Weak Asymptotics in the 3-dim Frobenius Problem

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3-dim Frobenius Problem

Let $\mathsf{S}\left(\mathbf{d}^3\right) \subset Z_+$ be the additive numerical semigroup

$$\mathsf{S}\left(\mathbf{d}^3\right) = \left\{s \mid s = \sum_{j=1}^{3} d_j x_j,\ x_j \in Z_+ \cup \{0\}\right\}$$

finitely generated by a minimal set of positive integers $\mathbf{d}^3 = (d_1, d_2, d_3)$ such that

$$3 \le d_1 < d_2 < d_3\ , \quad \gcd(d_1, d_2, d_3) = 1$$

The smallest integer $C\left(\mathbf{d}^3\right)$

$$C\left(\mathbf{d}^3\right) := \min\left\{s \in \mathsf{S}\left(\mathbf{d}^3\right) \mid s + Z_+ \cup \{0\} \subset \mathsf{S}\left(\mathbf{d}^3\right)\right\}$$

is called *the conductor* **of $\mathsf{S}\left(\mathbf{d}^3\right)$.**

The number

$$F\left(\mathbf{d}^3\right) = C\left(\mathbf{d}^3\right) - 1$$

is referred to as *the Frobenius number*

Denote by $\Delta\left(\mathbf{d}^3\right)$ the complement of $\mathsf{S}\left(\mathbf{d}^3\right)$ in Z_+, i.e.

$$\Delta\left(\mathbf{d}^3\right) = Z_+ \setminus \mathsf{S}\left(\mathbf{d}^3\right)$$

The number of gaps, or *a genus*,

$$G\left(\mathbf{d}^3\right) := \#\left\{\Delta\left(\mathbf{d}^3\right)\right\} < \infty$$

The number of non-gaps

$$\widetilde{G}\left(\mathbf{d}^3\right) := \#\left\{\mathsf{S}\left(\mathbf{d}^3\right) \cap \left[0; F\left(\mathbf{d}^3\right)\right]\right\}$$

so that

$$G\left(\mathbf{d}^3\right) + \widetilde{G}\left(\mathbf{d}^3\right) = C\left(\mathbf{d}^3\right)$$

The semigroup $\mathsf{S}\left(\mathbf{d}^3\right)$ is called *symmetric* **iff for any integer s holds**

$$s \in \mathsf{S}\left(\mathbf{d}^3\right) \quad \Longleftrightarrow \quad F\left(\mathbf{d}^3\right) - s \notin \mathsf{S}\left(\mathbf{d}^3\right)$$

Otherwise $\mathsf{S}\left(\mathbf{d}^3\right)$ is called *non-symmetric***.**

$$2G\left(\mathbf{d}^3\right) = C\left(\mathbf{d}^3\right) \quad \textbf{if} \ \ \mathsf{S}\left(\mathbf{d}^3\right) \ \ \textbf{is symmetric,}$$
$$\mathbf{2G}\left(\mathbf{d}^3\right) > C\left(\mathbf{d}^3\right) \quad \textbf{otherwise}$$

Denote by $p\left(\mathbf{d}^3\right)$ a fraction of the segment $\left[0; F\left(\mathbf{d}^3\right)\right]$ which is occupied by the semigroup $\mathsf{S}\left(\mathbf{d}^3\right)$

$$p\left(\mathbf{d}^3\right) = \frac{\widetilde{G}\left(\mathbf{d}^3\right)}{C\left(\mathbf{d}^3\right)}$$

Note that for every symmetric semigroup it holds

$$p\left(\mathbf{d}^3\right) \overset{\substack{symmetric \\ semigroup}}{=} \frac{1}{2}$$

Polynomial Ring Associated with Semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$

Let $R = \mathsf{k}\left[X_1, X_2, X_3\right]$ **be a polynomial ring in 3 variables over a field** k **of characteristic 0 and**

$$\pi \quad : \quad \mathsf{k}\left[X_1, X_2, X_3\right] \longmapsto \mathsf{k}\left[z^{d_1}, z^{d_2}, z^{d_3}\right]$$

be the projection induced by $\pi\left(X_i\right) = z^{d_i}$**. Denote**

$$\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^3\right)\right] := \mathsf{k}\left[z^{d_1}, z^{d_2}, z^{d_3}\right]$$

