Unitary Numerical Semigroups

and

Perfect Bricks

Iberian Meeting on Numerical Semigroups

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Notation

$S = < a_1, a_2, \cdots, a_n >$ = a numerical semigroup

$e(S) =$ the multiplicity of S = the smallest element of $S \backslash \{0\}$

$g(S) =$ the Frobenius number of S

$H(S) =$ the holes of S = $\{z \in \mathbb{Z} \,|\, z \notin S \text{ and } g(S) - z \notin S\}$

$I = (b_1, b_2, \cdots, b_k)$ = a non-principal relative ideal of S

$\mu_S(I)$ = the number of elements in the minimal generating set of I

$S - I$ = the dual of I in S = $\{z \in \mathbb{Z} \mid z + I \subseteq S\}$

$I + (S - I)$ = ideal sum of I and $S - I$

= $\{a + b \mid a \in I \text{ and } b \in S - I\}$

The following inequality is always true:

$$\mu_S(I)\mu_S(S-I) \geq \mu_S(I+(S-I)).$$

Question: Under what conditions is this equation true?

$$\mu_S(I)\mu_S(S-I) = \mu_S(I+(S-I))$$

Definition: If S is a numerical semigroup and I is a relative ideal of S such that $\mu_S(I) = n$ and $\mu_S(S-I) = m$ and $\mu_S\big(I+(S-I)\big) = mn$, then we say (S, I) is an $\boldsymbol{n \times m}$ **brick**.

Definition: If (S, I) is an $n \times m$ brick such that $I+(S-I) = S\backslash\{0\}$, then we say (S, I) is an $\boldsymbol{n \times m}$ **perfect brick**.

Original Motivation

The original motivation for examining the inequality

$$\mu_S(I)\mu_S(S - I) \geq \mu_S(I + (S - I))$$

comes from the study of torsion in tensor products of modules.

Let $(R, \boldsymbol{m})$ be a one-dimensional local Noetherian domain with quotient field K. Let I be a non-principal fractional ideal of R and let

$$I^{-1} = \{c \in K \mid cI \subseteq R\}.$$

If $\mu_R(I)\mu_R(I^{-1}) > \mu_R(II^{-1})$, then the tensor product $I \otimes_R I^{-1}$ has non-zero torsion. If $\mu_R(I)\mu_R(I^{-1}) = \mu_R(II^{-1})$, then other techniques must be employed to investigate torsion in $I \otimes_R I^{-1}$.

The natural relationship between the inequality

$$\mu_R(I)\mu_R(I^{-1}) \geq \mu_R(II^{-1})$$

and

$$\mu_S(I)\mu_S(S-I) \geq \mu_S(I+(S-I))$$

where $S = \upsilon(R)$, means that the latter inequality can reveal information about torsion in the tensor product of an ideal with its inverse.

For more details on this investigation see:

M. Auslander, *Modules over unramified regular local rings*, (1961)

P. Constapel, *Vanishing of Tor and torsion in tensor products*, (1996)

C. Huneke and R. Wiegand, *Tensor products of modules and the rigidity of Tor,* (1994)

C. Huneke and R. Wiegand, *Tensor products of modules, rigidity and local cohomology, (1997)*

Partial Results

(1996) If S is a numerical semigroup such that $e(S) \leq 7$ and I is a non-principal relative ideal of S, then $\mu_S(I)\mu_S(S-I) > \mu_S(I+(S-I))$.

(2002) If S is a numerical semigroup such that $e(S) \leq 8$ and I is a non-principal relative ideal of S, then $\mu_S(I)\mu_S(S-I) > \mu_S(I+(S-I))$.

(1997) If S is a numerical semigroup such that $e(S) \leq 9$ and I is a relative ideal of S such that $\mu_S(I) = 2 = \mu_S(S-I)$, then $\mu_S(I)\mu_S(S-I) > \mu_S(I+(S-I))$.

(1999) If S is a numerical semigroup such that $1 \leq |H(S)| \leq 3$ and I is a non-principal relative ideal of S, then $\mu_S(I)\mu_S(S-I) > \mu_S(I+(S-I))$.

The First Bricks

P. Constapel (1994)

$$S = < 14, 15, 20, 21 > \ , I = (0,1),$$
$$S - I = (14, 20),$$
$$I + (S - I) = (14, 15, 20, 21)$$

This is a 2×2 perfect brick.

