Proportionally modular Diophantine inequalities and proportionally modular numerical semigroups

A talk based on a joint work with J.C. Rosales

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Proportionally modular Diophantine inequalities

$$ax \bmod b \leq cx$$

a, b, c positive integers.

- a: factor of the inequality
- b: modulus of the inequality
- c: proportion of the inequality

Set of solutions: $\{x \in \mathbb{Z} : ax \bmod b \leq cx\}$.
(Non-negative solutions)

Equivalence: Two proportionally modular Diophantine inequalities are equivalent if they have the same set of solutions.

System of proportionally modular Diophantine inequalities

$$\left.\begin{array}{c} a_1 x \bmod b_1 \leq c_1 x \\ \vdots \\ a_m x \bmod b_m \leq c_m x \end{array}\right\}$$

a_i, b_i, c_i $(1 \leq i \leq n)$ positive integers.

$$\left.\begin{array}{l} \text{Set of solutions} \\ \text{Equivalence} \end{array}\right\} \text{Analogous definitions}$$

Problem: To get an equivalent system in which the value for all b_i is the same one, which in addition is a prime integer.

Answer: Yes

Tool: Theory of proportionally modular numerical semigroups

- Intervals (Submonoids generated by closed intervals)

Gift: New characterization of numerical semigroups with a Toms' decomposition.

Tool: Theory of proportionally modular numerical semigroups

- Quotients of numerical semigroups

Proportionally modular numerical semigroup

$$\mathrm{S}(a,b,c) = \{x \in \mathbb{Z} : ax \bmod b \leq cx\}$$

$\mathrm{S}(a,b,c)$ is a numerical semigroup.

Lemma.

1. $\mathrm{S}(a,b,c) = \mathrm{S}(a \bmod b, b, c)$.
2. If $c \geq a$ then $\mathrm{S}(a,b,c) = \mathbb{N}$.

We reduce our study to the case $1 \leq c < a < b$.

Submonoids generated by closed intervals

Let T be the submonoid of $(\mathbb{R}_0^+,+)$ generated by the closed interval $[\alpha,\beta]$, where $\alpha,\beta \in \mathbb{R}$ and $0 \leq \alpha < \beta$.

$$\mathrm{S}([\alpha,\beta]) = T \cap \mathbb{N}$$

$\mathrm{S}([\alpha,\beta])$ is a numerical semigroup.

Lemma (R-GS-GG-UB'03). Let $\alpha,\beta \in \mathbb{R}$ be such that $0 < \alpha < \beta$, and let x be a positive integer. Then $x \in \mathrm{S}([\alpha,\beta])$ if and only if there exists a positive integer k such that $\frac{x}{k} \in [\alpha,\beta]$. Therefore, $x \notin \mathrm{S}([\alpha,\beta])$ if and only if there exists an integer n for which $\frac{x}{n+1} < \alpha < \beta < \frac{x}{n}$.

Lemma **(R-GS-GG-UB'03).** Let a, b and c be positive integers such that $c < a < b$. Then $\mathrm{S}(a,b,c) = \mathrm{S}\left(\left[\frac{b}{a}, \frac{b}{a-c}\right]\right)$.
Conversely, if a_1, a_2, b_1 and b_2 are positive integers such that $\frac{b_1}{a_1} < \frac{b_2}{a_2}$, then $\mathrm{S}\left(\left[\frac{b_1}{a_1}, \frac{b_2}{a_2}\right]\right) = \mathrm{S}(a_1 b_2, b_1 b_2, a_1 b_2 - a_2 b_1)$.

Problem: To change the interval $I = \left[\frac{b}{a}, \frac{b}{a-c}\right]$ without changing the semigroup.

Example. Let $n \in \mathbb{N} \setminus \{0\}$.

- $n \in \mathrm{S}([n, n+1]) = \mathrm{S}\left(\left[\frac{n(n+1)}{n+1}, \frac{n(n+1)}{(n+1)-1}\right]\right)$
- $n \notin \mathrm{S}([n+\delta, n+1-\delta]), \forall \delta \in (0, \frac{1}{2})$.