Then $\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^3\right)\right]$ **is a 1-dim Cohen-Macaulay graded subring of** $\mathsf{k}\left[X_1, X_2, X_3\right]$**.**

The *type* **of the ring** $\mathsf{k}\left[\mathsf{S}\left(\mathbf{d}^3\right)\right]$

$$t\left(\mathsf{S}\left(\mathbf{d}^3\right)\right) = \#\left\{\mathsf{S}'\left(\mathbf{d}^3\right)\right\},$$

$$\mathsf{S}'\left(\mathbf{d}^3\right) = \left\{x \in Z \mid x \notin \mathsf{S}\left(\mathbf{d}^3\right), x + s \in \mathsf{S}\left(\mathbf{d}^3\right)\right\}$$

for all $s \in \mathsf{S}\left(\mathbf{d}^3\right) \setminus \{0\}$

Lemma (Herzog, 1970)

$$t\left(\mathsf{S}\left(\mathbf{d}^3\right)\right) = \begin{cases} 1\,, & \textbf{if } \mathsf{S}\left(\mathbf{d}^3\right) \textbf{ is symmetric}, \\ 2\,, & \textbf{if } \mathsf{S}\left(\mathbf{d}^3\right) \textbf{ is non-symmetric} \end{cases}$$

Theorem (Fröberg, Gottlieb, Häggkvist, 1987)

$$G\left(\mathbf{d}^3\right) \leq \widetilde{G}\left(\mathbf{d}^3\right) t\left(\mathsf{S}\left(\mathbf{d}^3\right)\right) \; .$$

Theorem (Brown, Curtis, 1991, and Brown, Herzog, 1992)

$$G\left(\mathbf{d}^3\right) = \widetilde{G}\left(\mathbf{d}^3\right) , \qquad \textbf{iff}$$
$$\mathbf{d}^3 \ \textbf{is symmetric},$$

$$G\left(\mathbf{d}^3\right) = 2\widetilde{G}\left(\mathbf{d}^3\right) , \qquad \textbf{iff}$$
$$\mathbf{d}^3 = \{3, 3k+1, 3k+2\},\ k \geq 1,$$

$$G\left(\mathbf{d}^3\right) = 2\widetilde{G}\left(\mathbf{d}^3\right) - 1 , \quad \textbf{iff}$$
$$\mathbf{d}^3 = \{3, 3k+2, 3k+4\},\ \{4, 5, 11\},\ \{4, 7, 13\}$$

$$G\left(\mathbf{d}^3\right) = 2\widetilde{G}\left(\mathbf{d}^3\right) - u ,\ \ 1 < u < \widetilde{G}\left(\mathbf{d}^3\right) , \quad \textbf{iff}$$
$$\mathbf{d}^3 = \{?,\ ?,\ ?\}$$

Wilf's Question (1978)

Is it true that for fixed m the following inequality holds

$$p\left(\mathbf{d}^m\right) = \frac{\widetilde{G}\left(\mathbf{d}^m\right)}{C\left(\mathbf{d}^m\right)} \geq \frac{1}{m}$$

with equality only for

$$\mathbf{d}^m = (m, m+1, \ldots, 2m-1) .$$

Dobbs, Mattews (2003) proved WQ for $m = 3$

Weak Asymptotics

Two sequences of real numbers $A(k)$ **and** $B(k),\ k \in Z_+$**, are said to have the same** *weak asymptotics*, **or to have asymptotically equal** *average growth rates*, **or to have the same** *Cesáro asymptotics*,

$$A(k) \stackrel{Cesáro}{\equiv} B(k) \quad \textbf{if} \quad \lim_{N\to\infty} \frac{\Sigma_{k=1}^{N} A(k)}{\Sigma_{k=1}^{N} B(k)} = 1$$

E.g.

$$\sin^2\left(\frac{\pi k}{2}\right) \stackrel{Cesáro}{\equiv} \frac{1}{2}\,, \qquad \phi(k) \stackrel{Cesáro}{\equiv} \frac{k}{\zeta(2)}$$

