On Tiling Rectangles and Tori

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(joint work with D. Labrousse)

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Diophantine Frobenius Problem

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Example : If $a_1 = 3$ and $a_2 = 8$ then

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So, $g(3, 8) = 13$.

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Applications

Petri nets (Chrzastowski-Wachtel and Raczunas, 1993) The problem of finding a formula for the least weight in *conservative weights circuits* and the Frobenius problem are equivalent.

Hypohamiltonian graphs (Skupién, 1992) Construction of an infinite family of hypohamiltonian graphs via a modular version of the Frobenius problem.

Random vectors generator (Vizvári, 1994) A method to generate a random vector without cyclic drawbacks

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(Apéry, 1945) Classification plane of algebraic branches
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- Discrete Optimisation problems (Knapsack problem)
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- Rödseth, 1978
- Davison, 1994
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When $n \geq 4$

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Theorem (Kannan, 1992) There is a polynomial time algorithm to compute $g(a_1, \ldots, a_n)$ when $n \geq 2$ is fixed.

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Semi-explicit formulas for $g(a, b, c)$

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A *Fibonacci semigroup* is a semigroup generated by *Fibonacci* numbers $F_{i_1}, \ldots, F_{i_r}$, $3 \leq i_1 < \cdots < i_r$ with $(F_{i_1}, \ldots, F_{i_r}) = 1$.

Theorem (Marin, R.A. and Revuelta, 2007) Let $i, k \geq 3$ be integers and let $r = \lfloor \frac{F_i - 1}{F_k} \rfloor$. Then,

$$g(F_i, F_{i+2}, F_{i+k}) = \begin{cases} (F_i - 1)F_{i+2} - F_i(rF_{k-2} + 1) & \text{if } r = 0 \text{ or } r \geq 1 \text{ and } \\ & F_{k-2}F_i < (F_i - rF_k)F_{i+2}, \\ (rF_k - 1)F_{i+2} - F_i((r-1)F_{k-2} + 1) & \text{otherwise.} \end{cases}$$

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Gaps

Theorem (R.A., 2007) Let $a, k \geq 1$ be integers and let $S = \langle a, a+1, \ldots, a+k \rangle$ be a semigroup with gaps $l_1 < \cdots < l_{N(S)}$. Let $v_m = (m+1)(a-1) - k\left(\frac{m(m+1)}{2}\right)$, $v_{-1} = 0$ and $r = \left\lfloor \frac{a-2}{k} \right\rfloor$. Then,

$$N(S) = v_r \text{ and } l_i = t_i(a+k) + i - v_{t_i - 1}$$

for each $i = 1, \ldots, N(S)$ where t_i is the smallest integer such that $v_{t_i} \geq i$.

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Theorem (R.A., 2007) Let p, q be positive integers with $(p, q) = 1$. Let $g_k(\langle p, q\rangle)$ the number of gaps of $\langle p, q\rangle$ in the interval $[pq-(k+1)(p+q), \ldots, pq-k(p+q)]$, for each $0 \leq k \leq \left\lfloor \frac{pq}{p+q} \right\rfloor - 1$. Then,

$$g_k(\langle p, q\rangle) = \begin{cases} 1 & \text{if } k = 0 \\ 2(k+1) + \left\lfloor \frac{kq}{p} \right\rfloor + \left\lfloor \frac{kp}{q} \right\rfloor & \text{if } 1 \leq k \leq \left\lfloor \frac{pq}{p+q} \right\rfloor - 1. \end{cases}$$

Tiling Problem

Let $R(a,b)$ and $T(a,b)$ be the 2-dimensional rectangle and torus respectively. R (or T) can be *tiled* with *bricks* $R_1,\ldots,R_n$ if R (or T) can be filled entirely with copies of R_i (rotations are allowed).

