

# Results on Numerical Semigroups with Applications to Algebraic Geometry Codes

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*Iberian Meeting on Numerical Semigroups*  
*Universidade do Porto*  
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## 1 Semigroup Families

- 0. Preliminaries
- A. Acute
- B. Symmetric
- C. Arf
- D. Classification

## 2 $\nu, \tau$ and Improved Codes

- A.  $\nu$  and  $\tau$
- B. Improved Codes
- C. Increasingness of  $\nu$  and  $\tau$
- D. Relation Between  $\nu$  and  $\tau$

## 3 Further on Semigroups

- A. Characterization
- B. Counting

## Definition

A *numerical semigroup* is a subset  $\Lambda$  of  $\mathbb{N}_0$  satisfying

- $0 \in \Lambda$
- $\Lambda + \Lambda \subseteq \Lambda$
- $|\mathbb{N}_0 \setminus \Lambda|$  is finite (*genus* :=  $g := |\mathbb{N}_0 \setminus \Lambda|$ )

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The third condition implies that there exist

- **Conductor** := the unique integer  $c$  with  $c - 1 \notin \Lambda$ ,  $c + \mathbb{N}_0 \subseteq \Lambda$
- **Frobenius number** := the largest gap =  $c - 1$
- **Dominant** := the non-gap previous to  $c$ .

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The inclusion  $\Lambda \subseteq \mathbb{N}_0$  implies that there exists

- **Enumeration** := the unique bijective increasing map  $\lambda : \mathbb{N}_0 \rightarrow \Lambda$   
( $\Lambda = \{\lambda_0 = 0 < \lambda_1 < \lambda_2 \dots\}$ )

## Consider

- $\chi$  a projective curve without singularities
- $P$  a rational point in  $\chi$
- $A$  the ring of functions on  $\chi$  with poles only at  $P$
- $\Lambda = \{-v_p(f) : f \in A \setminus \{0\}\}$  the pole orders of  $A$  at  $P$

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It holds:

- $0 = -v_P(1) \in \Lambda$
- $(-v_P(f)) + (-v_P(g)) = (-v_P(fg)) \in \Lambda$  for all  $f, g \in A$
- $|\mathbb{N}_0 \setminus \Lambda| = \text{genus of } \chi \text{ (finite)}$

## Example

### Klein quartic

$$\chi : x^3y + y^3 + x = 0$$

$P$  : rational point with affine coordinates  $x = 0, y = 0$

$i$	$\lambda_i$
0	0
1	3
2	5
3	6
4	7
5	8
6	9
7	10
8	11
$\vdots$	$\vdots$

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## Example

### Hermitian Curve Over $\mathbb{F}_4$

$$\chi : x^5 = y^4 + y$$

$P$  : the unique point at infinity

$i$	$\lambda_i$
0	0
1	4
2	5
3	8
4	9
5	10
6	12
7	13
$\vdots$	$\vdots$

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## Definition

A numerical semigroup is *ordinary* if it is equal to

$$\{0\} \cup \{i \in \mathbb{N}_0 : i \geq c\},$$

for some non-negative integer  $c$ .

A numerical semigroup is *acute* if it is ordinary or if its last interval of gaps is smaller than or equal to the previous one.

## Example

The Weierstrass semigroup at point  $P$  of the Klein quartic is acute.

$i$	$\lambda_i$
0	0
1	3
2	5
3	6
4	7
5	8
6	9
7	10
8	11
9	12
$\vdots$	$\vdots$

← 2 gaps

← 1 gap

## Example

The Weierstrass semigroup at point  $P$  of the Hermitian curve is acute.

$i$	$\lambda_i$
0	0
1	4
2	5
3	8
4	9
5	10
6	12
7	13
8	14
$\vdots$	$\vdots$

← 2 gaps

← 1 gap

## Definition

A numerical semigroup with conductor  $c$  and genus  $g$  is *symmetric* if  $c = 2g$ .

