On the homology of semigroup rings

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Let S be a numerical semigroup, $g(S)$ the Frobenius number, $T(S) = \{n \in \mathbb{Z}; n+s \in S \text{ if } s > 0\}$, $k[[S]]$ (or $k[S]$) the semigroup ring.

Lemma 1 *For $s \in S \setminus \{0\}$ we have $n \in T(S)$ if and only if $\overline{t^{n+s}} \in \mathrm{Soc}(k[[S]]/(t^s))$. Hence* $\dim_k \mathrm{Soc}(k[[S]]/(t^s)) = |T(S)|$ *(the CM-type of* $k[[S]]$*).*

Proof $n \in T(S)$ iff $t^n \notin k[[S]]$ and $t^n m \subset k[[S]]$ iff $t^{n+s} \notin t^s k[[S]]$ and $t^{n+s} m \subset t^s k[[S]]$ iff $\overline{t^{n+s} m} = \bar{0}$ and $\overline{t^{n+s}} \neq 0$ in $k[[S]]/(t^s)$ iff $\overline{t^{n+s}} \in \mathrm{Soc}(k[[S]]/(t^s))$.

Let $S = \langle n_1, \ldots, n_k \rangle$ and let c_i be the smallest positive integer such that $c_i n_i \in \langle n_1, \ldots, \widehat{n}_i, \ldots, n_k \rangle$, and suppose that $c_i n_i = \sum_{j \neq i} r_{ij} n_j$.

Theorem 1 (Herzog) *Assume* $S = \langle n_1, n_2, n_3 \rangle$ *and not symmetric. Then* $k[[S]] \simeq k[[X_1, X_2, X_3]]/I$ *where* $I =$

$$(X_1^{c_1} - X_2^{r_{12}} X_3^{r_{13}}, X_2^{c_2} - X_1^{r_{21}} X_3^{r_{23}}, X_3^{c_3} - X_1^{r_{31}} X_2^{r_{32}})$$

and $r_{ij} < c_j$ *for all* i, j.

Theorem 2 (Bresinsky) *Let* $S = \langle n_1, n_2, n_3, n_4 \rangle$ *to be symmetric but not a complete intersection. Then* $k[[S]] \simeq k[[X_1, X_2, X_3, X_4]]/I$ *where*

$$I = (X_1^{c_1} - X_3^{r_{13}} X_4^{r_{14}}, X_2^{c_2} - X_1^{r_{21}} X_4^{r_{24}},$$

$$X_3^{c_3} - X_1^{r_{31}} X_2^{r_{32}}, X_4^{c_4} - X_2^{r_{42}} X_3^{r_{43}},$$

$X_3^{r_{43}} X_1^{r_{21}} - X_2^{r_{32}} X_4^{r_{14}})$ *and* $r_{ij} < c_j$ *for all* i, j.

Theorem 3 *Assume* $S = \langle n_1, n_2, n_3 \rangle$ *and not symmetric. Then there is a minimal* $R = k[[X_1, X_2, X_3]]$*-resolution of* $k[[S]]$*:*

$$0 \to R[-n_2c_2 - n_3r_{13}] \oplus R[-n_3c_3 - n_2r_{12}] \to$$

$$R[-n_1c_1] \oplus R[-n_2c_2] \oplus R[-n_3c_3] \to R \to k[[S]] \to 0.$$

Proof It is easy to get a minimal $k[[X_2, X_3]]$-resolution of

$$T = k[[S]]/(t^{n_1}) = k[[X_2, X_3]]/(X_2^{r_{12}}X_3^{r_{13}}, X_2^{c_2}, X_3^{c_3}),$$

and then lift it.

Corollary 1 *Assume* $S = \langle n_1, n_2, n_3 \rangle$ *and not symmetric. Then* $g(S) = \max\{n_2r_{12} + n_3c_3 - (n_1 + n_2 + n_3), n_2c_2 + n_3r_{13} - (n_1 + n_2 + n_3)\}$.

Proof $\mathrm{Soc}(T) = (X_2^{r_{12}-1}X_3^{c_3-1}, X_2^{c_2-1}X_3^{r_{13}-1})$.

