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On the Buchsbaumness of the associated graded ring of semigroup rings

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1. Setup

$g_1 < g_2 < \cdots < g_n \in \mathbb{N}, \qquad GCD(g_1, \ldots g_n) = 1$
$S = \langle g_1, \ldots, g_n \rangle := \{n_1 g_1 + \cdots + n_n g_n \mid n_i \in \mathbb{N}, i = 1, \ldots, n\}$

$R = k[[S]] := k[[t^{g_1}, \ldots, t^{g_n}]]$ (or $R := k[t^{g_1}, \ldots, t^{g_n}]_{(t^{g_1}, \ldots, t^{g_n})}$)

R is a one-dimensional, local domain, with maximal ideal $\mathfrak{m} = (t^{g_1}, \ldots, t^{g_n})$ and quotient field $Q = k((t))$.
If we denote by $v : k((t)) \longrightarrow \mathbb{Z} \cup \infty$ the natural valuation, we get $v(R) = \{v(r) \mid r \in R \setminus \{0\}\} = S$.

The associated graded ring with respect to $\mathfrak{m}$ will be denoted by

$$G(\mathfrak{m}) := \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1} \qquad \mathcal{M} := \bigoplus_{i \geq 1} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

2. Problem, definition and first remarks

When is $G(\mathfrak{m})$ a Buchsbaum ring?

Definition. (Stückrad-Vogel) $G(\mathfrak{m})$ is Buchsbaum if $\mathcal{M} \cdot H^0_{\mathcal{M}} = 0$.

Remarks. • As $H^0_{\mathcal{M}} = (\cup_{k\geq 1}(0 :_{G(\mathfrak{m})} \mathcal{M}^k))$,

$G(\mathfrak{m})$ is Buchsbaum $\Longleftrightarrow \mathcal{M} \cdot (\cup_{k\geq 1}(0 :_{G(\mathfrak{m})} \mathcal{M}^k)) = 0$.

• $G(\mathfrak{m})$ is Buchsbaum $\Longleftrightarrow (0 :_{G(\mathfrak{m})} \mathcal{M}) = (0 :_{G(\mathfrak{m})} \mathcal{M}^k), \ \forall k \geq 1$.

• $(0 :_{G(\mathfrak{m})} \mathcal{M}) = (0) \Longleftrightarrow (0 :_{G(\mathfrak{m})} \mathcal{M}^k) = (0), \ \forall k \geq 1$.
• $G(\mathfrak{m})$ Cohen Macaulay (C-M) $\Longleftrightarrow (0 :_{G(\mathfrak{m})} \mathcal{M}) = (0)$
• $G(\mathfrak{m})$ C-M $\Longrightarrow G(\mathfrak{m})$ Buchsbaum

3. Some references

The property for $G(\mathfrak{m})$ to be Cohen-Macaulay has been largely studied, while,
not much is known about the Buchsbaum property of $G(\mathfrak{m})$ (except for the case that $G(\mathfrak{m})$ is Cohen-Macaulay).

General case (R Noetherian, local ring, of dimension d):

- Goto, *Buchsbaum rings of maximal embedding dimension*, (1982)
- Goto, *Noetherian local rings with Buchsbaum associated graded rings*, (1984)

The semigroup ring case (with S 3-generated):

- Sapko, *Associated graded rings of numerical semigroup rings*, (2001)

4. More remarks

Let x be any element s.t. $v(x) = g_1$ and $\overline{x}$ its image in $G(\mathfrak{m})$. The reduction number r of $\mathfrak{m}$ is the minimal natural number such that $\mathfrak{m}^{r+1} = x\mathfrak{m}^r$.

- $\mathfrak{m}^r/\mathfrak{m}^{r+1} \simeq \mathfrak{m}^{r+i}/\mathfrak{m}^{r+i+1}$, as R-module, $\forall\ i \geq 1$.