Herzinger (1996)

$$S = < 10, 14, 15, 21 > \ , I = (0,1),$$
$$S - I = (14, 20),$$
$$I + (S - I) = (14, 15, 20, 21)$$

This is a 2×2 brick (not perfect).

P. Garcia-Sanchez and I. Garcia-Garcia (2001)

$$S = < 14, 15, 20, 21, 25 > \ , I = (0,1),$$
$$S - I = (14, 20),$$
$$I + (S - I) = (14, 15, 20, 21)$$

This is a 2×2 brick (not perfect and not symmetric).

2005 – work with Stephen Wilson, Nandor Sieban, and Jeff Rushall (Northern Arizona University).

A systematic computer search revealed thousands of bricks. A subset of these bricks showed similarities that motivated the following definitions:

Definition: Let $S = < a_1, a_2, a_3, a_4 >$ be numerical semigroup of embedding dimension 4.

(i) We say S is **balanced** provided $a_1 + a_4 = a_2 + a_3$. If S is balanced we define the **common sum** of S to be $CS(S) = a_1 + a_4 = a_2 + a_3$.

(ii) If S is balanced we define the **common quotient** of S to be

$$CQ(S) = \frac{CS(S)}{\gcd(a_1,a_4)\gcd(a_2,a_3)}.$$

(iii) If S is balanced and $CQ(S) = 1$, then we say S is **unitary**.

Examples

$$S = < 14, 15, 20, 21 >$$
$$CS(S) = 14 + 21 = 15 + 20 = 35$$
$$\gcd(a_1, a_4) = 7 \quad \text{and} \quad \gcd(a_2, a_3) = 5$$
$$CQ(S) = 1$$

S is unitary

$$S = < 12, 15, 25, 28 >$$
$$CS(S) = 12 + 28 = 15 + 25 = 40$$
$$\gcd(a_1, a_4) = 4 \quad \text{and} \quad \gcd(a_2, a_3) = 5$$
$$CQ(S) = 2$$

S is balanced but not unitary

$$S = < 10, 14, 15, 21 >$$

S is not balanced

Note: All three of these numerical semigroups form 2×2 bricks with an appropriate relative ideal I. However, only the first one is perfect.

Theorem (2005 – Herzinger, Wilson, Sieben, Rushall)

If $S = < a_1, a_2, a_3, a_4 >$ is a unitary numerical semigroup and $n = a_2 - a_1 = a_4 - a_3$ and $I = (0, n)$, then (S, I) is a 2×2 perfect brick.

This proves the existence of an infinite family of 2×2 perfect bricks.

How does this relate back to torsion in tensor products?

In the process of proving the above theorem, we also proved that unitary numerical semigroups are symmetric. Now, a result by C. Huneke and R. Wiegand states

If $\mu_R(I) = 2 = \mu_R(I^{-1})$ and R is a one-dimensional, local, Gorenstein domain, then $I \otimes_R I^{-1}$ has non-zero torsion.

Since unitary numerical semigroups are symmetric, we conclude that the tensor product corresponding to the relative ideals I and $S - I$ all have non-zero torsion.

We conjectured that the converse of our theorem was also true. That is, every 2×2 perfect brick comes from a unitary numerical semigroup. This was proved in 2007.

Theorem (2007 – Holcomb, Herzinger)

If (S, I) is a 2×2 perfect brick, then S is a unitary numerical semigroup.

The investigation of unitary numerical semigroups and 2×2 perfect bricks created several questions that are unanswered.

Open Questions

1. Several examples of bricks of higher dimensions have been discovered:

$S = < 27, 28, 42, 43, 48, 49 >, I = (0,1)$ - a 2×3 perfect brick

$S = < 24, 25, 29, 30, 42, 43, 62, 63 >, I = (0,1)$ - a 2×4 perfect brick

$S = < 25, 26, 27, 36, 37, 38, 44, 45, 46 >, I = (0,1,2)$ - a 3×3 perfect brick

Are there notions of "balanced" and "unitary" that apply to these higher dimensional bricks? Are these notions unique to 2×2 bricks or are they just a special case of a more general concept yet to be discovered?