$\phi(k)$ and $\zeta(k)$ are Euler and Riemann functions

Weak Asymptotics at Large Integers

Let us replace k **by a neighborhood** $U_{N,r}(k)$ **of length** $2r$ **of a scaled integer** $Nk, N \in Z_+$**. Replace the values of** $A(k)$ **and** $B(k)$ **by the arithmetic means** $A_{N,r}(k)$ **and** $B_{N,r}(k)$**, respectively**

$$A_{N,r}(k) = \frac{1}{2r}\sum_{j=-r}^{r} A(Nk+j), \quad B_{N,r}(k) = \frac{1}{2r}\sum_{j=-r}^{r} B(Nk+j),$$

where $Nk + j \in U_{N,r}(k)$**. Two sequences of real numbers** $A(k)$ **and** $B(k),\ k \in Z_+$**, are said to have the same** *weak asymptotics at large* k

$$A(k) \stackrel{Cesáro}{\equiv} B(k) \quad \textbf{if} \quad \lim_{\substack{r,N\to\infty \\ r(N)/N\to 0}} \frac{A_{N,r}(k)}{B_{N,r}(k)} = w(k) = 1$$

Weak Asymptotics in Semigroups $\mathsf{S}\left(\mathbf{d}^3\right)$ at typical large vectors $\mathbf{d}^3$

Arnold's recipe (V. Arnold, 1999)

Let $\mathsf{S}\left(\mathbf{d}^3\right)$ be a semigroup, i.e. a generating set (d_1, d_2, d_3) is minimal. Replace the vector $\mathbf{d}^3$ by a cubic neighborhood $U_{N,r}\left(\mathbf{d}^3\right)$ of edge r of a scaled vector $N\mathbf{d}^3 \in Z_+^3,\ N \in Z_+$ such that

$$1 \ll r \ll N\ , \quad r(N)/N \to 0 \quad \textbf{when} \quad N \to \infty$$

Denote by $\mathbf{j}^3$ a vector (j_1, j_2, j_3). Replace the value $A\left(\mathbf{d}^3\right)$ by the arithmetic mean $A_{N,r}\left(\mathbf{d}^3\right)$ of the functions $A\left(N\mathbf{d}^3+\mathbf{j}^3\right)$ at the vectors $N\mathbf{d}^3+\mathbf{j}^3 \in U_{N,r}\left(\mathbf{d}^3\right)$ whose components $Nd_i + j_i, j_i \in Z_+,\ -r \le j_i \le r$, satisfy two constraints:

C1.

$$\gcd(Nd_1 + j_1, Nd_2 + j_2, Nd_3 + j_3) = 1$$

otherwise a semigroup has infinite complement $\Delta\left(\mathbf{d}^3\right)$

C2.

$$\{Nd_1 + j_1, Nd_2 + j_2, Nd_3 + j_3\}$$

is a minimal generating set, otherwise $N\mathbf{d}^3+\mathbf{j}^3$ does not generate the 3-dim semigroup

Call the vector $N\mathbf{d}^3+\mathbf{j}^3$ *typical* if its components satisfy both constraints, C1 and, C2

Denote by $M_{N,r}\left(\mathbf{d}^3\right)$ an entire set of typical vectors, $M_{N,r}\left(\mathbf{d}^3\right) \subset U_{N,r}\left(\mathbf{d}^3\right)$

$$M_{N,r}\left(\mathbf{d}^3\right) = \left\{ N\mathbf{d}^3 + \mathbf{j}^3 \;\middle|\; \begin{array}{c} -r \le j_i \le r,\ 1 \ll r \ll N \\ \\ \mathbf{C1},\ \mathbf{C2}\ \ are\ \ satisfied \end{array} \right\}$$

Denote by $\#\left\{M_{N,r}\left(\mathbf{d}^3\right)\right\}$ a cardinality. Notice that

$$N\mathbf{d}^3 \notin M_{N,r}\left(\mathbf{d}^3\right) \quad \rightarrow \quad \#\left\{M_{N,r}\left(\mathbf{d}^3\right)\right\} < (2r)^3$$