Example If $S = \langle 7, 10, 13\rangle$ we get the resolution

$$0 \to R[-59] \oplus R[-62] \to R[-49] \oplus R[-20] \oplus R[-52]$$

$$\to R \to k[[S]] \to 0.$$

This gives $g(S) = 62 - (7 + 10 + 13) = 32$ and $T(S) = \{29, 32\}$. The Hilbert series of $k[S]$ is

$$H_{k[S]} = \frac{1 - z^{49} - z^{20} - z^{52} + z^{59} + z^{62}}{(1 - z^7)(1 - z^{10})(1 - z^{13})}.$$

The conductor $k[S] : k[t]$ is $C = t^{33}k[t]$. The Hilbert series of $k[S]/C$, is $H_{k[S]/C} = H_{k[S]} - \frac{z^{33}}{1-z} = 1 + z^7 + z^{10} + z^{13} + z^{14} + z^{17} + z^{20} + z^{21} + z^{23} + z^{24} + z^{26} + z^{27} + z^{28} + z^{30} + z^{31}$. Thus the length of $k[S]/C$ is $l(k[S]/C) = H_{k[S]/C}(1) = 15$ and $l(k[t]/k[S]) = 33 - 15 = 18$.

Theorem 4 *Assume* $S = \langle n_1, n_2, n_3, n_4\rangle$ *to be symmetric but not a complete intersection. Then there is a minimal* $R = k[[X_1, X_2, X_3, X_4]$*- resolution of* $k[[S]]$*:* $0 \to R[-n_2c_2 - n_3c_3 - n_4r_{14}] \to R^5 \to R^5 \to R \to k[[S]] \to 0$.

Proof It is easy to get a $k[[X_2, X_3, X_4]]$-resolution of $T = k[[S]]/(t^{n_1}) = k[[X_2, X_3, X_4]]/I$ where $I = (X_3^{r_{13}}X_4^{r_{14}}, X_2^{c_2}, X_3^{c_3}, X_4^{c_4} - X_2^{r_{42}}X_3^{r_{43}}, X_2^{r_{32}}X_4^{r_{14}})$, and then lift it.

Corollary 2 *Assume* $S = \langle n_1, n_2, n_3, n_4\rangle$ *to be symmetric but not a complete intersection. Then* $g(S) = n_2c_2 + n_3c_3 + n_4r_{14} - (n_1 + n_2 + n_3 + n_4)\}$.

Proof $\mathrm{Soc}(T) = (X_2^{c_2-1}X_3^{c_3-1}X_4^{r_{14}-1})$.

Example Let $S = \langle 5, 7, 9, 11 \rangle$. We get $g(S) = 13$. In the same way one calculates that

$$H_{k[S]} - z^{14}/(1-z) = 1 + z^5 + z^7 + z^9 + z^{10} + z^{11} + z^{12}$$

so the elements in S below the conductor are 0,5,7,9,10,11,12.

A semigroup S is symmetric if and only if its semigroup ring is Gorenstein. I will now discuss when it is even a complete intersection. A Gorenstein ring of codimension ≤ 2 is a complete intersection, so for semigroup rings in at most three variables Gorenstein and complete intersection are the same thing.

There is a general procure to construct new complete intersections from old. Namely, let $S = \langle n_1, \ldots, n_k \rangle$ be a complete intersection semigroup, and let $T = \langle dn_1, \ldots, dn_k, a_1n_1 + \cdots + a_kn_k \rangle$, where $\sum a_i > 1$ and $(d, a_1n_1 + \cdots + a_kn_k) = 1$. Then T is a complete intersection semigroup (Watanabe).