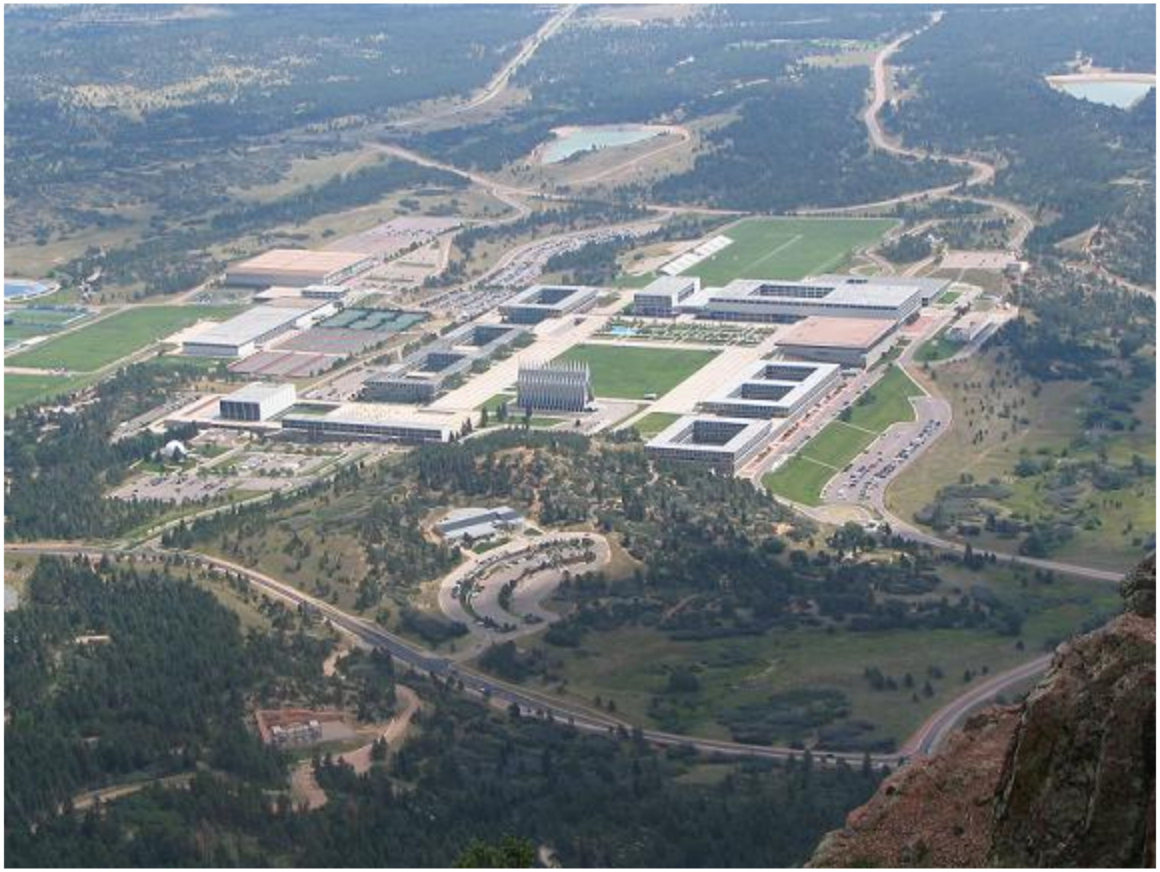
Are there other infinite families of $n \times m$ bricks that can be classified using similar notions?

2. In every example of a 2×2 brick, (S, I) that has been discovered, the ideal I is reflexive. That is $\big(S - (S - I)\big) = I$. In the case of 2×2 perfect bricks this is clear, but it is even true when S is not symmetric. Is this always true?

3. Are there any bricks with multiplicity 9? I have never found one.

4. Are there any bricks with $|H(S)| = 4$? I have never found one and I have not been able to prove that none exist.

5. Let I and J be non-principal relative ideals of a numerical semigroup S such that $\mu_S(I)\mu_S(J) = \mu_S(I + J)$. Under what conditions is it true that $(I + J) - I = J$? I have been able to prove that if $J = S - I$ and $\mu_S(I)\mu_S(J) = \mu_S(I + J)$, then $(I + J) - I = J$. In other words, this equality of relative ideals holds for bricks. I also have some examples of non-bricks where this holds.



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