Lemma. Let $I = [\alpha,\beta]$ be given with $\alpha,\beta \in \mathbb{R}^+$. There exists $\varepsilon > 0$ such that $\mathrm{S}([\alpha,\beta]) = \mathrm{S}([\alpha-\delta,\beta+\delta])$ for every $\delta \in [0,\varepsilon)$.

Proof.

($\subseteq$) $I \subseteq J \Rightarrow \mathrm{S}(I) \subseteq \mathrm{S}(J)$.

($\supseteq$)
- $\exists x \in \mathbb{N}\setminus\{0\}$ s.t. $x \in \mathrm{S}([\alpha - \frac{\varepsilon}{n+2}, \beta + \frac{\varepsilon}{n+2}])$ $\forall n \in \mathbb{N}$.
- $\forall n \in \mathbb{N}$ $\exists k_n \in \mathbb{N}\setminus\{0\}$ s.t. $\alpha - \frac{\varepsilon}{n+2} < \frac{x}{k_n} < \beta + \frac{\varepsilon}{n+2}$.
- $\alpha - \frac{\varepsilon}{n+2} \geq \frac{\alpha}{2} \Rightarrow k_n \leq \lceil \frac{2x}{\alpha} \rceil$.
- $k_n = k$ $\forall n$ (by passing to a subsequence).
- $\alpha - \frac{\varepsilon}{n+2} \leq \frac{x}{k} \leq \beta + \frac{\varepsilon}{n+2}$, $\forall n \in \mathbb{N}$.
- $x \in \mathrm{S}([\alpha,\beta])$.

* $\forall x \notin \mathrm{S}([\alpha,\beta])$ $\exists \varepsilon_x > 0$ s.t. $x \notin \mathrm{S}([\alpha-\delta,\beta+\delta])$ $\forall \delta \in [0,\varepsilon_x)$.

* $\varepsilon = \min\{\varepsilon_x \mid x \in \mathbb{N}\setminus \mathrm{S}([\alpha,\beta])\}$.

Lemma. Let a, b and c be positive integers such that $c < a < b$. Then

$$\mathrm{S}(a,b,c) = \mathrm{S}\left(\left[\frac{(b-a)b+1}{(b-a)a+1}, \frac{(a-c)b+1}{(a-c)^2}\right]\right).$$

Corollary. Let a, b and c be positive integers such that $c < a < b$. Let $\alpha, \beta \in \mathbb{R}$ be such that

$$\frac{(b-a)b+1}{(b-a)a+1} \leq \alpha \leq \frac{b}{a} \leq \frac{b}{a-c} \leq \beta \leq \frac{(a-c)b+1}{(a-c)^2}.$$

Then $\mathrm{S}(a,b,c) = \mathrm{S}([\alpha,\beta])$.

Proportionally modular Diophantine inequalities

Proposition. Let S be a proportionally modular numerical semigroup. Then there exists $N \in \mathbb{N} \setminus \{0\}$ such that for every $n \in \mathbb{N}$, $n \geq N$, S is the set of integer solutions of a proportionally modular Diophantine inequality with modulus n.

Proof.

* $S = \mathbb{N} \Rightarrow S = \mathrm{S}(1, n, 1)$, $\forall n \in \mathbb{N}$, $n \geq 1$.
* $S \neq \mathbb{N} \Rightarrow \exists a, b$ $(1 < a < b)$ s.t. $S = \mathrm{S}([a, b])$.
  - $\exists \varepsilon > 0$ s.t. $S = \mathrm{S}([a - \varepsilon, b + \varepsilon])$ and $1 < a - \varepsilon$.
  - $\exists N \in \mathbb{N} \setminus \{0\}$ such that $n \geq N \Rightarrow \exists a_1, a_2 \in \mathbb{N}$ s.t. $a - \frac{\varepsilon}{2} \leq \frac{n}{a_1} \leq a$, $b \leq \frac{n}{a_2} \leq b + \frac{\varepsilon}{2}$.
  - $\mathrm{S}([a, b]) \subseteq \mathrm{S}\left(\left[\frac{n}{a_1}, \frac{n}{a_2}\right]\right) \subseteq \mathrm{S}\left(\left[a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right]\right) = \mathrm{S}([a, b])$.
  - $a_2 < a_1 < n \Rightarrow S = \mathrm{S}([\frac{n}{a_1}, \frac{n}{a_2}]) = \mathrm{S}(a_1, n, a_1 - a_2)$.