For 3-generated semigroups we get all symmetric semigroups in this way (Herzog, Watanabe). For complete intersections it is easy to use the Hilbert series to determine the Frobenius number. A complete intersection $k[x_1, \ldots, x_k]/(f_1, \ldots, f_{k-1})$ has Hilbert series

$$\frac{\prod_{i=1}^{k-1}(1 - z^{r_i})}{\prod_{i=1}^{k}(1 - z^{n_i})},$$

where $\deg f_i = r_i, \deg x_i = n_i$. With the lemma above, this gives that the Frobenius number is $g = \sum r_i - \sum n_i$. Note how this generalizes the 2-generated case $S = \langle n_1, n_2 \rangle$, where $k[S] = k[x_1, x_2]/(x_1^{n_2} - x_2^{n_1})$ and $g(S) = n_1 n_2 - n_1 - n_2$.

For a symmetric semigroup in three variables, $S = \langle dn_1, dn_2, a_1n_1 + a_2n_2 \rangle$, we have

$$k[S] = k[x_1, x_2, x_3]/(x_1^{n_2} - x_2^{n_1}, x_1^{a_1}x_2^{a_2} - x_3^d).$$

Thus $g(S) = dn_1n_2 + d(a_1n_1 + a_2n_2) - (dn_1 + dn_2 + a_1n_1 + a_2n_2)$. In general we have, if $S = \langle a, dS_1 \rangle$, where $S_1 = \langle n_1, \ldots, n_k \rangle$ is a complete intersection, $a = \sum a_in_i$, and $(a, d) = 1$, it follows easily that

$$g(S) = dg(S_1) + (d - 1) \sum_{i=1}^{k} a_in_i.$$

Proof If the relations in S_1 are of degrees $r_1, \ldots, r_{k-1}$, the relations in S are of degrees $dr_1, \ldots, dr_{k-1}$ and $d(a_1n_1 + \cdots + a_kn_k)$. Thus $g(S) = \sum dr_i + d(a_1n_1 + \cdots + c_kn_k) - \sum dn_i - (a_1n_1 + \cdots + a_kn_k) = d(\sum r_i - \sum n_i) + (d - 1)(a_1n_1 + \cdots + a_kn_k) = dg(S_1) + (d-1) \sum_{i=1}^{k} a_in_i$.

As an example, let $n_1, \ldots, n_k$ be pairwise relatively prime and let S be generated by

$$\{\frac{\prod_{i=1}^{k} n_i}{n_j}; j = 1, \ldots, k\}.$$

In this case we can show more, not only $k[S]$ is a complete intersection, but also $\mathrm{gr}_m(k[S])$ is (Barucci-Fröberg). One easily gets $g(S) = (k-1)N - \sum_{j=1}^{k} N/n_j$, where $N = \prod_{i=1}^{k} n_i$.

Example If $S = \langle 2\cdot 3\cdot 5, 2\cdot 3\cdot 7, 2\cdot 5\cdot 7, 3\cdot 5\cdot 7\rangle = \langle 30, 42, 70, 105\rangle$, then $g(S) = 3 \cdot 210 - (105 + 70 + 42 + 30) = 383$.

For semigroups generated by four elements, there is also another kind of complete intersection semigroups. These have relations $x_1^{a_1} - x_2^{a_2}, x_3^{a_3} - x_4^{a_4}, x_1^{b_1} x_2^{b_2} - x_3^{b_3} x_4^{b_4}$. This is the case when $S = \langle n_1, n_2, n_3, n_4 \rangle$, $d = (n_1, n_2)$, $d' = (n_3, n_4)$, $dd' = b_1 n_1 + b_2 n_2 = b_3 n_3 + b_4 n_4$. Here $g(S) = \mathrm{lcm}(n_1, n_2) + \mathrm{lcm}(n_3, n_4) + dd' - (n_1 + n_2 + n_3 + n_4)$.

As an example, if $S = \langle 14, 21, 15, 20 \rangle$, then $g(S) = 42 + 60 + 35 - (14 + 21 + 15 + 20) = 67$.

The Poincaré series

For a local ring (A, m, k) (or a graded k-algebra), M an A-module, let $P_M(z) = \sum \dim_k \mathrm{Tor}_i^A(k, M) z^i$. It was for a long time an open question (Serre, Kaplansky, Shafarevich) if $P_k(z)$ was always rational. A first counterexample was given by Anick. There are in fact counterexamples even for semigroup rings (Fröberg-Roos),

$k[[t^{18}, t^{24}, t^{25}, t^{26}, t^{28}, t^{30}, t^{33}]]$ is one.