## Example:

The Weierstrass semigroup at point  $P$  of the Hermitian curve is symmetric.

Its conductor is  $c = 12$  and its genus is  $g = 6$ .

$i$	$\lambda_i$
0	0
1	4
2	5
3	8
4	9
5	10
6	12
7	13
8	14
9	15
10	16
$\vdots$	$\vdots$

← 3 gaps

← 2 gaps

← 1 gap

←  $c = 12$

## Lemma

A numerical semigroup  $\Lambda$  is symmetric if and only if for any non-negative integer  $i$ ,

$$i \notin \Lambda \iff c - 1 - i \in \Lambda.$$

$i$	$\lambda_i$	
0	0	11-10
		11-9
		11-8
1	4	
2	5	
<hr/>		
		11-5
		11-4
3	8	
4	9	
5	10	
		11-0
6	12	
$\vdots$	$\vdots$	

## Lemma

*All symmetric semigroups are acute.*

## Proof

*Let  $\Lambda$  be a non-ordinary symmetric semigroup.*

*Since  $1 \notin \Lambda$ , by the previous Lemma  $c - 2 \in \Lambda$ .*

*Thus, the last interval of gaps consists of one gap ( $c - 1$ ).*

*The semigroup must therefore be acute.*



## Definition

A numerical semigroup  $\Lambda$  is *Arf* if for any  $a, b, c \in \Lambda$  with  $a \geq b \geq c$  we have  $a + b - c \in \Lambda$ .

## Example

The Weierstrass semigroup at point  $P$  of the Klein quartic is Arf.

$i$	$\lambda_i$
0	0
1	3
2	5
3	6
4	7
5	8
6	9
7	10
$\vdots$	$\vdots$

$$7 + 5 - 3 = 9 \in \Lambda$$



## Lemma

All Arf semigroups are acute.

## Proof

Let  $\Lambda$  be a non-ordinary Arf semigroup.

Consider  $c, c', d, d'$  as in the example, where  $c', c' + 1, \dots, d$  is the last interval of non-gaps before the conductor.

$i$	$\lambda_i$	
0	0	← $d'$
1	3	← $c'$
		← $d$
2	5	← $c$
3	6	
4	7	
5	8	
6	9	
7	10	
$\vdots$	$\vdots$	

$$d \geq c' > d' \implies d + c' - d' \in \Lambda.$$

$$\left. \begin{array}{l} d + c' - d' \in \Lambda \\ d + c' - d' > d \end{array} \right\} \implies d + c' - d' \geq c \implies c - d \leq c' - d'. \quad \square$$

## Theorem

- *The set of acute semigroups is a proper subset of the whole set of numerical semigroups.*
- *It properly includes*
  - *Symmetric and pseudo-symmetric semigroups,*
  - *Arf semigroups,*
  - *Semigroups generated by an interval.*

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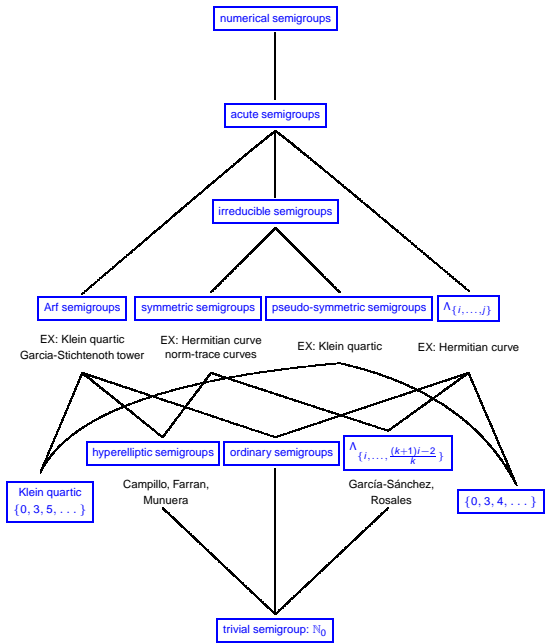
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## Definition