- $\mathcal{M}$ is generated by $\mathfrak{m}/\mathfrak{m}^2$, hence
$$(0 :_{G(\mathfrak{m})} \mathcal{M}^i) = (0 :_{G(\mathfrak{m})} \mathfrak{m}^i/\mathfrak{m}^{i+1}).$$

- $(0 :_{G(\mathfrak{m})} \mathcal{M}) \subseteq \cdots \subseteq (0 :_{G(\mathfrak{m})} \mathcal{M}^r) = \cdots = (0 :_{G(\mathfrak{m})} \mathcal{M}^k) = \ldots$ $(\forall k \geq r)$

- $G(\mathfrak{m})$ is Buchsbaum (not C-M) $\Longleftrightarrow$ $0 \neq (0 :_{G(\mathfrak{m})} \mathcal{M}) = (0 :_{G(\mathfrak{m})} \mathcal{M}^r)$

5. Graded description of $(0:_{G(\mathfrak{m})}\mathcal{M}^k)$

Recall that: $G(\mathfrak{m})$ is Buchsbaum $\iff (0:_{G(\mathfrak{m})}\mathcal{M}) = (0:_{G(\mathfrak{m})}\mathcal{M}^r)$

(1) $$(0:_{G(\mathfrak{m})}\mathcal{M}^k) = \bigoplus_{h\geq 1} \frac{(\mathfrak{m}^{h+k+1}:_R \mathfrak{m}^k)\cap \mathfrak{m}^h}{\mathfrak{m}^{h+1}}$$

Remark. For every h, $(\mathfrak{m}^{h+1}:_R \mathfrak{m})\cap \mathfrak{m}^h \subseteq (\mathfrak{m}^{h+r+1}:_R \mathfrak{m}^r)\cap \mathfrak{m}^h$

This means that each direct summand of $(0:_{G(\mathfrak{m})}\mathcal{M})$ is contained in the corresponding direct summand of $(0:_{G(\mathfrak{m})}\mathcal{M}^r)$. Hence:

$G(\mathfrak{m})$ is Buchsbaum $\iff (\mathfrak{m}^{h+1}:_R \mathfrak{m})\cap\mathfrak{m}^h = (\mathfrak{m}^{h+r+1}:_R \mathfrak{m}^r)\cap\mathfrak{m}^h$ $\forall\ h\geq 1.$

Let $R' = B(\mathfrak{m}) = \cup_{n\geq 1}(\mathfrak{m}^n :_Q \mathfrak{m}^n) = (\mathfrak{m}^r :_Q \mathfrak{m}^r) = x^{-r}\mathfrak{m}^r$ (the last one is an equality of R-modules). Hence:

$$(\mathfrak{m}^{h+r+1} :_R \mathfrak{m}^r) \cap \mathfrak{m}^h = (\mathfrak{m}^{h+r+1} :_Q \mathfrak{m}^r) \cap \mathfrak{m}^h = x^{h+1}R' \cap \mathfrak{m}^h$$

(2) $$(0 :_{G(\mathfrak{m})} \mathcal{M}^r) = \bigoplus_{h\geq 1} \frac{xR' \cap x^{-h}\mathfrak{m}^h}{x^{-h}\mathfrak{m}^{h+1}} x^h$$

Proposition.

(3) $$(0 :_{G(\mathfrak{m})} \mathcal{M}^r) = \bigoplus_{h=1}^{r-2} \frac{xR' \cap x^{-h}\mathfrak{m}^h}{x^{-h}\mathfrak{m}^{h+1}} x^h$$

(All direct summands are zero for $h \geq r-1$, since $x^{-h}\mathfrak{m}^{h+1} = xR'$, $\forall\ h \geq r-1$.) Hence:

$G(\mathfrak{m})$ is Buchsbaum $\iff (\mathfrak{m}^{h+1} :_Q \mathfrak{m}) \cap \mathfrak{m}^h = (\mathfrak{m}^{h+r+1} :_Q \mathfrak{m}^r) \cap \mathfrak{m}^h$ $\forall\ h = 1, \ldots, r-2.$

The direct summands of $(0 :_{G(\mathfrak{m})} \mathcal{M})$ and $(0 :_{G(\mathfrak{m})} \mathcal{M}^r)$ corresponding to $h = r - 2$, are equal:

Lemma.

$$\frac{(\mathfrak{m}^{2r-1} :_Q \mathfrak{m}^r) \cap \mathfrak{m}^{r-2}}{\mathfrak{m}^{r-1}} = \frac{(\mathfrak{m}^r :_Q \mathfrak{m}) \cap \mathfrak{m}^{r-2}}{\mathfrak{m}^{r-1}}$$

Corollary. (Goto) Let r be the reduction number of R. If $r \leq 3$ then $G(\mathfrak{m})$ is Buchsbaum.