Corollary. Let S be a proportionally modular numerical semigroup. Then there exist $a, b, c \in \mathbb{N} \setminus \{0\}$ such that $c < a < b$, b is a prime number, and $S = \mathrm{S}(a, b, c)$.

System of proportionally modular Diophantine inequalities

$$\left.\begin{array}{c} a_1 x \bmod b_1 \leq c_1 x \\ \vdots \\ a_n x \bmod b_n \leq c_n x \end{array}\right\}$$

Remark. The set of nonnegative integer solutions is the numerical semigroup $S = \bigcap_{i=1}^{n} \mathrm{S}(a_i, b_i, c_i)$.

Remark. If $A = \prod_{i=1}^{n} a_i$, $B = \prod_{i=1}^{n} b_i$ and $C = \prod_{i=1}^{n} c_i$, then

$$\mathrm{S}(a_i, b_i, c_i) = \mathrm{S}\left(A, \frac{Ab_i}{a_i}, \frac{Ac_i}{a_i}\right) = \mathrm{S}\left(\frac{Ba_i}{B_i}, B, \frac{Bc_i}{b_i}\right) = \mathrm{S}\left(\frac{Ca_i}{c_i}, \frac{Cb_i}{c_i}, C\right).$$

Consequence. There exists an equivalent system with the same factor (modulus or proportion) for all the inequalities in the system.

Corollary. Every system of proportionally modular Diophantine inequalities is equivalent to a system of proportionally modular Diophantine inequalities in which all the inequalities have the same modulus, which in addition is a prime integer.

Quotients of numerical semigroups

$$\frac{S}{d} = \{x \in \mathbb{N} \mid dx \in S\}$$

$\frac{S}{d}$ is a numerical semigroup.

Lemma (R-UB'06). Let $n_1, n_2, d \in \mathbb{N} \setminus \{0\}$ such that n_1, n_2 are relatively primes. Then $\frac{\langle n_1, n_2 \rangle}{d}$ is a proportionally modular numerical semigroup. Conversely, every proportionally modular numerical semigroup can be represented in this form.

Lemma (R-GS'08). Let $a_1, a_2, b_1, b_2 \in \mathbb{N} \setminus \{0\}$ be such that $1 < \frac{b_1}{a_1} < \frac{b_2}{a_2}$. If $\gcd\{b_1, b_2\} = 1$, then

$$\mathrm{S}\left(\left[\frac{b_1}{a_1}, \frac{b_2}{a_2}\right]\right) = \frac{\langle b_1, b_2 \rangle}{a_1 b_2 - a_2 b_1}.$$

Proposition. Let S be a proportionally modular numerical semigroup. Then there exists $N \in \mathbb{N} \setminus \{0\}$ such that for every $n \in \mathbb{N}$, $n \geq N$, we can represent S as a quotient of $\langle n, n+1 \rangle$ by a positive integer.

Proof.

