Delta Sets of Numerical Semigroups

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Numerical Monoids

Definition

A Numerical Monoid is an additive submonoid of $\mathbb{N}_0$*.*
We say that a numerical monoid M *is generated by* $S = \{n_1, \ldots, n_t\}$ *if* $M = \{a_1 n_1 + \cdots a_t n_t : \ a_i \geq 0, \ 1 \leq i \leq t\}$*.*

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The set of generating elements $\{n_1, \ldots, n_t\}$ *of a numerical monoid* S *is called minimal if there does not exist another generating set* $\{m_1, \ldots, m_k\}$ *of* S *with* $k < t$.

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Example: $\langle 4, 6, 9 \rangle = \{0, 4, 6, 8, 9, 10, 12, 13, \ldots\}$

Factorization

In a minimally generated numerical monoid with minimal generating set $S = \{n_1, \ldots, n_t\}$, the irreducible elements are $n_1, \ldots, n_t$.
A factorization of $m \in M$ is a set $\{a_1, \ldots, a_t\}$ of nonnegative integers such that

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The set of lengths of m with respect to a generating set S is

$$\mathcal{L}^S(m) = \{a_1 + \cdots + a_t : \ a_i \geq 0, \ m = a_1 n_1 + \cdots + a_t n_t\}.$$

Delta Sets

Definition

Let $m \in M$ and $\mathcal{L}^S(m) = \{x_1, \ldots, x_k\}$ with $x_1 < x_2 < \cdots < x_k$. Then the Delta Set of m is

$$\Delta^S(m) = \{x_i - x_{i-1} : \; 2 \leq i \leq k\}.$$

The Delta Set of M with respect to S is

$$\Delta^S(M) = \bigcup_{m \in M} \Delta^S(m).$$

Note: When I write $\mathcal{L}(M)$ or $\Delta(M)$ without specifying a generating set, assume that S is the minimal generating set.

An Example

Examples: Let $M = \langle 7, 9, 12 \rangle$.
We can factor: $21 = 1 \cdot 9 + 1 \cdot 12 = 3 \cdot 7$.
So $\mathcal{L}(21) = \{2, 3\}$ and $\Delta(21) = \{1\}$. In fact, $\Delta(M) = \{1\}$.

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Let $M = \langle 7, 10, 12 \rangle$.
We can factor: $42 = 3 \cdot 10 + 1 \cdot 12 = 6 \cdot 7$.
So $\mathcal{L}(42) = \{4, 6\}$ and $\Delta(42) = \{2\}$. In fact, $\Delta(M) = \{1, 2\}$.

The Monoid of Trades and $\max(\Delta(M))$

Let $M = \langle n_1, \ldots, n_t \rangle$. Then we let

$$H_M = \{(x_1, \ldots, x_t, y_1, \ldots, y_t) : x_1 n_1 + \cdots + x_t n_t = y_1 n_1 + \cdots + y_t n_t\}$$

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Let $M = \langle n_1, n_2, n_3 \rangle$. Let k_1 be the minimum positive integer such that $k_1 n_1 \in \langle n_2, n_3 \rangle$. Let k_3 be the minimum positive integer such that $k_3 n_3 \in \langle n_1, n_2 \rangle$.

Theorem

$\max(\Delta^S(M)) = \max(\Delta^S(k_1 n_1) \cup \Delta^S(k_3 n_3))$.

$\min(\Delta^S(M))$

Let M be a primitive numerical monoid with generating set $S = \{n_1, \ldots, n_t\}$.

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We can compute $\min(\Delta^S(M))$ and $\max(\Delta^S(M))$ without much trouble.
What kinds of sets can occur as $\Delta^S(M)$?

Intervals

Let $M = \langle n_1, n_2 \rangle$. Then $\Delta(M) = \{n_2 - n_1\}$.

Theorem

Let $M = \langle n, n + d, \cdots, n + kd \rangle$ with $(n, d) = 1$. Then $\Delta(M) = \{d\}$.