Corollary. Let $e = g_1$ be the multiplicity of R. If $e \leq 4$ then $G(\mathfrak{m})$ is Buchsbaum.

6. A characterization

Propostion A. (i) $G(\mathfrak{m})$ is **NOT** C-M $\Longleftrightarrow$
$\exists\ \alpha \in R'$ and $h \in \{1, \ldots, r-2\}$ such that $\alpha x^{h+1} \in \mathfrak{m}^h \setminus \mathfrak{m}^{h+1}$
(there is an element, $\overline{\alpha x^{h+1}}$, in $(0 :_{G(\mathfrak{m})} \mathcal{M}^r)$)

(ii) Assume that $G(\mathfrak{m})$ is Buchsbaum not C-M;
if α and h are as in (i), then $\alpha x^{h+2} \in \mathfrak{m}^{h+2}$
(any $\overline{\alpha x^{h+1}} \in (0 :_{G(\mathfrak{m})} \mathcal{M}^r)$ is also in $(0 :_{G(\mathfrak{m})} \mathcal{M}) \subseteq (0 :_{G(\mathfrak{m})} \overline{x}))$

More precisely:

Theorem B. Let $G(\mathfrak{m})$ be not C-M. Then
$G(\mathfrak{m})$ is Buchsbaum $\Longleftrightarrow \forall\ \alpha \in R'$ such that $\alpha x^{h+1} \in \mathfrak{m}^h \setminus \mathfrak{m}^{h+1}$,
for some $h \in \{1, \cdots, r-2\}$, then $\alpha x^{h+1}\mathfrak{m} \subseteq \mathfrak{m}^{h+2}$.

7. Some well-known facts

• $H \subset \mathbb{Z}$, $H \neq \emptyset$ is a relative ideal of a semigroup S if $H + S \subseteq H$ and $H + s \subseteq S$ ($\exists\ s \in S$).

• If H and L are relative ideals of S, then $H + L$, nH and $H - L := \{n \in \mathbb{Z} \mid n + L \subseteq H\}$ are also relative ideals of S.

• The ideal $M = \{s \in S \mid s \neq 0\}$ is called the maximal ideal of S.

• The reduction number r of M is the minimal natural number such that $(r + 1)M = g_1 + rM$.

• If I and J are fractional ideals of R, then $v(I)$ and $v(J)$ are relative ideals of $S = v(R)$;

• if I and J are monomial ideals, then $v(I \cap J) = v(I) \cap v(J)$, $v(I^n) = n \cdot v(I)$ and $v(I :_Q J) = v(I) - v(J)$;

• if $J \subseteq I$, then $l_R(I/J) = |v(I) \setminus v(J)|$.

The blow up of $S=\langle g_1,g_2,\ldots,g_n\rangle$ is the numerical semigroup

$$S'=\bigcup_{n\geq 1}(nM-nM)=(rM-rM)=rM-rg_1=\langle g_1,g_2-g_1,\ldots,g_n-g_1\rangle$$

8. Translation at semigroup level

$$v((\mathfrak{m}^{h+r+1}:_Q\mathfrak{m}^r)\cap\mathfrak{m}^h)=(((r+h+1)M-rM)\bigcap hM)=$$
$$=((S'+(h+1)g_1)\bigcap hM)$$
$$v((\mathfrak{m}^{h+1}:_Q\mathfrak{m})\cap\mathfrak{m}^h)=((h+2)M-M)\bigcap hM)$$

$G(\mathfrak{m})$ C-M $\Longleftrightarrow ((h+1)g_1+S')\bigcap hM=(h+1)M \quad \forall\, h=1,\ldots,r-2$

$G(\mathfrak{m})$ Buchs., not C-M $\Longleftrightarrow ((S'+(h+1)g_1)\bigcap hM)\setminus(h+1)M=$
$=(((h+2)M-M)\bigcap hM)\setminus(h+1)M\neq\emptyset$

$\Longleftrightarrow \forall\ \alpha\in S'$ such that $\alpha+(h+1)g_1\in hM\setminus(h+1)M$, for some $h\in\{1,\cdots,r-2\}$, then $\alpha+(h+1)g_1+M\subseteq(h+2)M$.