Given a numerical semigroup  $\Lambda$  define its  $\nu$  sequence as

$$\nu_i = \#\{j \in \mathbb{N}_0 : \lambda_i - \lambda_j \in \Lambda\}$$

## Example Klein quartic

$i$	$\lambda_i$	$\nu_i$
0	0	1
1	3	2
2	5	2
3	6	3
4	7	2
5	8	4
6	9	4
7	10	5
8	11	6
9	12	7
10	13	8
$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$

## Definition

Given a numerical semigroup  $\Lambda$  define its  $\tau$  sequence as

$$\tau_i = \max\{j \in \mathbb{N}_0 : \text{exists } k \text{ with } j \leq k \leq i \text{ and } \lambda_j + \lambda_k = \lambda_i\}$$

## Example Klein quartic

$i$	$\lambda_i$	$\tau_i$	
0	0	0	$0 + 0 = 0$
1	3	0	$0 + 3 = 3$
2	5	0	$0 + 5 = 5$
3	6	1	$3 + 3 = 6$
4	7	0	$0 + 7 = 7$
5	8	1	$3 + 5 = 8$
6	9	1	$3 + 6 = 9$
7	10	2	$5 + 5 = 10$
8	11	2	$5 + 6 = 11$
9	12	3	$6 + 6 = 12$
10	13	3	$6 + 7 = 13$
$\vdots$	$\vdots$	$\vdots$	
$\vdots$	$\vdots$	$\vdots$	

## Definition

Consider

- $\chi$  a projective curve without singularities over  $\mathbb{F}_q$
- $P$  a rational point in  $\chi$
- $A$  the ring of functions on  $\chi$  with poles only at  $P$
- $\Lambda$  the Weierstrass semigroup at  $P$
- $\lambda$  enumeration of  $\Lambda$  ( $\Lambda = \{\lambda_0 = 0 < \lambda_1 < \lambda_2 \dots\}$ )
- $\{f_i : i \in \mathbb{N}_0, f_i \in A, -v_P(f) = \lambda_i\}$ .
- $P_1, P_2, \dots, P_n$  rational points in  $\chi$  different from  $P$
- $ev : A \longrightarrow \mathbb{F}_q^n, f \mapsto (f(P_1), \dots, f(P_n))$ .

Define

$$C_i = \{ev(f) : f \in A, -v_P(f) \leq \lambda_i\}^\perp \quad \text{for } i \in \mathbb{N}_0 \text{ (standard)}$$

$$C_W = \langle ev(f_i) : i \in W \rangle^\perp \quad \text{for } W \subseteq \mathbb{N}_0$$

## Definition

The points  $P_{i_1}, \dots, P_{i_t}$  are *generically distributed* if no element  $f \in A$  with  $-v_P(f) < \lambda_t$  vanishes in all of them.

*Generic errors* are those errors whose non-zero positions correspond to generically distributed points.

Generic errors of weight  $t$  can be a very large portion of all possible errors of weight  $t$  [Hansen, 2001].

By restricting the errors to be corrected to generic errors the decoding requirements will be weaker and we still will be able to correct almost all errors.

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# Correction Capability of Berlekamp-Massey-Sakata Algorithm with Majority Voting

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## Theorem

[Feng, Rao, 1995]

All error vectors of weight  $t$  can be corrected by  $C_W$  if  $W$  contains all  $i$  with  $\lfloor \frac{\nu_i - 1}{2} \rfloor < t$ .

## Theorem

[Bras-Amorós, O'Sullivan, 2006]

All generic error vectors of weight  $t$  can be corrected by  $C_W$  if  $W$  contains all  $i$  with  $\tau_i < t$ .