Define a mean average

$$A_{N,r}\left(\mathbf{d}^3\right) = \frac{1}{\#\left\{M_{N,r}\left(\mathbf{d}^3\right)\right\}} \sum_{\substack{j_i=-r \\ i=1,2,3}}^{r} A\left(N\mathbf{d}^3 + \mathbf{j}^3\right)$$

Say that two numerical functions $A\left(\mathbf{d}^3\right)$ and $B\left(\mathbf{d}^3\right)$ have the same *weak asymptotics at the typical large* $\mathbf{d}^3$

$$\mathsf{A}\left(\mathbf{d}^3\right) \overset{Ces\acute{a}ro}{\equiv} \mathsf{B}\left(\mathbf{d}^3\right) \quad \mathbf{if} \quad \lim_{\substack{r,N\rightarrow\infty \\ r(N)/N\rightarrow 0}} \frac{A_{N,r}\left(\mathbf{d}^3\right)}{B_{N,r}\left(\mathbf{d}^3\right)} = w(\mathbf{d}^3) = 1$$

1. Arnold's Conjectures

<u>**Conjecture**</u> $\#1999-8$

Explore the statistics of $C\left(\mathbf{d}^3\right)$ for typical large vectors $\mathbf{d}^3$. Conjecturally,

$$C\left(\mathbf{d}^3\right) \overset{\textit{Cesáro}}{\equiv} \sqrt{2}\sqrt{d_1 d_2 d_3}$$

<u>**Conjecture**</u> $\#2003-5$

The mean values $C_{N,r}\left(\mathbf{d}^3\right)$ have a growth rate (*probably provided by conjectured formula***)**

$$C\left(\mathbf{d}^3\right) \overset{\textit{Cesáro}}{\equiv} const \cdot \sqrt{d_1 d_2 d_3}$$

<u>**Conjecture**</u> $\#1999-9$

Determine $p\left(\mathbf{d}^3\right)$ for large vectors $\mathbf{d}^3$. Conjecturally, this fraction is asymptotically equal to $1/3$

$$\widetilde{G}\left(\mathbf{d}^3\right) \overset{\textit{Cesáro}}{\equiv} \frac{1}{3}\, C\left(\mathbf{d}^3\right)$$

<u>**Conjecture**</u> $\#1999-10$
(*It implies <u>Conjecture</u> # 1999-9*)

Find the typical density $\sigma_3(s)$ of filling the segment $[0; F\left(\mathbf{d}^3\right)]$ asymptotically for large $\mathbf{d}^3$. Conjecturally,

$$\sigma_3(s) = \left(\frac{s}{C\left(\mathbf{d}^3\right)}\right)^2, \ \textbf{where} \quad \int_0^C \sigma_3(s)ds = \widetilde{G}\left(\mathbf{d}^3\right)$$

Statistics of Numerical Semigroups $\mathsf{S}\left(N\mathbf{d}^3+\mathbf{j}^3\right)\ ,\quad N\to\infty$

Represent a set $M_{N,r}\left(\mathbf{d}^3\right)$ as follows,

$$M_{N,r}\left(\mathbf{d}^3\right)=\widehat{M_{N,r}}\left(\mathbf{d}^3\right)\setminus\widetilde{M_{N,r}}\left(\mathbf{d}^3\right)$$

$$\widehat{M_{N,r}}\left(\mathbf{d}^3\right)=\left\{N\mathbf{d}^3+\mathbf{j}^3\;\middle|\;\begin{array}{c}-r\le j_i\le r,\ 1\ll r\ll N\\ \\ \mathbf{C1}\ \ is\ \ satisfied\end{array}\right\}$$

$$\widetilde{M_{N,r}}\left(\mathbf{d}^3\right)=\left\{N\mathbf{d}^3+\mathbf{j}^3\;\middle|\;\begin{array}{c}-r\le j_i\le r,\ 1\ll r\ll N\\ \overline{\mathbf{C2}}\ \ is\ \ satisfied\\ \mathbf{C1}\ \ is\ \ satisfied\end{array}\right\}$$

