Catenary degree in numerical. monoids (An application to numerical monoids generated by arithmetic sequences)

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This is a joint work with

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Set of factorizations

- Let $S = \langle n_1, \dots, n_p \rangle$ be a numerical monoid minimally generated by $n_1 < n_2 < \dots < n_p$.
- For every $s \in S$, we define the set of factorizations of s

$$\mathrm{Z}(s) = \{(a_1, \dots a_p) \in \mathbb{N}^p : a_1 n_1 + \dots + a_p n_p = s\}$$

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Support

Let $a = (a_1, \ldots a_p) \in \mathbb{Z}^p$ We define the support of a as:

$$\mathrm{supp}(a) = \{i : a_i \neq 0\}$$

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Distance

- For $z = (z_1, \ldots, z_p) \in \mathbb{Z}^p$ we define its norm:

$$|z| = \max\{\sum_{z_i > 0} z_i, \sum_{z_i < 0} |z_i|\}$$

  The length of a factorization $a \in \mathbb{N}^p$ is $|a|$.
- For $a, b \in \mathbb{N}^p$ we define the distance between a and b as

$$d(a, b) = |a - b|$$

Distance is the main tool in this work. The catenary degree tries to control the distance between all different factorizations of the elements of S.

Catenary degree

Given $s \in S$ and $a, b \in \mathrm{Z}(s)$, then an N-chain of factorizations from a to b is a sequence $a = a^1, \ldots, a^t = b$, $a^i \in \mathrm{Z}(s)$ such that $\mathrm{d}(a^i, a^{i+1}) \leq N$ for all i.

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The catenary degree of s, $\mathrm{c}(s)$, is the minimal $N \in \mathbb{N} \cup \{\infty\}$ such that for any two factorizations $a, b \in \mathrm{Z}(s)$, there is an N-chain from a to b.

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The catenary degree of S is $\mathrm{c}(S) = \sup\{c(s) : s \in S\}$.

The catenary degree of an element

Let $S = \langle 6, 9, 11 \rangle$

$$Z(42) = \{(0,1,3),(1,4,0),(4,2,0),(7,0,0)\}$$

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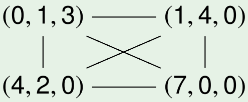

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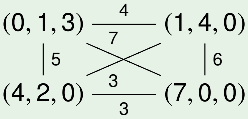

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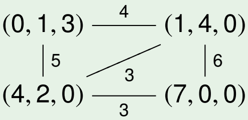

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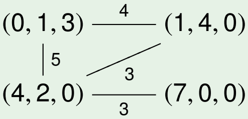

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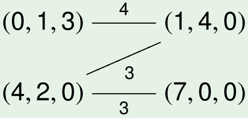

So

$$C(42) = 4$$

The Tool

$\mathcal{R}$-classes

Let $s \in S$ and $a, b \in \mathrm{Z}(s)$ be two factorizations of s. We say that both factorizations are related $a\mathcal{R}b$ if there exists a sequence $a = a^1, \ldots, a^t = b$, $a^i \in \mathrm{Z}(s)$ such that $supp(a^i) \cap supp(a^{i+1}) \neq \emptyset$.

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Proposition

The distance between elements a and b in different $\mathcal{R}$-clases is $d(a, b) = \max\{|a|, |b|\}$.
We can control the elements with more than one $\mathcal{R}$-class.

Apéry set

- For every n_i we define the Apéry set of S respect to n_i as

$$Ap(S, n_i) = \{s \in S : s - n_i \notin S\}$$

- $Ap(S, n_i) = \{w(0), w(1), \dots, w(n_{i-1})\}$ with $w(j)$ the least element in S congruent with j modulo n_i

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Important Result

The elements of S with more than one $\mathcal{R}$-class are of the form $w + n_j$ where $w \in Ap(S, n_1)$ and $j = 2, \ldots, p$.