# Correction Capability Optimized Codes

## Codes for correcting $t$ errors of any kind

Take  $W$  equal to

$$\tilde{R}(t) = \{i \in \mathbb{N}_0 : \lfloor \frac{\nu_i - 1}{2} \rfloor < t\} \text{ (Feng–Rao improved codes)}$$

$$R(t) = \{i \in \mathbb{N}_0 : i \leq i(t)\}, \text{ where } i(t) = \max \tilde{R}(t) \text{ (standard)}$$

## Codes for correcting $t$ generic errors

Take  $W$  equal to

$$\tilde{R}^*(t) = \{i \in \mathbb{N}_0 : \tau_i < t\}$$

$$R^*(t) = \{i \in \mathbb{N}_0 : i \leq i^*(t)\}, \text{ where } i^*(t) = \max \tilde{R}^*(t) \text{ (standard)}$$

Increasingness of  $\nu$   $\longleftrightarrow$  Compare  $\tilde{R}(t)$  and  $R(t)$

Increasingness of  $\tau$   $\longleftrightarrow$  Compare  $\tilde{R}^*(t)$  and  $R^*(t)$

Compare  $\nu$  and  $\tau$   $\longleftrightarrow$  Compare  $\tilde{R}^*(t)$  and  $\tilde{R}(t)$

# Hermitian Codes Redundancy ( $\mathbb{F}_{7^2}$ )

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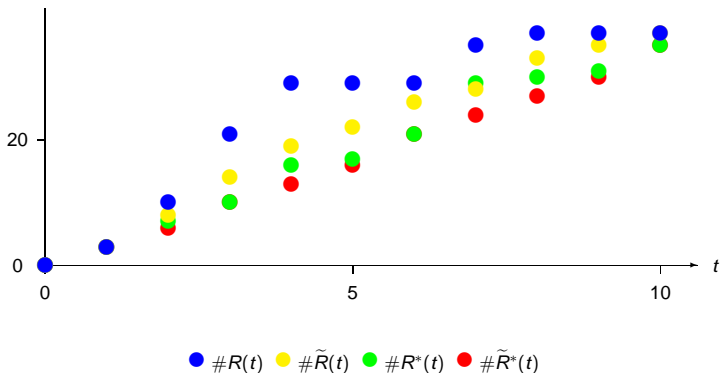
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The  $\nu$  sequence of the ordinary semigroup  $\{0\} \cup [c, \infty)$  is

$$1, 2, \binom{c}{2}, 2, 3, 4, 5, 6, 7, 8, 9, 10, \dots$$

## Proposition

If  $\nu$  is non-decreasing then  $\Lambda$  is acute.

## Proposition

Let  $\Lambda$  be the non-ordinary acute semigroup

$$\{0\} \cup [c^{(k)}, d^{(k-1)}] \cup \dots \cup [c'', d'] \cup [c', d] \cup [c, \infty)$$

and let  $m = \min\{\lambda^{-1}(c + c' - 2), \lambda^{-1}(2d)\}$ . Then

- $\nu_m > \nu_{m+1}$
- $\nu_i \leq \nu_{i+1}$  for all  $i > m$ .

In fact, the unique  $m$  satisfying the previous conditions is  $m = \min\{\lambda^{-1}(c + c' - 2), \lambda^{-1}(2d)\}$  if and only if  $\Lambda$  is either acute or  $2d - c + 1 \notin \Lambda$  [Munuera, Torres, Oneto, Tamone].

## Corollary

- *The unique semigroup for which  $\nu$  is strictly increasing is  $\mathbb{N}_0$ .*
- *The only numerical semigroups for which  $\nu$  is non-decreasing are ordinary-semigroups.*

## Corollary

- *$\tilde{R}(t) = R(t)$  for all  $t \in \mathbb{N}_0$  if and only if the associated numerical semigroup is ordinary.*

The  $\tau$  sequence of  $\mathbb{N}_0$  is

$$0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \dots$$

The  $\tau$  sequence of the semigroup  $\{0\} \cup [c, \infty)$  with  $c > 0$  is

$$0, (c+1), 0, 1, 1, 2, 2, 3, 3, 4, 4, \dots$$

## Proposition

*For a non-ordinary semigroup with conductor  $c$ , genus  $g$  and dominant  $d$  (non-gap previous to  $c$ ) let  $m = \lambda^{-1}(2d)$ . Then*

- $\tau_m = c - g - 1 > \tau_{m+1}$
- $\tau_i \leq \tau_{i+1}$  for all  $i > m$ .