$\overline{C2}$: $\{\mathbf{Nd}_1+j_1,Nd_2+j_2,Nd_3+j_3\}$ **is not minimal**

<u>Lemma</u> (LGF, 2005)

$$\lim_{\substack{r,N\to\infty\\ r/N\to 0}}\ \frac{\#\left\{\widehat{M_{N,r}}\left(\mathbf{d}^3\right)\right\}}{(2r)^3}=\frac{1}{\zeta(3)}\simeq 0.8319$$

$$\lim_{\substack{r,N\to\infty\\ r/N\to 0}}\ \frac{\#\left\{\widetilde{M_{N,r}}\left(\mathbf{d}^3\right)\right\}}{(2r)^3}=0$$

$M_{N,r}\left(\mathbf{d}^m\right)$ **is a union of symmetric** $M_{N,r}^{sym}\left(\mathbf{d}^3\right)$ **and non-symmetric** $M_{N,r}^{nsym}\left(\mathbf{d}^3\right)$ **semigroups**

$$M_{N,r}\left(\mathbf{d}^3\right)=M_{N,r}^{sym}\left(\mathbf{d}^3\right)\ \cup\ M_{N,r}^{nsym}\left(\mathbf{d}^3\right)$$

$$\#\left\{M_{N,r}\left(\mathbf{d}^3\right)\right\}=\#\left\{M_{N,r}^{sym}\left(\mathbf{d}^3\right)\right\}+\#\left\{M_{N,r}^{nsym}\left(\mathbf{d}^3\right)\right\}$$

Theorem (Herzog, 1970, and Watanabe, 1973)

A semigroup $\mathsf{S}(d_1, d_2, d_3)$ is symmetric iff its minimal generating set has a presentation with at least 2 relatively not prime elements

$$\gcd(d_1, d_2) = b\ ,\quad \gcd(d_3, b) = 1\ ,\quad d_3 \in \mathsf{S}\left(\frac{d_1}{b}, \frac{d_2}{b}\right)$$

Lemma (LGF, 2005)

Let $\mathsf{S}(\mathbf{d}^3)$ and $\mathsf{S}(N\mathbf{d}^3 + \mathbf{j}^3)$ be semigroups and $\{Nd_1 + j_1, Nd_2 + j_2, Nd_3 + j_3\}$ be a minimal generating set such that

$$-r \le j_1, j_2, j_3 \le r\ ,\quad 1 \ll r \ll N\ .$$

If $r(N)/N \to 0$ when $r, N \to \infty$ then

$$\lim_{\substack{r,N\to\infty \\ r/N\to 0}} \frac{\#\left\{M_{N,r}^{sym}\left(\mathbf{d}^3\right)\right\}}{(2r)^3} = 0$$

Corollary (LGF, 2005)

Almost all numerical semigroups $\mathsf{S}(N\mathbf{d}^3 + \mathbf{j}^3)$ are non-symmetric

$$\lim_{\substack{r,N\to\infty \\ r/N\to 0}} \frac{\#\left\{M_{N,r}^{nsym}\left(\mathbf{d}^3\right)\right\}}{(2r)^3} = \frac{1}{\zeta(3)}$$

Matrix $\mathcal{R}_3$ of Minimal Relations for Non-Symmetric Semigroups

Lemma (Johnson, 1960)

$$\mathcal{R}_3 \cdot \mathbf{d}^3 = \mathbf{0} \ , \quad \mathcal{R}_3 = \begin{pmatrix} u_1+w_1 & -u_2 & -w_3 \\ -w_1 & u_2+w_2 & -u_3 \\ -u_1 & -w_2 & u_3+w_3 \end{pmatrix}$$

$$u_i, w_i \in Z_+ \ , \quad i=1,2,3$$

$$\begin{aligned} d_1 &= u_2u_3 + w_2w_3 + u_2w_3 \ , \quad \gcd(u_1, w_2, u_3+w_3) = 1, \\ d_2 &= u_3u_1 + w_3w_1 + u_3w_1 \ , \quad \gcd(u_2, w_3, u_1+w_1) = 1, \\ d_3 &= u_1u_2 + w_1w_2 + u_1w_2 \ , \quad \gcd(u_3, w_1, u_2+w_2) = 1 \end{aligned}$$