$\mathcal{R}$-classes in $S = \langle 6, 9, 11 \rangle$

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The factorizations and $\mathcal{R}$ clases of $18, 33, 29, 36 \in \langle 6, 9, 11 \rangle$ are:

$$\begin{array}{ll} \mathrm{Z}(18) = \{(0,2,0)\}, \cup\{(3,0,0)\} & 2\ \mathcal{R} - \text{classes} \\ \mathrm{Z}(33) = \{(0,0,3)\}, \cup\{(1,3,0),(4,1,0)\} & 2\ \mathcal{R} - \text{classes} \\ \mathrm{Z}(29) = \{(0,2,1),(3,0,1)\} & 1\ \mathcal{R} - \text{class} \\ \mathrm{Z}(36) = \{(6,0,0),(0,4,0),(3,2,0)\} & 1\ \mathcal{R} - \text{class} \end{array}$$

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The Apéry set with respect n_1 is:

$$Ap(S, 6) = \{0, 31, 20, 9, 22, 11\}$$

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Using Apéry set for $\langle 6, 9, 11 \rangle$

$9 + Ap(S,6)$	=	9	40	29	18	31	20
$11 + Ap(S,6)$	=	11	42	31	20	33	22

$66 \in \langle 6, 9, 11\rangle$

The factorizations of $66 \in \langle 6, 9, 11\rangle$ are

$$Z(66) = \{(0,0,6),(1,3,3),(2,6,0),(4,1,3),(5,4,0),(8,2,0),(11,0,0)\}$$

We are going to connect them using 18's and 33's factorizations.
Remember:

$$Z(18) = \{{\color{red}(0,2,0)},{\color{blue}(3,0,0)}\}$$
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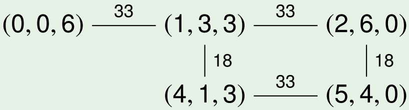

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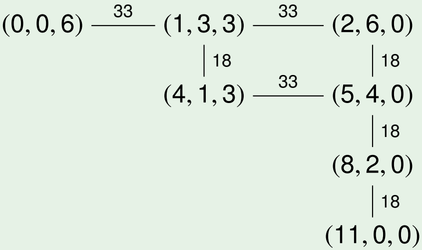

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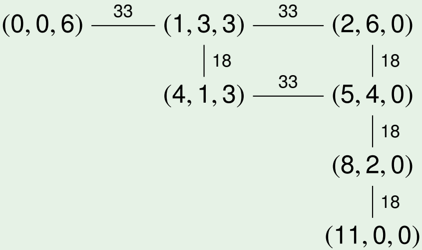

We can see that distances between 66's factorizations are the same as 18's and 33's factorizations.

Catenary degree (Chapman, García-Sánchez, Ll., Ponomarenko, Rosales 2006)

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- We have proved that

$$\mathrm{c}(S) = \max\{r(s) \mid s \in S, k_s > 1\}$$

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The above is equivalent to say that $bn_1 - n_1 - n_i \notin S$

We see next examples in which this bound is reached.

Generated by arithmetic sequences

¿From now on, we consider $S = \langle a, a + d, \ldots, a + cd \rangle$ with $1 \leq c < a$ and $gcd(a, d) = 1$.

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Main theorem

$$c(S) = \left\lceil \frac{a}{c} \right\rceil + d$$

Where $\lceil q \rceil = \min\{n \in \mathbb{N} : n \geq q\}$

Sketch of the proof

First, we control membership to S and the factorizations of the elements in the Apéry set of the multiplicity

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Every element in S can be written as $ka + rd$.

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So we can assume that $r < a$. This will be the key for the following.
If $r < a$,

$$n = ka + rd \in S \Leftrightarrow 0 \leq r \leq kc$$

In the minimal case n is multiple of a. In the maximal case n is multiple of $a + cd$.

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Factorizations of elements with more than one $\mathcal{R}$-class

Let $n = w + (a + jd)$ with $w \in Ap(S, a)$ and $j \in \{1, \dots c\}$. If $z \in Z(n)$ then $|z| \leq \left\lceil \frac{a}{c} \right\rceil + d + 1$

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Hence

$$\mathrm{c}(S) \leq \left\lceil \frac{a}{c} \right\rceil + d + 1$$

Besides recall that that

$$min\{k \in \mathbb{N} - \{0\} : ka \in < a + d, \dots, a + cd >\} \leq \mathrm{c}(S)$$

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Thus

$$\left\lceil \frac{a}{c} \right\rceil + d \leq \mathrm{c}(S) \leq \left\lceil \frac{a}{c} \right\rceil + d + 1$$

The bound is tight

Now to finish the proof of main theorem. We only need to prove that $c(S) \neq \left\lceil \frac{a}{c} \right\rceil + d + 1$

A new result. Generalized arithmetic sequences

Omidali has been to proof that if $S = \langle a, ha + d, \ldots, ha + cd \rangle$ then:

$$c(S) = \left\lceil \frac{a}{c} \right\rceil h + d$$