## Corollary

- 1 *The unique numerical semigroups with non-decreasing  $\tau$  sequence are ordinary semigroups.*

The  $\tau$  sequence of  $\mathbb{N}_0$  is

$$0, 0, 1, 1, 2, 2, 3, 3, 4, 4, 5, 5, \dots$$

The  $\tau$  sequence of the semigroup  $\{0\} \cup [c, \infty)$  with  $c > 0$  is

$$0, (c+1), 0, 1, 1, 2, 2, 3, 3, 4, 4, \dots$$

## Proposition

*For a non-ordinary semigroup with conductor  $c$ , genus  $g$  and dominant  $d$  (non-gap previous to  $c$ ) let  $m = \lambda^{-1}(2d)$ . Then*

- $\tau_m = c - g - 1 > \tau_{m+1}$
- $\tau_i \leq \tau_{i+1}$  for all  $i > m$ .

## Corollary

- ②  $\tilde{R}^*(t) = R^*(t)$  for all  $t \in \mathbb{N}_0$  if and only if the associated numerical semigroup is ordinary.

## Proposition

- $\tau_i \geq \lfloor \frac{\nu_i - 1}{2} \rfloor$  for all  $i \in \mathbb{N}_0$
- $\tau_i = \lfloor \frac{\nu_i - 1}{2} \rfloor$  for all  $i \geq 2c - g - 1$
- $\tau_i = \lfloor \frac{\nu_i - 1}{2} \rfloor$  for all  $i \in \mathbb{N}_0$  if and only if  $\Lambda$  is Arf.

## Corollary

- 1  $\tilde{R}^*(t) \subseteq \tilde{R}(t)$  for all  $t \in \mathbb{N}_0$ .
- 2  $\tilde{R}^*(t) = \tilde{R}(t)$  for all  $t \in \mathbb{N}_0$  if and only if the associated numerical semigroup is Arf.

## Theorem

*A numerical semigroup is completely determined by its  $\tau$  sequence.*

## Proof

*We can construct a numerical semigroup  $\Lambda$  from its  $\tau$  sequence as follows:*

- *Let  $k$  be the minimum integer such that for all  $i \in \mathbb{N}_0$ ,*
  - $\tau_{k+2i} = \tau_{k+2i+1}$
  - $\tau_{k+2i+2} = \tau_{k+2i+1} + 1$
- *Set*
  - $c = k - \tau_k + 1$
  - $g = k - 2\tau_k$

*This determines  $\lambda_i$  for all  $i \geq c - g$*

- *For  $i = c - g - 1$  to  $1$ ,  $\lambda_i = \frac{1}{2} \min\{\lambda_j : \tau_j = i\}$*





## Theorem

A numerical semigroup is completely determined by its  $\nu$  sequence.

## Proof

We can construct a numerical semigroup  $\Lambda$  from its  $\nu$  sequence as follows:

- If  $\nu_i = i + 1$  for all  $i \in \mathbb{N}_0$  then  $\Lambda = \mathbb{N}_0$
- Otherwise let  $k = \max\{j : \nu_j = \nu_{j+1}\}$  (it exists and it is unique)
- Let  $g = k + 2 - \nu_k$  and  $c = \frac{k+g+2}{2}$ 
  - $0 \in \Lambda, 1, c - 1 \notin \Lambda$
  - For all  $i \geq c, i \in \Lambda$
- For  $i = c - 2$  to  $i = 2$ ,
  - Define  $\tilde{D}(i) = \{l \in \Lambda^c : c - 1 + i - l \in \Lambda^c, i < l < c - 1\}$
  - $i \in \Lambda$  if and only if  $\nu_{c-1+i-g} = c + i - 2g + \#\tilde{D}(i)$



## Definition

Given a numerical semigroup  $\Lambda$  with enumeration  $\lambda$  define the binary operation

$$i \oplus j = \lambda^{-1}(\lambda_i + \lambda_j).$$

Equivalently,

$$\lambda_{i \oplus j} = \lambda_i + \lambda_j.$$

## Theorem

A numerical semigroup is completely determined by the  $\oplus$  operation.