Example : Tiling $R(13,13)$ with $R(2,2), R(3,3)$ and $R(5,5)$

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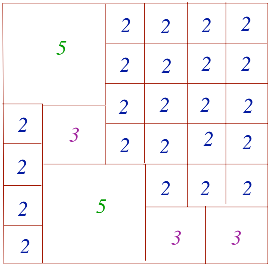

Question : Does there exist a function $C_R = C_R(x, y, u, v)$ (resp. $C_T = C_T(x, y, u, v)$) such that for all integers $a, b \geq C_R$ (resp. $a, b \geq C_T$) the rectangle $R(a, b)$ (resp. torus $T(a, b)$) can be tiled with copies of the rectangles $R(x, y)$ and $R(u, v)$ for given positive integers x, y, u and v

The special case when $x = 4, y = 6, u = 5$ and $v = 7$ was posed in the 1991 William Mowell Putnam Examination (Problem B-3).

Theorem (Klosinski, Alexanderson and Larson, 1992)$R(a, b)$ can be tiled with $R(4, 6)$ and $R(5, 7)$ if $a, b \geq 2214$.

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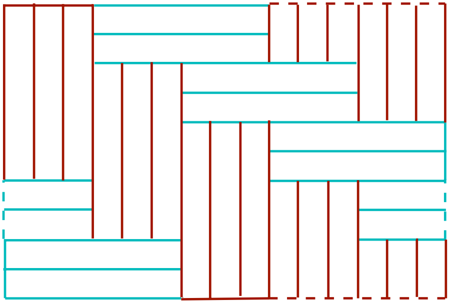
Example : Tiling $T(15, 10)$ with $R(1, 6)$



Theorem (Fricke, 1995) $R(a, b)$ can be tiled with $R(x, x)$ and $R(y, y)$ if and only if either a and b are both multiple of x or a and b are both multiple of y or one of the numbers a, b is a multiple of xy and the other can be expressed as a nonnegative integer combination of x and y.

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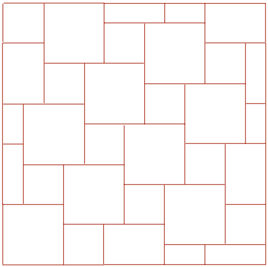

Theorem (Labrousse and R.A., 2007) Let u, v, x and y be positive integers. Then, there exists $C_T(x, y, u, v)$ such that $T(a, b)$ can be tiled with $R(x, y)$ and $R(u, v)$ if and only if $\gcd(xy, uv) = 1$.

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$$a, b \geq \min\{n_1(uv + xy) + 1, n_2(uv + xy) + 1\}$$

where $n_1 = \max\{vx, uy\}$ and $n_2 = \max\{ux, vy\}$.