Definition A graded algebra R is Koszul if $P_k(z) = 1/H_R(-z)$.

Our example uses a classification of Roos-Sturmfels of monomial curves in P^n (toric rings). For graded rings $R = k[x_1, \dots, x_n]/I$ the following are well known:

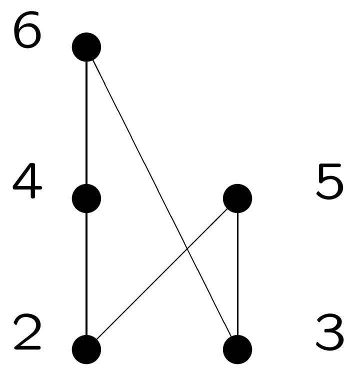
I has a quadratic Gröbner basis implies that R is Koszul implies that I is generated by quadrics. It was for some time open if the converse of these implications were true in the toric case. With an extensive computer search (more than 70000 cases) they found counterexamples to both statements. There is a ring with quadratic I which is not Koszul in P^5, and there is a Koszul algebra without quadratic Gröbner basis in P^7.

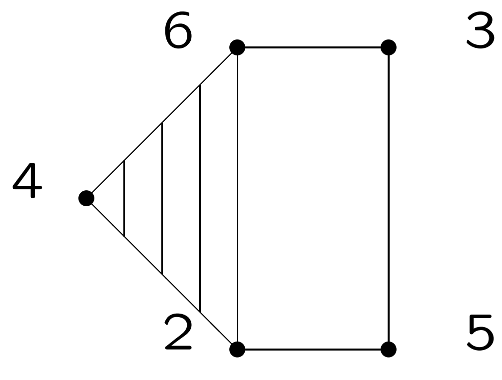
Roos-Sturmfels uses a result by Laudal-Sletsjöe:

$\mathrm{Tor}_i^{k[M]}(k,k)_\lambda = \tilde{H}_{i-2}(\Delta(\lambda),k)$, if M is a semigroup and $\Delta(\lambda)$ is the poset $(0,\lambda)$ considered as a simplicial complex.

Then it is not so hard to see that $k[M]$ is Koszul if and only if all intervals are Cohen-Macaulay (i.e. have CM Stanley-Reisner rings).

Example: $R = k[t^2, t^3]$, $\mathrm{Tor}_3^R(k,k)_8 \simeq H_1((0,8),k)$





If A is a commutative k-algebra, k a field, the module of derivations $\mathrm{Der}_k(A) \subseteq \mathrm{Hom}_k(A, A)$ is the set

$$\{\rho \in \mathrm{Hom}_k(A, A) \mid \rho(ab) = a\rho(b) + \rho(a)b\}.$$

Theorem 5 (Eriksson, Eriksen) *Let S be a numerical semigroup and $T(S) = \{a_1, \ldots, a_h\}$. The module of derivations of $k[S]$ is the left $k[S]$-module generated by*

$$\{t\partial\} \cup \{t^{a_i+1}\partial, i = 1, \ldots, h\}$$

where $\partial = \frac{\partial}{\partial t}$. In particular the number of generators is $|T(S)| + 1$.

Theorem 6 (Fröberg-Micale) *Let S be a numerical semigroup and $T(S) = \{a_1, \ldots, a_h\}$, and let $A = k[S]$. Then $P_{\mathrm{Der}_k(A)}(z) = 1 + hP_k(z)$, so $P_{\mathrm{Der}_k(A)}(z)$ is rational if and only if $P_k(z)$ is rational.*

Theorem 7 *Let $A = k[S]$. Then $P_k(z)$ is rational in the following cases:*

- *S is 3-generated.*
- *S is n-generated and symmetric, $n \leq 5$.*
- *S is a complete intersection.*
- $S = \langle a, a+l, \ldots, a+ld \rangle$, $2d \geq a-1$, $\gcd(a, l) = 1$.