* $S = \mathbb{N} \Rightarrow S = \frac{\langle n,n+1 \rangle}{n}$, $\forall n \in \mathbb{N}$, $n \geq 1$.
* $S \neq \mathbb{N} \Rightarrow \exists a, b$ $(1 < a < b)$ s.t. $S = \mathrm{S}([a,b])$.
    - $\exists \varepsilon > 0$ s.t. $S = \mathrm{S}([a-\varepsilon, b+\varepsilon])$ and $1 < a - \varepsilon$.
    - $\exists N \in \mathbb{N} \setminus \{0\}$ such that $n \geq N \Rightarrow \exists a_1, a_2 \in \mathbb{N}$ s.t. $a - \frac{\varepsilon}{2} \leq \frac{n}{a_1} \leq a$, $b \leq \frac{n+1}{a_2} \leq b + \frac{\varepsilon}{2}$.
    - $\mathrm{S}([a,b]) \subseteq \mathrm{S}\left(\left[\frac{n}{a_1}, \frac{n+1}{a_2}\right]\right) \subseteq \mathrm{S}\left(\left[a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right]\right) = \mathrm{S}([a,b])$.
    - $1 < \frac{n}{a_1} \Rightarrow S = \mathrm{S}([\frac{n}{a_1}, \frac{n+1}{a_2}]) = \frac{\langle n,n+1 \rangle}{(n+1)a_1 - na_2}$.

Corollary. Let $a, d \in \mathbb{N} \setminus \{0\}$. Then $\frac{\langle a, a+1 \rangle}{d}$ is a proportionally modular numerical semigroup. Conversely, every proportionally modular numerical semigroup can be represented in this form.

Toms' decompositions

Definition (Toms'03). A numerical semigroup S has a Toms' decomposition if there exist $q_1, \ldots, q_n, m_1, \ldots, m_n, L \in \mathbb{N} \setminus \{0\}$ fulfilling that

1. $\gcd\{q_i, m_i\} = \gcd\{L, q_i\} = \gcd\{L, m_i\} = 1$, for all $i \in \{1, \ldots, n\}$,
2. $S = \dfrac{\langle q_1, m_1 \rangle}{L} \cap \cdots \cap \dfrac{\langle q_n, m_n \rangle}{L}$.

Theorem (R-GS'08). Every system proportionally modular numerical semigroup admits a Toms' decomposition.

Lemma (R-GS-GG-UB'03). Let $n_1, n_2, u, v \in \mathbb{N} \setminus \{0\}$ such that $un_2 - vn_1 = 1$. Then

$$\langle n_1, n_2 \rangle = \{x \in \mathbb{N} \mid un_2 x \bmod n_1 n_2 \leq x\}.$$

Corollary. Let $a, d_1, \ldots, d_n \in \mathbb{N} \setminus \{0\}$. Then

$$S = \{x \in \mathbb{N} \mid \{d_1 x, \ldots, d_n x\} \subseteq \langle a, a+1 \rangle\}$$

is a numerical semigroup having a Toms' decomposition. Conversely, every numerical semigroup having a Toms' decomposition can be represented in this form.

Corollary. Let $a, d_1, \ldots, d_n \in \mathbb{N} \setminus \{0\}$. Then

$$S = \left\{ x \in \mathbb{N} \mid \begin{array}{c} (a+1)d_1 x \bmod a(a+1) \leq d_1 x, \\ \vdots \\ (a+1)d_n x \bmod a(a+1) \leq d_n x. \end{array} \right\}$$

is a numerical semigroup having a Toms' decomposition. Conversely, every numerical semigroup having a Toms' decomposition can be represented in this form.

- A.M. Robles-Pérez and J.C. Rosales, *Equivalent proportionally modular Diophantine inequalities*. Arch. Math. 90 (2008), 24–30.
- J.C. Rosales and P.A. García-Sánchez, Numerical semigroups having a Toms decomposition, Canad. Math. Bull. 51 (2008), 134–139.
- J.C. Rosales, P.A. García-Sánchez, J.I. García-García and J.M. Urbano-Blanco, *Proportionally modular Diophantine inequalities*. J. Number Theory 103 (2003), 281–294.
- J.C. Rosales and J.M. Urbano-Blanco, *Proportionally modular Diophantine inequalities and full semigroups.* Semigroup Forum 72 (2006), 362–374.
- A. Toms, Strongly perforated K_0-groups of simple C^*-algebras. Canad. Math. Bull. 46 (2003) 457–472.