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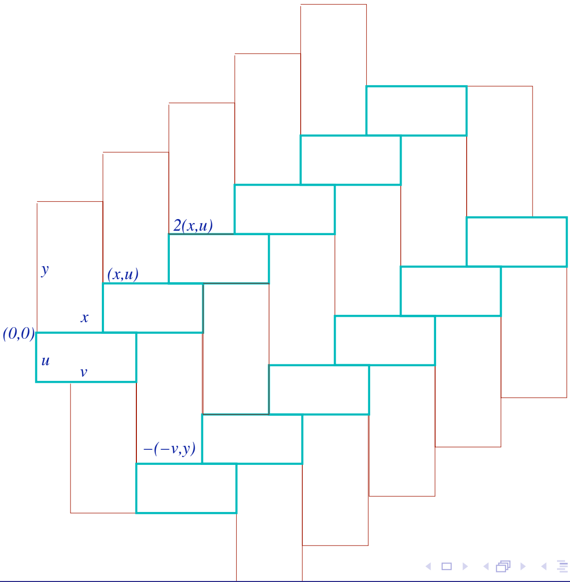
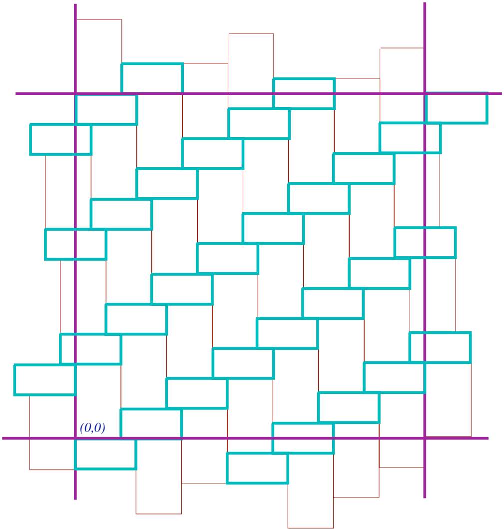
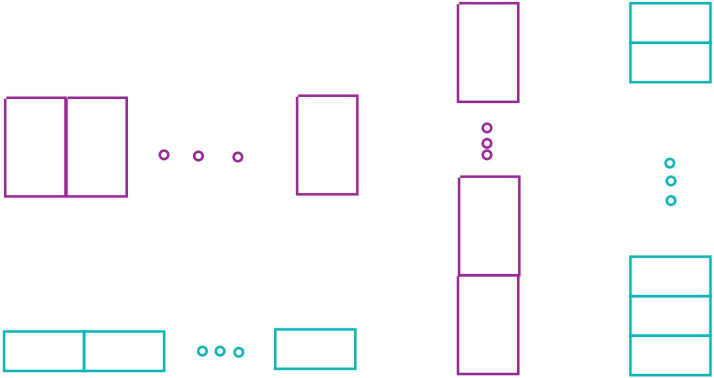
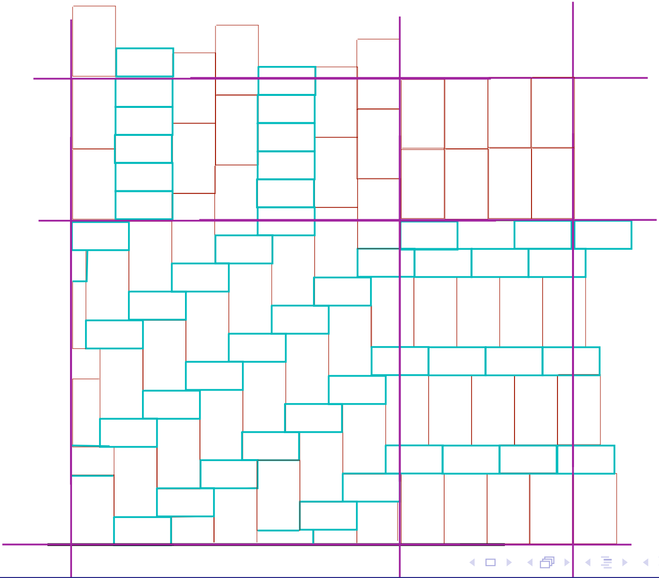

2(x,u)
y
(x,u)
x
(0,0)
u
v
−(−v,y)


(0,0)





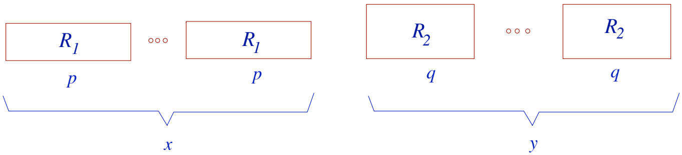
Theorem (Labrousse and R.A., 2007)

