FACTORIZATION IN NUMERICAL SEMIGROUPRINGS

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$\mathbf{S}$ add abelian semigroup with 0

cancellative	$x, y, z \in S$	$x + z = y + z \Rightarrow x = y$
torsion free	$x, y \in S, n \geq 1$	$nx = ny \Rightarrow x = y$
reduced	$x, y \in S$	$x + y = 0 \;\Rightarrow\; x = y = 0.$

$\mathbf{D}$ integral domain with id
$\mathbf{D[S]}$ semigroup ring

Gilmer-Parker Theorem 1974

$D[S]$ factorial $\Leftrightarrow$ D factorial & S factorial
UFD free

In particular S numerical monoid $\neq \mathbb{N}$
$D[S]$ is never factorial.

Question Does the semigroup ring over a numerical monoid posses any good factorization properties at all?

Cale factorization in semigroup M with mult
comm, $1 \in M$, cancell, $M^\times$ units

M **inside factorial** with **base** $Q \subset M \setminus M^\times$
if every $x \in M \setminus M^\times$ has a unique factorization

$$x^{m(x)} = u \prod_{q \in Q} q^{x(q)}.$$

$m(x) \geq 1$ minimal unit $x(q) \geq 1$ for finitely many q

tame base $e(q)\frac{x(q)}{m(x)} \in \mathbb{N}$ all x, all q, $e(q) \in \mathbb{N}\setminus\{0\}$.

M **Cale semigroup**/monoid inside factorial
& tame base

domain R **Cale domain**
multiplicative semigroup R^* Cale semigroup

Cale factorization a substitute for
„factorization of ideals into prime ideals".
K. Lisbon 1988, Chapman/Halter-Koch/K 2002,
Chapman/K 2005, K/Maney/Ponomarenko 2007.

Theorem K. 2004

$D[S]$ Cale $\;\Leftrightarrow\;$ D Cale & S Cale
& $\overline{D[S]} = \overline{D}[\overline{S}]$

root-closure of semigroup M with mult
$\overline{M} = \{x \in$ quot $M \mid x^n \in M$, some $n \geq 1\}$.

Corollary K field, S numerical semigroup $\neq \mathbb{N}$
$K[S]$ Cale $\Leftrightarrow \overline{K[S]} = K[\mathbb{N}]$

Proof $\quad K$ Cale since $K = K^{\times}, \overline{K} = K$
$S =< A >$, A finite, $gcd\ A = 1$, $a = \min A$

Cale with $Q = \{a\}$ $\quad \underset{\frac{a}{d}}{\underset{|}{m(x)}}x \;=\; a\underset{\frac{x}{d}}{\underset{|}{x(q)}}$, $d = gcd(x,a)$

$\overline{S} = \{x \in \mathbb{Z} \mid nx \in S,$ some $n \geq 1\} = \mathbb{N}$. $\square$

Lemma K field, S numerical semigroup $\neq \mathbb{N}$
$\overline{K[S]} = K[\mathbb{N}] \Leftrightarrow K$ finite

Proof $\Rightarrow t \in \mathbb{N} \backslash S$. $1 + X^t \in K[\mathbb{N}] \subseteq \overline{K[S]}$
$\Rightarrow$ some n $1 + nX^t + \ldots = (1 + X^t)^n \in K[S]$
$\Rightarrow$ $n = 0$ in $K \rightarrow$ char $K > 0$
$\Leftarrow$ $K[\mathbb{N}]$ factorial, hence, root-closed,
$\overline{K[S]} \subseteq K[\mathbb{N}]$.

Conversely, let $f = \sum r_m X^m \in K[\mathbb{N}], p = \text{char}\, K > 0$.
$\Rightarrow$ $f^{p^k} = \sum r_m^{p^k} X^{mp^k}$ all k.
S numerical implies $mp^k \in S$ for k big enough
$\Rightarrow f^{p^k} \in K[S]$ for k big enough
$\Rightarrow f \in \overline{K[S]}$. $\square$

Theorem K field, S numerical semigroup $\neq \mathbb{N}$
$K[S]$ Cale $\Leftrightarrow K$ finite

Proof Corollary & Lemma $\square$

Elasticity in numerical semigroup rings
M semigroup with mult, comm, $1 \in M$, cancell, atomic

elasticity

$$\begin{aligned}\rho(M) &= \sup\{\frac{m}{n} \mid x_1 \dots x_m = y_1 \dots y_n, \\ &\quad x_i, y_j \text{ atoms}, m, n \geq 1\}\end{aligned}$$

Theorem
D.F. Anderson/Chapman/Inman/Smith 1993
K field, S numerical semigroup $\neq \mathbb{N}$

(i) $\rho(K[S])$ finite $\Leftrightarrow K$ finite

(ii) For $S = < 2, 3 >, \rho(K[S]) = \frac{1}{2}(d(K) + 2)$
$d(K) =$ Davenport constant of $(K, +)$
$= n(p - 1) + 1, p =$ char $K, |K| = p^n$.

Theorem D.F. Anderson/Jenkins 19??
K field, $S = < n, n + 1, \dots, 2n - 1 >, n \geq 1$

$$\rho(K[S]) = \frac{1}{2n}(nd(G_n(K)) + 3n - 2)$$

$G_n(K)$ additive finite group associated to K.