Conductor $C\left(\mathbf{d}^3\right)$ and Genus $G\left(\mathbf{d}^3\right)$

Theorem (LGF, 2004/6)

$$\begin{aligned} C\left(\mathbf{d}^3\right) &= 1 + \prod_{i=1}^{3}(u_i+w_i) - A_2 - B_2 - D + max\{A_3, B_3\} \\ 2G\left(\mathbf{d}^3\right) &= 1 + \prod_{i=1}^{3}(u_i+w_i) - A_2 - B_2 - D + A_3 + B_3 \end{aligned}$$

$$2G\left(\mathbf{d}^3\right) - C\left(\mathbf{d}^3\right) = min\{A_3, B_3\}$$

where

$$\begin{aligned} &A_2 = u_1u_2 + u_3u_1 + u_2u_3 \ , \quad A_3 = u_1u_2u_3 \ , \\ &B_2 = w_1w_2 + w_3w_1 + w_2w_3 \ , \quad B_3 = w_1w_2w_3 \ , \\ &D = u_1w_2 + u_2w_3 + u_3w_1 \ . \end{aligned}$$

New Scaling in the $Z_+^3 \times Z_+^3$ Lattice

Consider two tuples $\mathbf{u}^3$ and $\mathbf{w}^3$

$$\mathbf{u}^3 = (u_1, u_2, u_3) \ , \quad \mathbf{w}^3 = (w_1, w_2, w_3)$$

Their union is a tuple in the 6–dim cubic lattice

$$\mathbf{u}^3 \cup \mathbf{w}^3 = (u_1, u_2, u_3, w_1, w_2, w_3) \in Z_+^3 \times Z_+^3$$

A mapping $Z_+^3 \times Z_+^3 \longmapsto Z_+^3$ is defined by three homogeneous functions of the 2-nd order

$$d_i = f_i(u_1, u_2, u_3, w_1, w_2, w_3), \quad N^2\mathbf{d}^3 = f(N\mathbf{u}^3, N\mathbf{w}^3)$$

Replace a scaling in Z_+^3 lattice, $N^2 d_i \in Z_+, \ N \in Z_+$, by the scaling in $Z_+^3 \times Z_+^3$ lattice, $Nu_i, Nw_i \in Z_+$ and define

$$A_{N,r}\left(\mathbf{u}^3 \cup \mathbf{w}^3\right) = \left\{\left(N\mathbf{u}^3 + \mathbf{j}^3\right) \cup \left(N\mathbf{w}^3 + \mathbf{k}^3\right) | \mathcal{C}\right\}$$

where

$$\mathcal{C} = \left\{ \begin{array}{c} \gcd\left(D_{1,N}\left(\mathbf{j}^3, \mathbf{k}^3\right), D_{2,N}\left(\mathbf{j}^3, \mathbf{k}^3\right), D_{3,N}\left(\mathbf{j}^3, \mathbf{k}^3\right), \right) = 1 \ , \\ \mathbf{j}^3 = (j_1, j_2, j_3) \ , \quad \mathbf{k}^3 = (k_1, k_2, k_3) \ , \\ -r \le j_i, k_i \le r \ , \quad 1 \ll r \ll N \end{array} \right\}$$

$$D_{i,N}\left(\mathbf{j}^3, \mathbf{k}^3\right) = d_i\left(N\mathbf{u}^3 + \mathbf{j}^3, N\mathbf{w}^3 + \mathbf{k}^3\right), \quad i = 1, 2, 3$$

A cardinality $\#\left\{A_{N,r}\left(\mathbf{u}^3 \cup \mathbf{w}^3\right)\right\}$

$$\#\left\{A_{N,r}\left(\mathbf{u}^3 \cup \mathbf{w}^3\right)\right\} \simeq \frac{(2r)^6}{\zeta(3)}$$

The averaging will be performed on the set $A_{N,r}\left(\mathbf{u}^3 \cup \mathbf{w}^3\right)$ instead of the set $M_{N,r}\left(\mathbf{d}^3\right)$

Asymptotics $\mathsf{K}\left(\mathbf{d}^3\right)$ and $\mathsf{P}\left(\mathbf{d}^3\right)$