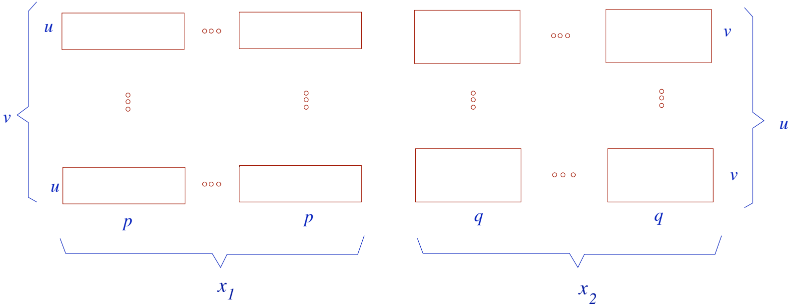
Let $R^i(a^i_1, \ldots, a^i_n)$ $i = 1, \ldots, m$ be rectangles. If

a) $\gcd(a^{i_1}_1, \ldots, a^{i_k}_1) = 1$ for all $\{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$

b) $\gcd(e, f) = 1$ for all $\{e, f\} \subset \{a^1_j, \ldots, a^m_j\}$ with $2 \leq j \leq n$

then all sufficiently large rectangle can be tiled with $R^1, \ldots, R^m$.


R_1
R_1
p
p
x
R_2
R_2
q
q
y



$B(R_1, R_2) = (t, uv)$ $t > g(p, q)$

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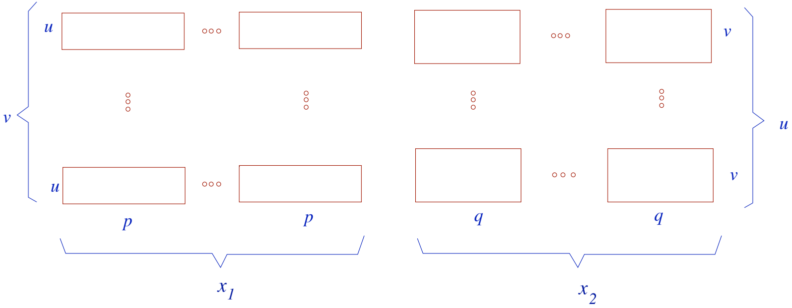

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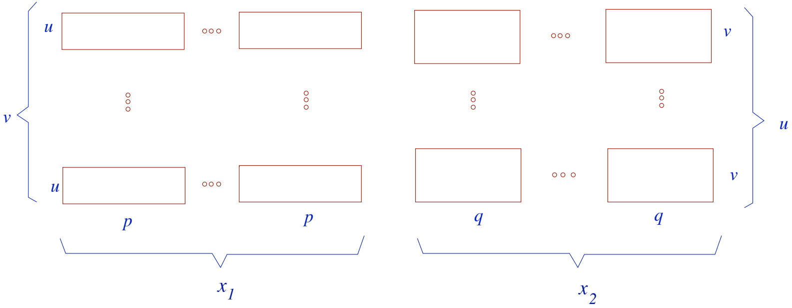

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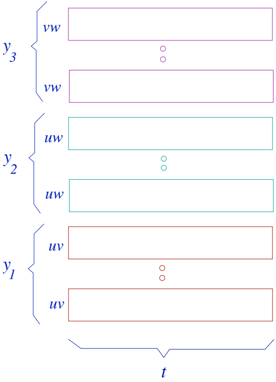

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$$a > g(A_1, \ldots, A_{n+1}) = nP - \sum_{i=1}^{n+1} A_i$$

where $A_i = P/a_i$ with $P = \prod_{j=1}^{n+1} a_j$.

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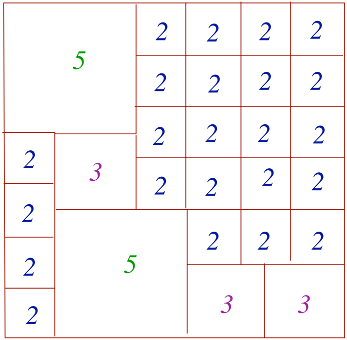
Tiling $R(13,13)$ with $R(2,2)$, $R(3,3)$ and $R(5,5)$



Tiling $R(17,17)$ with $R(2,2)$, $R(3,3)$ and $R(7,7)$