9. The Apery set

Fix $\overline{s} \in S$ and set $\omega_i := min\{s \in S \mid s \equiv i \pmod{\overline{s}}\}$. $\omega_0 = 0$
We call the Apery set S with respect of $\overline{S}$, the set

$$\text{Ap}_{\overline{s}}(S) = \{\omega_0, \ldots, \omega_{\overline{s}-1}\}$$

We will compare the Apery sets of S and S', with respect to g_1.
We fix the following notations:
$\text{Ap}_{g_1}(S) = \{\omega_0, \ldots, \omega_{g_1-1}\}$

$\text{Ap}_{g_1}(S') = \{\omega'_0, \ldots, \omega'_{g_1-1}\}$

Definition. (Barucci-Fröberg) For each $i = 0, 1, \ldots, g_1 - 1$ let:
a_i be the only integer such that $\omega_i' + a_i g_1 = \omega_i$;
$b_i = \max\{l \mid \omega_i \in lM\}$.

Remark. $b_0 = a_0 = 0$ and $1 \leq b_i \leq a_i$.

Theorem. (Barucci-Fröberg) $G(\mathfrak{m})$ is C-M $\iff$ $a_i = b_i$ for each $i = 0, 1, \ldots, g_1 - 1$.
Semigroup level: $a_i > b_i \iff$
$\exists\ s \equiv i \pmod{g_1}$ such that $s \in ((h+1)g_1 + S') \cap hM \setminus (h+1)M$.

Idea of the proof of $(\Rightarrow)$:
(Recall that $G(\mathfrak{m})$ C-M $\iff ((h+1)g_1 + S') \bigcap hM = (h+1)M$)
Assume $a_i > b_i$; let $s' = \omega_i' + (a_i - b_i - 1)g_1 \in S'$.
Then $\omega_i' + a_i g_1 = s' + (b_i + 1)g_1 \in b_i M \setminus (b_i + 1)M$;
for $h = b_i$ we get the thesis.
In particular, $t^{\omega_i} \in (0 :_{G(\mathfrak{m})} \mathcal{M}^r)$.

10. An example

Let $S = \langle 13, 16, 23, 31, 41, 51, 56 \rangle$ ($r = 5$, $R = k[[S]]$)

$S' = \langle 3, 10 \rangle = \{0, 3, 6, 9, 10, 12, 13, 15, 16, 18, \longrightarrow\}$

$\mathrm{Ap}_{13}(S) = \{0, 79, 41, 16, 56, 31, 32, 46, 47, 48, 23, 63, 51\}$

$\mathrm{Ap}_{13}(S') = \{0, 27, 15, 3, 30, 18, 6, 20, 21, 9, 10, 24, 12\}$.

$41 \in M \setminus 2M \Longrightarrow b_2 = \max\{l \mid \omega_2 \in lM\} = 1$

$41 - 15 = 2 \cdot 13 \Longrightarrow a_2 = 2 \Longrightarrow t^{41} \in (0 :_{G(\mathfrak{m})} \mathcal{M}^r)$.

Hence $G(\mathfrak{m})$ is not C-M. We will see that $G(\mathfrak{m})$ is not even Buchsbaum.

11. More invariants

We define two more families of invariants for S. Recall that:

$$\begin{array}{ccccccc} M - g_1 & \subseteq & 2M - 2g_1 & \subseteq & \cdots & \subseteq & rM - rg_1 \\ \cup| & & \cup| & & & & || \\ M - M & \subseteq & 2M - 2M & \subseteq & \cdots & \subseteq & rM - rM \end{array}$$

For each $i = 0, 1, \ldots, g_1 - 1$ set:

$c_i := \min\{n \mid \omega_i' \in nM - ng_1\}$

$d_i := \min\{n \mid \omega_i' \in nM -_{\mathbb{Z}} nM\}$.