Denote by $V\left(\mathbf{d}^3\right) = d_1 d_2 d_3$. Define the asymptotics

$$\mathsf{K}\left(\mathbf{d}^3\right) = \lim_{\substack{r,N\to\infty \\ r(N)/N\to 0}} \frac{C_{N,r}\left(\mathbf{d}^3\right)}{\sqrt{V_{N,r}\left(\mathbf{d}^3\right)}}$$

$$\mathsf{P}\left(\mathbf{d}^3\right) = \lim_{\substack{r,N\to\infty \\ r(N)/N\to 0}} \frac{\widetilde{G}_{N,r}\left(\mathbf{d}^3\right)}{C_{N,r}\left(\mathbf{d}^3\right)}$$

Straightforward calculation gives

$$\mathsf{K}\left(\mathbf{d}^3\right) = \frac{(1+\rho_1)(1+\rho_2)(1+\rho_3) + max\{1, \rho_1\rho_2\rho_3\}}{\sqrt{(1+\rho_2\rho_3+\rho_2)(1+\rho_3\rho_1+\rho_3)(1+\rho_1\rho_2+\rho_1)}}$$

$$\mathsf{P}\left(\mathbf{d}^3\right) = \frac{1}{2}\left(1 - \frac{min\{1, \rho_1\rho_2\rho_3\}}{(1+\rho_1)(1+\rho_2)(1+\rho_3) + max\{1, \rho_1\rho_2\rho_3\}}\right)$$

where

$$\rho_i = \frac{u_i}{w_i}\ , \quad i = 1, 2, 3\ , \quad 0 < \rho_i < \infty$$

Conjectures are refuted

Conjecture #1999 − 8

$$\mathsf{K}\left(\mathbf{d}^3\right) > \sqrt{3}$$

Moreover, if $\rho_1 = \rho_2 = \rho \gg 1$, $\rho_3 = \rho^{-2} \ll 1$, **then**