Questions

- explicit formula for $\rho(S), S$ num semigroup
- role of Cale structure in $K[S], K$ finite.

Embedding numerical semigroups

S,T semigroups $S \hookrightarrow T$ if $S \subseteq T$ and for $x, y \in S$

$$x \,|_S\, y \quad \Leftrightarrow \quad x \,|_T\, y.$$

For S numerical $S \subset \mathbb{N}$ but **not** $S \hookrightarrow \mathbb{N}$ for $S \neq \mathbb{N}$
Question Good embedding $S \hookrightarrow T$ for S?

Lemma $S \subset T, T$ factorial

(i) $S \hookrightarrow T \quad \Rightarrow \quad S = \overline{S}$

(ii) If for each $t \in T$ exists $n_t \in \mathbb{N}$ s.t. $n_t t \in S$:
$S \hookrightarrow T \quad \Leftarrow \quad S = \overline{S}$

(iii) S numerical semigroup, T factorial semigroup

$$S \hookrightarrow T \Rightarrow S = \mathbb{N}.$$

Proof (i) $x \in \overline{S},\ x = y - z, y,\ z \in S,\ nx \in S$ (add)

$$\Rightarrow \ nz \,|_T\, ny. \quad T \text{ factorial } \Rightarrow z \,|_T\, y$$
$$\Rightarrow \ z \,|_S\, y \quad \Rightarrow \quad x \in S.$$

(ii) $y, z \in S,\ z \,|_T\, y \Rightarrow x = y - z \in T \Rightarrow nx \in S$
$\Rightarrow x \in \overline{S} = S \Rightarrow z \,|_S\, y.$

(iii) follows from (i) by $\overline{S} = \mathbb{N}$. $\square$

But S numerical can be embedded into $K[S]$:
$\varphi : S \to K[S], s \mapsto X^s$
φ homomorphism $(S,+) \to (K[S],\cdot)$, injective
$x \mid_S y \Leftrightarrow \varphi(x) \mid_{K[S]} \varphi(y)$ for $x, y \in S$

„ $\Rightarrow$ “ $y = x + z \Rightarrow \varphi(y) = \varphi(x) \cdot \varphi(z)$
$\Rightarrow \varphi(x) \mid_{K[S]} \varphi(y)$

„ $\Leftarrow$ „ $\varphi(x) \mid_{K[S]} \varphi(y), \varphi(x) = X^x, \varphi(y) = X^y$
$$\Rightarrow X^y = X^x \sum_{a \in S} r_a X^a = \sum_{a \in S} r_a X^{x+a}$$
$\Rightarrow r_a = 0$ if $x + a \neq y, r_a = 1$ if $x + a = y$.
$\Rightarrow x \mid_S y$.

By identification

numerical $S \hookrightarrow K[S]$ **Cale domain**
for **any finite** field K.

Embedding subsets of S in K[S]

S numerical semigroup, $K = \mathbb{F}_2$
$P(S)$ set of finite subsets of S
$\varphi : S \to \mathbb{F}_2[S], S \mapsto X^s$ can be extended
$\varphi : P(S) \to \mathbb{F}_2[S], A \mapsto \sum_{s \in A} X^s$
φ is a bijection which induces
„domain structure" on subsets.

For $A, B \in P(S)$ define
$A \oplus B = (A \cup B) \setminus (A \cap B)$

$$\begin{aligned} \varphi(A \oplus B) &= \varphi(A) + \varphi(B) \\ \varphi(A) + \varphi(B) &= \sum_{a \in A \setminus A \cap B} X^a + \sum_{b \in B \setminus A \cap B} X^b \\ &\quad + 2 \sum_{c \in A \cap B} X^c = \varphi(A \oplus B). \end{aligned}$$

Ex. $\quad A = \{s\} = s, B \subset S$

$$s \oplus B = \begin{cases} B \setminus s, & s \in B \\ B \cup s, & s \notin B \end{cases}$$

$$s \oplus t = \begin{cases} \emptyset, & s = t \\ s \cup t, & s \neq t \end{cases}$$

For $A, B \in P(S)$ define

$A \otimes B = \{a + b \mid a \in A, b \in B, (a, b)$ essentiell $\}$

(a, b) **essentiell** in $A \times B$ if

$\sharp\{(a', b') \in A \times B \mid a' + b' = a + b\}$ is odd

$$\begin{aligned}
\varphi(A \otimes B) &= \varphi(A) \cdot \varphi(B) \\
\varphi(A) \cdot \varphi(B) &= \left(\sum_{a \in A} X^a\right)\left(\sum_{b \in B} X^b\right) \\
&= \sum_{(a,b) \in A \times B} X^{a+b}, \text{ for } s(a, b) = a + b \\
&= \sum_{c \in A+B} \sum_{s^{-1}(c)} X^c = \sum_{c \in A+B} \sharp s^{-1}(c) X^c \\
&= \sum_{c \in A \otimes B} X^c = \varphi(A \otimes B).
\end{aligned}$$

Ex. $A = \{s\} = s, B \subset S$

$s \otimes B = s + B$, especially $s \otimes t = s + t$

powers $A^n = A \otimes \ldots \otimes A$

$S =< 2, 3 >, A = \{2, 3\}$

$A^2 = \{4, 6\},\ A^3 = \{6, 7, 8, 9\}, A^4 = \{8, 12\}.$