Proposition. $b_i \leq a_i \leq c_i \leq d_i$. Moreover, $b_i < a_i \iff a_i < c_i$.
In particular, if $b_i < a_i$ and $d_i = a_i + 1$, then $c_i = a_i + 1$.

Theorem C. $G(\mathfrak{m})$ is Buchsbaum $\Longrightarrow c_i - a_i \leq a_i - b_i, \ \forall \ i.$

(Easy case: $a_i = b_i + 1$. Recall that $t^{\omega_i} \in (0 :_{G(\mathfrak{m})} \mathcal{M}^r)$.
$G(\mathfrak{m})$ Buchsbaum $\Longrightarrow$
$t^{\omega_i} \in (0 :_{G(\mathfrak{m})} \mathcal{M}^r) = (0 :_{G(\mathfrak{m})} \mathcal{M}) \subseteq (0 :_{G(\mathfrak{m})} \overline{x}) \Longrightarrow$
$\omega_i + g_1 \in (b_i + 2)M$ i.e. $\omega_i' + (a_1 + 1)g_1 \in (a_1 + 1)M \Longrightarrow c_i = a_i + 1$.)

In the example, $15 + 3 \cdot 13 = 54 \notin 3M$, $15 + 4 \cdot 13 = 67 \notin 4M$ and $15 + 5 \cdot 13 = 80 \in 5M \Longrightarrow c_2 = \min\{n \mid 15 \in nM - ng_1\} = 5$.
Hence $c_2 - a_2 > a_2 - b_2$ and $G(\mathfrak{m})$ is not Buchsbaum.
Note that: $b_2 = 1$, $a_2 = 2$, $41 + 13 = 15 + 3 \cdot 13 \notin 3M$ hence $\overline{t^{41}} \notin (0 :_{G(\mathfrak{m})} \mathcal{M})$.

Remark. $R = k[[t^5, t^9, t^{22}]]$ shows that the condition in the last theorem is not sufficient.

Theorem D. Suppose $d_i = a_i + 1$ for every i such that $a_i > b_i$. Then $G(\mathfrak{m})$ is Buchsbaum.

(Let $\alpha \in S'$ such that $\alpha + (h+1)g_1 \in hM \setminus (h+1)M$ We need to show that $\alpha + (h+1)g_1 \in ((h+2)M - M)$

Easy case: $a_i = b_i + 1$ and $\alpha = \omega_i'$
We have $\omega_i' + (b_i + 1)g_1 \in b_i M \setminus (b_i + 1)M$ and
$d_i = b_i + 2$ i.e. $\omega_i' \in ((b_i + 2)M - (b_i + 2)M)$
$\Longrightarrow \beta := \alpha + (h+1)g_1 \in ((b_i + 2)M - M)$)

Remark. $R = k[[t^{10}, t^{17}, t^{23}, t^{82}]]$ shows that the condition in the last theorem is not necessary.

12. A further example

$S = \langle 12, 23, 66, 99, 100, 110, 121 \rangle$, $R = k[[S]]$.
$S' = \langle 11, 12, 54 \rangle$

$\mathrm{Ap}_{12}(S) = \{0, 121, 110, 99, 100, 89, 66, 115, 92, 69, 46, 23\}$.
$\mathrm{Ap}_{12}(S') = \{0, 109, 98, 87, 76, 65, 54, 55, 44, 33, 22, 11\}$.

$100 \in M \setminus 2M$ and $100 - 76 = 2 \cdot 12$, $\Longrightarrow b_4 = 1$ and $a_4 = 2$.
(Hence $G(\mathfrak{m})$ is not C-M.)

$76 + 2 \cdot 12 = 100 \notin 2M \Longrightarrow 76 + 2M \not\subseteq 2M$,
Moreover, $76 + 3M \subseteq 3M \Longrightarrow d_4 = \min\{n \mid 76 \in nM -_{\mathbb{Z}} nM\} = 3$

By Theorem D, $d_4 = a_4 + 1 \Longrightarrow G(\mathfrak{m})$ is Buchsbaum.
In fact $76 + 2 \cdot 12 + M \subseteq 3M$