## Theorem

*No numerical semigroup can be determined by a finite subset of*

- $\nu$  values
- $\tau$  values
- $\oplus$  values.

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Let  $n_g$  denote the number of numerical semigroups of genus  $g$ .

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- $n_0 = 1$ , since the unique numerical semigroup of genus 0 is  $\mathbb{N}_0$

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- $n_1 = 1$ , since the unique numerical semigroup of genus 1 is  $\mathbb{N}_0 \setminus \{1\}$

Let  $n_g$  denote the number of numerical semigroups of genus  $g$ .

- $n_0 = 1$ , since the unique numerical semigroup of genus 0 is  $\mathbb{N}_0$
- $n_1 = 1$ , since the unique numerical semigroup of genus 1 is  $\mathbb{N}_0 \setminus \{1\}$
- $n_2 = 2$ . Indeed the unique numerical semigroups of genus 2 are

$$\{0, 3, 4, 5, \dots\},$$

$$\{0, 2, 4, 5, \dots\}.$$

## Conjecture

$$\textcircled{1} \quad n_g \geq n_{g-1} + n_{g-2}$$

$$\textcircled{2} \quad \lim_{g \rightarrow \infty} \frac{n_{g-1} + n_{g-2}}{n_g} = 1$$

$$\textcircled{3} \quad \lim_{g \rightarrow \infty} \frac{n_g}{n_{g-1}} = \phi$$

At the moment it has not even been proved that  $n_g$  is increasing.



# Conjecture $n_g/n_{g-1} \rightarrow \phi$

$g$	$n_g$	$n_{g-1} + n_{g-2}$	$\frac{n_{g-1} + n_{g-2}}{n_g}$	$\frac{n_g}{n_{g-1}}$
0	1			
1	1			1
2	2	2	1	2
3	4	3	0.75	2
4	7	6	0.857143	1.75
5	12	11	0.916667	1.71429
6	23	19	0.826087	1.91667
7	39	35	0.897436	1.69565
8	67	62	0.925373	1.71795
9	118	106	0.898305	1.76119
10	204	185	0.906863	1.72881
11	343	322	0.938776	1.68137
12	592	547	0.923986	1.72595
13	1001	935	0.934066	1.69088
14	1693	1593	0.940933	1.69131
15	2857	2694	0.942947	1.68754
16	4806	4550	0.946733	1.68218
17	8045	7663	0.952517	1.67395
18	13467	12851	0.954259	1.67396
19	22464	21512	0.957621	1.66808
20	37396	35931	0.960825	1.66471
21	62194	59860	0.962472	1.66312
22	103246	99590	0.964589	1.66006
23	170963	165440	0.967695	1.65588
24	282828	274209	0.969526	1.65432
25	467224	453791	0.971249	1.65197
26	770832	750052	0.973042	1.64981
27	1270267	1238056	0.974642	1.64792
28	2091030	2041099	0.976121	1.64613
29	3437839	3361297	0.977735	1.64409
30	5646773	5528869	0.97912	1.64254
31	9266788	9084612	0.980341	1.64108
32	15195070	14913561	0.981474	1.63973
33	24896206	24461858	0.982554	1.63844
34	40761087	40091276	0.983567	1.63724
35	66687201	65657293	0.984556	1.63605
36	109032500	107448288	0.98547	1.63498
37	178158289	175719701	0.986312	1.63399
38	290939807	287190789	0.987114	1.63304
39	474851445	469098096	0.987884	1.63213
40	774614284	765791252	0.98861	1.63128
41	1262992840	1249465729	0.98929	1.63048
42	2058356522	2037607124	0.989919	1.62975
43	3353191846	3321349362	0.990504	1.62906
44	5460401576	5411548368	0.991053	1.62842
45	8888486816	8813593422	0.991574	1.62781
46	14463633648	14348888392	0.992067	1.62723
47	23527845502	23352120464	0.992531	1.62669
48	38260496374	37991479150	0.992969	1.62618
49	62200036752	61788341876	0.993381	1.6257
50	101090300128	100460533126	0.99377	1.62525