- $m(S) \leq 7$, where $m(S)$ is the multiplicity.

- S is of maximal embedding dimension.

- S is of maximal length of type ≥ 2 or of almost maximal length.

- S is a “monomial semigroup”.

Complete intersection semigroup rings were determined by Delorme.

Maximal embedding dimension means $edim(k[S] = m(S)$.

Maximal length (almost maximal length) means $l(\bar{R}/R) = l(R/C)t(R)$ ($l(\bar{R}/R) = l(R/C)t(R) - 1$, resp.), where $t(R) =$ CM-type(R).

A semigroup S is monomial if $v(R) = S$ implies $R \simeq k[S]$.

If $U = R/(x)$, x a nonzerodivisor in $m \setminus m^2$, then $P_k^R(z) = (1+z)P_k^U(z)$. We let $U = k[[S]]/(t^{m(S)})$.

If S is 3-generated, then U has embedding dimension 2, and U is either a complete or a so called Golod ring. $P_k^{k[[S]]}(z) = (1+z)P_k^U(z) = (1+z)/(1-z)^2$ or $(1+z)^3/(1-3z^2-2z^3)$.

Gorenstein rings of codimension at most 4 has rational Poincaré series (Avramov-Kustin-Miller).

If $S = \langle a, a+l, \ldots, a+ld \rangle$, $2d > a-1$, $\gcd(a,l) = 1$, then $U = k[[x_1, \ldots, x_d]]/I$, I generated in degree 2 and constitute a Gröbner basis in Degrevlex. Then U is a so called Koszul algebra, and $P_k^U(z) = 1/H_U(-z)$ and so rational. If $2d = a-1$, then $U = k[[x_1, \ldots, x_d]]/(I + m^3)$, I as above, and $P_k^U(z)$ is rational (Löfwall).

Kunz considers a classification of numerical semigroups. Fix an integer $m \geq 3$ and denote by H_m the set of all numerical semigroups H with $m \in H$. Using the Apéry set of H with respect to m, he associates to H_m a polyhedral cone $P_m \subset \mathbf{R}^{m-1}$ such that one has a bijection $H_m \to P_m \cap \mathbf{N}^{m-1}$. Then the disjoint decomposition of P_m into open faces leads to a classification of the numerical semigroups. It turns out that the semigroups of multiplicity m belonging to a fixed open face of P_m have the same Betti numbers, in particular they have the same Hilbert series, Cohen-Macaulay type, embedding dimension. If $m(S) \leq 7$ we use a classification of all possible U's. There are about 70 different cases. In all of these we can use the methods above.

If S is of maximal embedding dimension, then $U = k[[x_1, \ldots, x_d]]/(m^2)$, which is Koszul.

If S has maximal length, then either $k[[S]]$ is Gorenstein or of maximal embedding dimension (Brown-Herzog).

The rings of almost maximal length are classified (Brown-Curtis). They are either Golod or of maximal embedding dimension.

Let $\mathbb{C}[[x_1,\ldots,x_k]]/(f)$ be an analytically irreducible curve, i.e. the zero set of f, an irreducible power series. Then this curve can be parametrized as $(t^{n_1}, t^{n_2}+\cdots,\ldots,t^{n_k}+\cdots)$. The set of values is a semigroup which contains $\langle n_1,\ldots,n_k\rangle$, but is in general larger. If any curve with semigroup S is isomorphic to the semigroup ring $\mathbb{C}[[S]]$, then S is called a monomial semigroup. The monomial semigroups were classified by Pfister-Steenbrink, but an error in their proof was corrected by (Micale. They are either of maximal embedding dimension, Koszul, or 2-generated.

Differential operators

If R is a k-algebra, k a field, the ring of differential operators $D(R)$ on R is a subalgebra of $\text{Hom}_k(R, R)$ which can be defined recursively as follows.

$D^0(R) \simeq R$ consists of the multiplications θ_r with elements in R, $\theta_r(a) = ra$.

