerical semigroups A is a set of incomparable semigroups if

S, $\bar{S} \in A$ and $S \subseteq \bar{S}$ then $S = \bar{S}$

•Let $C(m)$ be the set of all C-semigroups with multiplicity m. Then

1) $C(m)$ is a set of incomparable semigroups

2) $\#C(m) = \#\{x \in \mathbb{N} : 2 \leq x < \frac{m}{2},\ gcd(m, x) = 1\}$

• Now we determine which intervals I such that $S(I)$ has multiplicity m.

• If $S(I)$ has multiplicity $m \Rightarrow \ \exists\, p \in \{1 \cdots, m-1\}$ such that $\frac{m}{p} \in I$ and $gcd(m,p) = 1$.

From this result we have that $\frac{m}{1} \in I$, or $\frac{m}{m-1} \in I$ or $\frac{m}{p} \in I$ with $p \in \{2, \cdots, m-2\}$

We study separately each cases

• I an interval of rational numbers > 1. Then $S(I)$ has multiplicity m if and only if the conditions holds:

1) $m \in I$ and $I \subseteq]m-1, \infty[$

2) $\frac{m}{m-1} \in I$ and $I \subseteq]1, \frac{m-1}{m-2}[$

3) there exist positive integers u, v, p such that $v < u < m-1$, $v < p < m-1$, $pu - mv = 1$, $\frac{m}{p} \in I$ and $I \subseteq]\frac{m-u}{p-v}, \frac{u}{v}[$

• I an interval of rational numbers > 1 such that I is maximal and $S(I)$ has multiplicity m then either:

$I =]m-1, \infty[$ or $I =]1, \frac{m-1}{m-2}[$ or $I =]\frac{m-u}{p-v}, \frac{u}{v}[$ and u, v, p with conditions above.

• Let I an interval of rational numbers > 1 such that I is maximal and $S(I)$ has multiplicity m then $S(I) = \{0, m, m+1, \rightarrow\}$ or $S(I)$ is a C-semigroup.

The following results characterize intervals I that generate a $P.M.$ numerical semigroups S(I) with $m(S) = m$ such that $m + 1 \notin S(I)$.

New characterization for C-semigroups.

• If S is C-semigroup then $m(S) \geq 5$.

• If S is a C-semigroup then $m(S) + 1 \notin S$.

• Suppose that $m \geq 3$ and I an interval of rational numbers.

1) If $I \subseteq]m-1, \infty[$. Then $S(I)$ has multiplicity m and $m+1 \notin S(I)$ if and only if $I \subseteq]m-1, m+1[$ and $m \in I$.

2) If $I \subseteq]1, \frac{m-1}{m-2}[$. Then $S(I)$ has multiplicity m and $m+1 \notin S(I)$ if and only if $I \subseteq]\frac{m+1}{m}, \frac{m-1}{m-2}[$ and $\frac{m}{m-1} \in I$.

• Let S be a proportional modular semigroup with $m(S) = m$ and $m \geq 3$ and $m+1 \notin S$.
Then $S \subseteq S(]m-1, m+1[)$ or S is contained in a C-semigroup.

$\Lambda(m) = \{S \in PM(m) \; : \; m+1 \notin S\}$

• If m is an integer ≥ 5. Then a numerical S is a maximal element of $\Lambda(m)$ if and only if S is either a C-semigroup with $m(S) = m$ or $S = S(]m-1, m+1[)$ if m is an even.

• The cases for $m \in \{2, 3, 4\}$ are study separately.

From this we have another characterization for C-semigroups.

• S is a numerical semigroup with $m \geq 5$. Then S is a C-semigroup if and only if S is a maximal element in $\Lambda(m)$ and $\{m+2, \cdots, 2m-2\} \cap S \neq \emptyset$.

As a consequence this results we can compute the number of maximal elements of $\Lambda(m)$.

- The number of maximal elements of $\Lambda(m)$ is

$$\#\left\{x \in \mathbb{N} \mid 2 \leq x < \frac{m}{2},\ gcd\{m, x\} = 1\right\},$$

if m is odd, and is

$$\#\left\{x \in \mathbb{N} \mid 2 \leq x < \frac{m}{2},\ gcd\{m, x\} = 1\right\} + 1,$$

if m is even.

- Example - We will obtain the maximal elements of $\Lambda(14)$. Clearly

$$\{x \in \mathbb{N} \mid 2 \leq x < 7,\ gcd\{14, x\} = 1\} = \{3, 5\}.$$

There exist two C-semigroups with $m(S) = 14$ which are $S\left(\left]\frac{9}{2}, \frac{5}{1}\right[\right)$ and $S\left(\left]\frac{11}{4}, \frac{3}{1}\right[\right)$ and another maximal element, which is $S(]13, 15[)$.

REFERENCES

J. C. Rosales, M.B.Branco and P. Vasco, Proportionally modular inealities and their multiplicity, submitted.

J. C. Rosales, P. A. García-Sánchez, J. I. García-García and J. M. Urbano-Blanco, Proportionally modular Diophantine inequalities, J. Number Theory 103 (2003), 281-294.

J. C. Rosales and J. M. Urbano-Blanco, Opened modular numerical semigroups, J. Algebra 306 (2006), 368-377.

J. J. Sylvester, Excursus on rational fractions and partitions, Amer J. Math. 5 (1882), 119-136.

J. J. Sylvester, Mathematical questions with their solutions, Educational Times 41 (1884), 21.