Results on  
Numerical  
Semigroups  
with  
Applications  
to  
Algebraic  
Geometry  
Codes

Maria  
Bras-Amorós

Semigroup  
Families

- 0. Preliminaries
- A. Acute
- B. Symmetric
- C. Arf
- D. Classification

$\nu$ ,  $\tau$  and  
Improved  
Codes

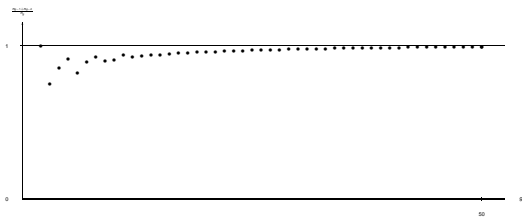
- A.  $\nu$  and  $\tau$
- B. Improved Codes
- C. Increasingness  
of  $\nu$  and  $\tau$
- D. Relation  
Between  $\nu$  and  $\tau$

Further on  
Semigroups

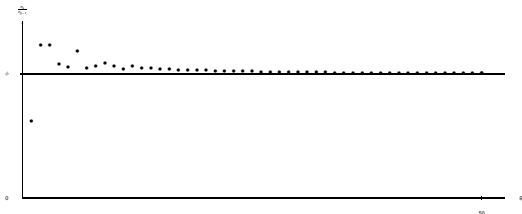
- A. Characterization
- B. Counting

# Conjecture $n_g/n_{g-1} \rightarrow \phi$

## Behavior of $\frac{n_{g-1}+n_{g-2}}{n_g}$



## Behavior of $\frac{n_g}{n_{g-1}}$



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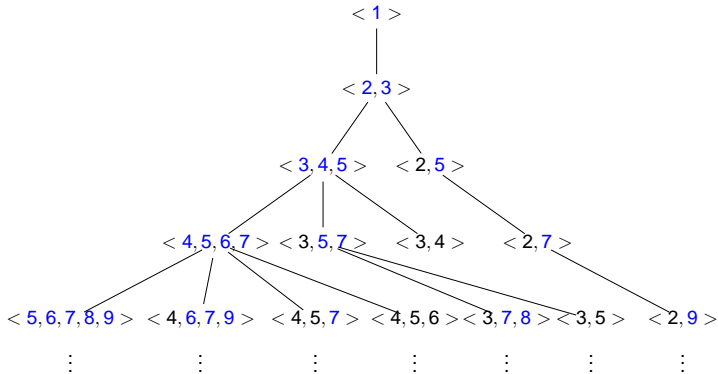
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Further on  
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# Tree of Numerical Semigroups

The father of a numerical semigroup  $\Lambda$  is  $\Lambda$  together with its Frobenius number.



For each numerical semigroup its children are obtained taking away one by one all generators larger than its Frobenius number.