$D^1(R) = \{\theta; [\theta, D^0(R)] \subseteq D^0(R)\}$. Thus $D^1(R) \setminus D^0(R)$ are the derivations. $[,]$ is the commutator.

In general $D^n(R) = \{\theta; [\theta, D^0(R)] \subseteq D^{n-1}(R)\}$ and $D(R) = \cup_{n \geq 0} D^n(R)$.

If $R = k[x_1, \ldots, x_n]$, then

$$D(R) = k[x_1, \ldots, x_n, \partial_1, \ldots, \partial_n],$$

the Weyl algebra.

Nakai's conjecture: $D(R)$ is generated by $D^1(R)$ if and only if R is regular.

Fact 1: If $R = k[S]$, then $D(R) \subseteq D(k[t, t^{-1}]) = k[t, t^{-1}, \partial]$.

Definition If $\theta = t^m \partial^n$, then $\deg(\theta) = m - n$. Then $D(k[t, t^{-1}])$ is graded, and $D(k[S])$ inherits this grading.

Fact 2: If $S = \langle s_1, \ldots, s_k \rangle$, then $\text{gr}(D(k[S]))$ is the subring of $k[t, y]$ generated by $\{t^{s_1}, \ldots, t^{s_k}\} \cup \{y^{s_1}, \ldots, y^{s_k}\} \cup \{ty\} \cup \{t^{v(n)+n} y^{v(n)}\}_{n \in \mathbb{Z} \setminus S}$, where $v(n) = \#\{s \in S; s + n \notin S\}$.

This gives that $\text{gr}D(k[S])$ is Noetherian, so $D(k[S])$ is left and right Noetherian.

A strong version of Nakai's conjecture is true: $D^1(k[S])^2 \neq D^2(k[S])$.

References

V. Barucci-R. Fröberg, *Associated graded rings of one-dimensional analytically irreducible rings*, J. Algebra **304** (2006), 349–358.

H. Bresinsky, *Symmetric semigroups of integers generated by* 4 *elements*, Manuscripta Math. **17**, (1975), 205–219.

W. C. Brown-F. Curtis, *Numerical Semigroups of Maximal or Almost Maximal Length*, Semigroup Forum **42** (1991), 219–235.

W. C. Brown-J. Herzog, *One dimensional local rings of maximal and almost maximal length*, J. Algebra **51**, (1992), 332–347.

C. Delorme, *Sous-monoides d'intersection complète de N*, Ann. Sci. Ècole Norm. Sup. **9**, (1976), 145–154.

E. Eriksen, *Differential operators on monomial curves*, J. Algebra **264**, (2003), 186–198.

A. Eriksson, *The ring of differential operators of a Stanley-Reisner ring*, Comm. Algebra **26**, (1998), 4007–4013.

R. Fröberg, *The Frobenius number of some semigroups*, Comm. Algebra **22**, (1994), 6021–6024.

R. Fröberg-V. Micale, *The Poincaré series of the module of derivations of affine monomial curves*, J. Algebra **280** (2004), 120–131.

R. Fröberg-J.-E. Roos, *An affine monomial curve with irrational Poincar-Betti series*, J. Pure Appl. Algebra **152**, (2000), 89–92.

J. Herzog, *Generators and relations of abelian semigroups and semigroup rings*, Manuscripta Math. **3**, (1970), 175–193.

E. Kunz, *Über die Klassifikation numerischer Halbgruppen*, Regensburger Mathematische Schriften **11**, Universität Regensburg, Fachbereich Mathematik, Regensburg, 1987.

A. Laudal-A. Sletsjöe, *Betti numbers of monoid algebras. Applications to 2-dimensional torus embeddings*, Math. Scand **56**, (1985), 145–162.

V. Micale, *On monomial semigroups*, Comm. Algebra **30**, (2002), 4687–4698.

J.-E. Roos-B. Sturmfels, *A toric ring with irrational Poincar-Betti series*, C.R. Acad. Sci. Paris **326** (1998),141–146.

K.-I. Watanabe, *Some examples of one dimensional Gorenstein domains*, Nagoya Math. J. **49**, 101–109.