$$\mathsf{K}\left(\mathbf{d}^3\right) = \sqrt{\rho} \gg 1$$

Conjecture #2003 − 5

$$\mathsf{K}\left(\mathbf{d}^3\right) \quad \textbf{is not a constant}$$

Conjecture #1999 − 9

$$\frac{4}{9} < \mathsf{P}\left(\mathbf{d}^3\right) < \frac{1}{2}$$

$$\mathsf{P}\left(\mathbf{d}^3\right) \quad \textbf{is not a constant}$$

Conjecture #1999 − 10

$$\sigma_3(s) \neq \left(\frac{s}{C\left(\mathbf{d}^3\right)}\right)^2$$

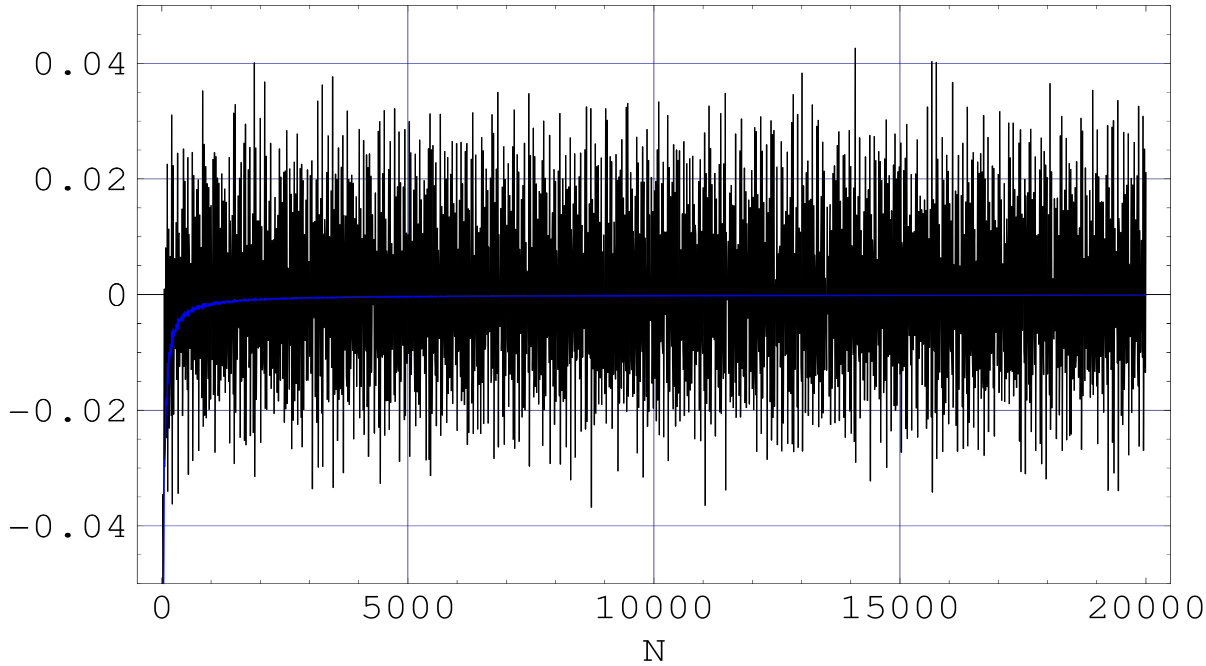

Figure 1: **Differential (*black zig-zag curve*) and cumulative (*blue curve*) density $P_2 - 1/\zeta(2)$ of symmetric tuples (d_1, d_2) in Z_+^2- lattice with edge N. (*Jointly with B. Rubinstein*)**

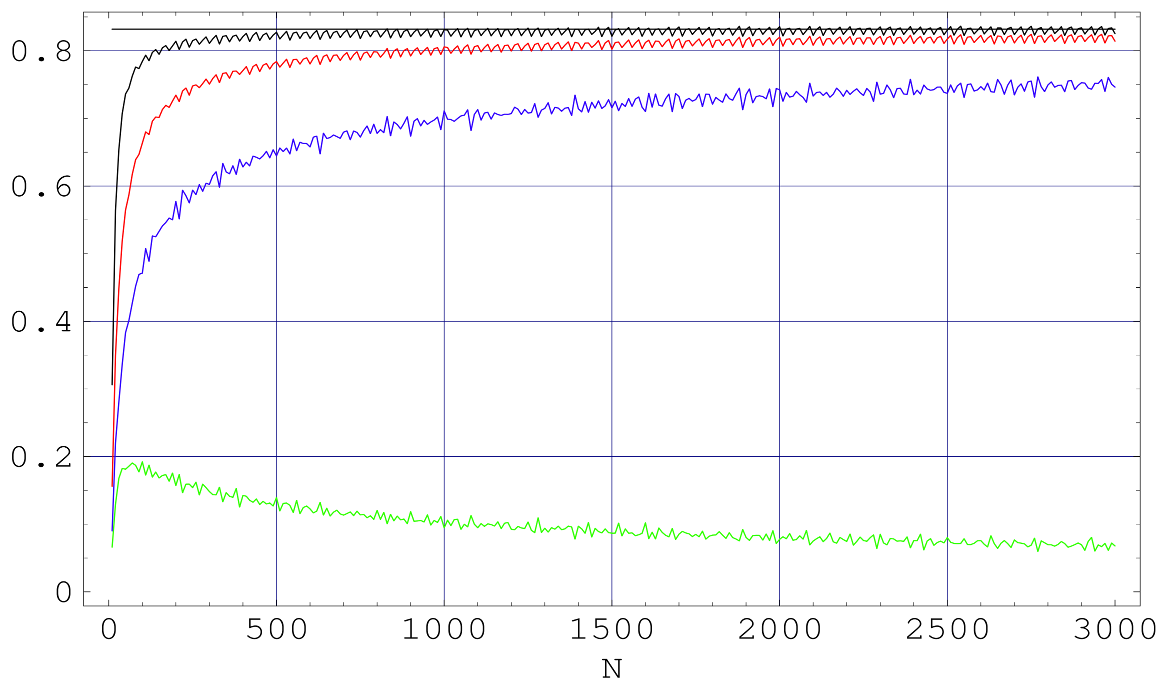

Figure 2: **Differential densities P_3 of 4 different distributions of the triples (d_1, d_2, d_3) in Z_+^3−lattice with edge N:** $\gcd(d_1, d_2, d_3) = 1$ **(*black*), symmetric and nonsymmetric triples (*red*), only nonsymmetric triples (*blue*), only symmetric triples (*green*).**
(*Jointly with B. Rubinstein*)