## Lemma

*The ordinary semigroup  $\Lambda = \{0, g + 1, g + 2, \dots\}$  has minimal set of generators  $\{g + 1, g + 2, \dots, 2g + 1\}$  and*

- $\Lambda \setminus \{g + 1\}$  has  $g + 2$  minimal generators larger than its Frobenius number.*
- $\Lambda \setminus \{g + 2\}$  has  $g$  minimal generators larger than its Frobenius number.*
- $\Lambda \setminus \{g + r\}$ , with  $r > 2$ , has  $g - r + 1$  minimal generators larger than its Frobenius number.*

## Lemma

Let  $\Lambda$  be a non-ordinary numerical semigroup. Suppose that  $\{\lambda_{i_1} < \lambda_{i_2} < \dots < \lambda_{i_k}\}$  are the minimal generators of  $\Lambda$  which are larger than the Frobenius number. Then the number of minimal generators of the numerical semigroup  $\Lambda \setminus \{\lambda_{i_j}\}$  which are larger than its Frobenius number is

- at least  $k - j$ ,
- at most  $k - j + 1$ .

**Lemma**

Let  $\Lambda$  be a non-ordinary numerical semigroup. Suppose that  $\{\lambda_{i_1} < \lambda_{i_2} < \dots < \lambda_{i_k}\}$  are the minimal generators of  $\Lambda$  which are larger than the Frobenius number. Then the number of minimal generators of the numerical semigroup  $\Lambda \setminus \{\lambda_{i_j}\}$  which are larger than its Frobenius number is

- at least  $k - j$ ,
- at most  $k - j + 1$ .

**Lemma**

For  $g \geq 3$ ,

$$2F_g \leq n_g \leq 1 + 3 \cdot 2^{g-3}.$$

# Some Known Degrees in the Tree

(Joint work with Stanislav Bulygin)

- Non-ordinary **hyperelliptic** numerical semigroups  $\langle 2, 2k + 1 \rangle$  have a single child.
- Non-ordinary non-hyperelliptic **symmetric** semigroups are leaves.
- Non-ordinary **pseudo-symmetric** semigroups  $(c = 2g - 1)$  with  $\lambda_1 \neq 3$  are leaves.
- Non-ordinary pseudo-symmetric semigroups with  $\lambda_1 = 3$  have at most one child if they are different than  $\langle 3, 5, 7 \rangle$ .
- Non-hyperelliptic **Arf** numerical semigroups have at least two children.

(Joint work with Stanislav Bulygin)

## Lemma

Let  $\Lambda = \{\lambda_0 = 0 < \lambda_1 < \lambda_2 < \dots\}$  be a numerical semigroup with genus  $g$  and conductor  $c$  and let  $\delta$  be the greatest common divisor of  $\lambda_0, \dots, \lambda_{c-g-1}$ . Then,

- 1  $\Lambda$  lies in an infinite chain if and only if  $\delta \neq 1$ .
- 2 If  $\delta \neq 1$  then  $\Lambda$  lies in infinitely many infinite chains if and only if  $\delta$  is not prime.
- 3 If  $\delta$  is a prime then the number of infinite chains in which  $\Lambda$  lies is the number of descendants of  $\{\frac{\lambda_0}{\delta}, \frac{\lambda_1}{\delta}, \dots, \frac{\lambda_{c-g-1}}{\delta}\} \cup \{l \in \mathbb{N}_0 : l \geq \lceil \frac{c}{\delta} \rceil\}$ .



The results explained in this talk summarize the work in the next articles

(<http://www.crisis-deim.urv.cat/~mbras/>):

**Acute semigroups, semigroup classifications, results on the  $\nu$  sequence,  $\tau$  sequence**

[1] *Acute Semigroups, the Order Bound on the Minimum Distance and the Feng-Rao Improvements*, IEEE Transactions on Information Theory, 2004.

[2] *A Note on Numerical Semigroups*, IEEE Transactions on Information Theory, 2007.

[3] *On Numerical Semigroups and the Redundancy of Improved Codes Correcting Generic Errors*, submitted.

**Counting of numerical semigroups**

[4] *Fibonacci-Like Behavior of the Number of Numerical Semigroups of a Given Genus*, Semigroup Forum, 2008.

[5] *Bounds on the Number of Numerical Semigroups of a Given Genus*, ArXiv, 2008.

[6] (with S. Bulygin) *Manuscript Notes*, 2008.