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Theorem

If $M = \langle n, n+d, (d+1)n - d \rangle$, then

$$\Delta(S) = \{d, 2d, \ldots, \left\lfloor \frac{n+d-1}{d+2} \right\rfloor d\}.$$

An Example: $\langle 5, 6, 19\rangle$

Let $M = \langle 5, 6, 19\rangle$.
We can factor: $38 = 4 \cdot 5 + 3 \cdot 6 = 2 \cdot 19$.
So $\mathcal{L}(38) = \{2, 7\}$ and $\Delta(38) = \{5\}$.

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We can also factor:

$$\begin{aligned} 63 \quad &= 0 \cdot 5 + 1 \cdot 6 + 3 \cdot 19 \quad = 5 \cdot 5 + 0 \cdot 6 + 2 \cdot 19 \\ &= 4 \cdot 5 + 4 \cdot 6 + 1 \cdot 19 \quad = 3 \cdot 5 + 8 \cdot 6 + 0 \cdot 19 \\ &= 9 \cdot 5 + 3 \cdot 6 + 0 \cdot 19. \end{aligned}$$

So $\mathcal{L}(63) = \{4, 7, 9, 11, 12\}$ and $\Delta(63) = \{1, 2, 3\}$.

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So $\mathcal{L}(63) = \{4, 7, 9, 11, 12\}$ and $\Delta(63) = \{1, 2, 3\}$.
In fact, $\Delta(M) = \{1, 2, 3, 5\}$.

Gaps in $\Delta(M)$

Theorem

If $M = \langle n, n+1, n^2-n-1 \rangle$, *then*

$$\Delta(M) = \{1, \ldots, n-2\} \cup \{2n-5\}.$$

So there are monoids M for which $\Delta(M)$ has arbitrarily large gaps.

In general it is much harder to say that something is not in $\Delta(M)$ than it is to say that something is in $\Delta(M)$.

Periodicity

It is easy to show that $\Delta^S(M)$ is finite. Therefore given a numerical monoid M and a generating set S there exists some $N \in \mathbb{N}$ such that $\Delta^S(M) = \bigcup_{m \in M, m \leq N} \Delta^S(m)$.

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Given a primitive numerical monoid $M = \langle n_1, \ldots, n_t \rangle$, we have for all $m \geq 2tn_2n_t^2$, that $\Delta^S(m) = \Delta^S(m + n_1 n_t)$.

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Corollary

If we let $N = 2tn_2n_t^2 + n_1 n_k$, we have $\Delta^S(M) = \bigcup_{m \in M, m \leq N} \Delta^S(m)$.

Changing S and $|\Delta^S(M)|$

If we fix a primitive numerical monoid M, but consider different generating sets, what kinds of sets can we get for $\Delta^S(M)$?

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Proposition

Let M be a primitive numerical monoid, and $S = \{n_1, \ldots, n_t\}$ be any generating set of M. For all $N \geq \lceil \frac{n_t}{n_1} \rceil n_t$, if we let $S' = \{m \in M | m \leq N\}$, then $\Delta^{S'}(M) = 1$.

$M = \langle n_1, n_2, in_1 + jn_2 \rangle$

Proposition

Let $M = \langle n_1, n_2 \rangle$ be a primitive numerical monoid, and $S = \{n_1, n_2, in_1 + jn_2\}$ with $i < n_2$. Then $i + j - 1 \in \Delta^S(M)$.

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We have $|\Delta^S(M)| = 1$ if and only if either $i + j - 1 = n_2 - n_1$, or $j = 0$ and there exists $l \leq \lceil \frac{n_2}{i} \rceil$ such that $l(i + j - 1) = n_2 - n_1$.

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Given a finite set T with $\min(T) = \gcd(T)$, does there exist a primitive numerical monoid M with generating set S such that $\Delta^S(M) = T$?

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Does there exist a numerical monoid with $\Delta^S(M) = \{1, k\}$ with $k \neq 1, 2$?
Does there exist a numerical monoid with $\Delta^S(M) = \{1, 3\}$?

A Partial Result

Let $M = \langle n_1, n_2 \rangle$ and $S = \{n_1, n_2, in_1 + jn_2\}$ for $i, j \in \mathbb{N}$.

Theorem

If $\Delta^S(M) = \{1, k\}$, then $k \in \{1, 2\}$.

We would like to extend this result to $M = \langle n_1, n_2, n_3 \rangle$, where M is minimally generated.