Proportionally modular Diophantine inequalities

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What we intend to study

- A *proportionally modular Diophantine inequality* is an expression of the form $ax \bmod b \leq cx$, with a, b, and c positive integers
- The set $S = \mathrm{S}(a, b, c)$ of solutions to such inequality is a numerical semigroup
- A numerical semigroup is *proportionally modular* if it is of this form

Some basic definitions

- If $A \subset \mathbb{N}$, we denote by $\langle A \rangle$ the submonoid of $(\mathbb{N}, +)$ generated by A,

$$\langle A \rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N}, \lambda_1, \ldots, \lambda_n \in \mathbb{N}, a_1, \ldots, a_n \in A\}$$

- $\langle A \rangle$ is a numerical semigroup if and if $\gcd(A) = 1$
- Every numerical semigroup S is finitely generated, and thus there exist positive integers $n_1, \ldots, n_p \in S$ such that $S = \langle n_1, \ldots, n_p \rangle$. If no proper subset of $\{n_1, \ldots, n_p\}$ generates S, we say that this set is a *minimal generating set* of S
- Minimal generating sets are unique and its elements are *minimal generator* of the semigroup

Example

$$\mathrm{S}(12,32,3) = \{x \in \mathbb{N} \mid 12x \bmod 32 \leq 3x\} = \{0,3,6,7,8,9,10,\rightarrow\}$$
$$= \langle 3,7,8\rangle$$

$12\times 1 \bmod 32 = 12 > 3\times 1$	$12\times 7 \bmod 32 = 20 \leq 3\times 7$
$12\times 2 \bmod 32 = 24 > 3\times 2$	$12\times 8 \bmod 32 = 0 \leq 3\times 8$
$12\times 3 \bmod 32 = 4 \leq 3\times 3$	$12\times 9 \bmod 32 = 12 \leq 3\times 9$
$12\times 4 \bmod 32 = 16 > 3\times 4$	$12\times 10 \bmod 32 = 24 \leq 3\times 10$
$12\times 5 \bmod 32 = 28 > 3\times 5$	$\vdots$
$12\times 6 \bmod 32 = 8 \leq 3\times 6$	

Some simplifications

- The inequality $ax \bmod b \leq cx$ has the same solutions as $(a \bmod b)x \bmod b \leq cx$
- If $c \geq a$, then $\mathrm{S}(a,b,c) = \mathbb{N}$
- Thus we can assume that $c < a < b$

Lemma

If $c < a < b$ are positive integers, $\mathrm{S}(a,b,c) = T \cap \mathbb{N}$, where T is the submonoid of $(\mathbb{Q},+)$ generated by $[\frac{b}{a}, \frac{b}{a-c}]$. Conversely, if a_1, a_2, b_1, b_2 are positive integers with $\frac{a_1}{b_1} < \frac{a_2}{b_2}$, and T is the submonoid of $(\mathbb{Q},+)$ generated by $[\frac{a_1}{b_1}, \frac{a_2}{b_2}]$, then $T \cap \mathbb{N} = \mathrm{S}(a_2 b_1, a_1 a_2, a_2 b_1 - a_1 b_2)$

- $T \cap \mathbb{N}$ is the *proportionally modular numerical semigroup associated to the interval* $[\frac{a_1}{b_1}, \frac{a_2}{b_2}]$, and we denote it by $\mathrm{S}([\frac{a_1}{b_1}, \frac{a_2}{b_2}])$

Lemma

If $c < a < b$ are positive integers, $\mathrm{S}(a,b,c) = T \cap \mathbb{N}$, where T is the submonoid of $(\mathbb{Q},+)$ generated by $[\frac{b}{a}, \frac{b}{a-c}]$. Conversely, if a_1, a_2, b_1, b_2 are positive integers with $\frac{a_1}{b_1} < \frac{a_2}{b_2}$, and T is the submonoid of $(\mathbb{Q},+)$ generated by $[\frac{a_1}{b_1}, \frac{a_2}{b_2}]$, then $T \cap \mathbb{N} = \mathrm{S}(a_2 b_1, a_1 a_2, a_2 b_1 - a_1 b_2)$

Example

- $\mathrm{S}(12,32,3) = \mathrm{S}([\frac{32}{12}, \frac{32}{9}]) = \mathrm{S}([\frac{8}{3}, \frac{32}{9}])$
- $T = \bigcup_{k\in\mathbb{N}}[k\frac{8}{3}, k\frac{32}{9}] = \{0\} \cup [\frac{8}{3}, \frac{32}{9}] \cup [\frac{16}{3}, \frac{64}{9}] \cup [8, \frac{32}{3}] \cup \cdots$
- $\mathrm{S}(12,32,3) = T \cap \mathbb{N} = \{0,3,6,7,8,9,10,\ldots\}$

First results

Lemma

A positive integer x belongs to $\mathrm{S}([\frac{a_1}{b_1}, \frac{a_2}{b_2}])$ if and only if there exists a positive integer y such that $\frac{a_1}{b_1} \leq \frac{x}{y} \leq \frac{a_2}{b_2}$

Lemma

If $a_2 b_1 - a_1 b_2 = 1$, then $\mathrm{S}([\frac{a_1}{b_1}, \frac{a_2}{b_2}]) = \langle a_1, a_2 \rangle$

Bézout sequence

A sequence $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_p}{b_p}$ of rational numbers is a *Bézout sequence* if $a_1, \ldots, a_p, b_1, \ldots, b_p$ are positive integers and $a_{i+1}b_i - a_ib_{i+1} = 1$ for all $i \in \{1, \ldots, p-1\}$
Its *length* is p, and $\frac{a_1}{b_1}$ and $\frac{a_p}{b_p}$ are its *ends*

Proposition

If $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_p}{b_p}$ is a Bézout sequence, $\mathrm{S}([\frac{a_1}{b_1}, \frac{a_p}{b_p}]) = \langle a_1, \ldots, a_p \rangle$

Example

Let us solve the Diophantine inequality

$$50x \bmod 131 \leq 3x$$

We know that the set of solutions is $S([\frac{131}{50}, \frac{131}{47}])$

As

$$\frac{131}{50} < \frac{76}{29} < \frac{21}{8} < \frac{8}{3} < \frac{11}{4} < \frac{25}{9} < \frac{39}{14} < \frac{131}{47}$$

is a Bézout sequence,

$$S\left([\frac{131}{50}, \frac{131}{47}]\right) = \langle 131, 76, 21, 8, 11, 25, 39 \rangle = \langle 8, 11, 21, 25, 39 \rangle$$

Problems:

1) If $\frac{a}{b} < \frac{c}{d}$, is there a Bézout sequence with ends $\frac{a}{b}$ and $\frac{c}{d}$?
2) If so, find a procedure to compute such a sequence
3) Characterize proportionally modular numerical semigroups in terms of their minimal generators

Proposition

Let a_1, a_2, b_1 and b_2 be positive integers with $\frac{a_1}{b_1} < \frac{a_2}{b_2}$ and $\mathrm{mcd}\{a_1, b_1\} = 1 = \mathrm{mcd}\{a_2, b_2\}$. Then there exists a Bézout sequence of length less than or equal to $a_2 b_1 - a_1 b_2 + 1$ and with ends $\frac{a_1}{b_1}$ and $\frac{a_2}{b_2}$

Corollary

$\mathrm{S}([\frac{a_1}{b_1}, \frac{a_2}{b_2}])$ has embedding dimension less than or equal to $a_2 b_1 - a_1 b_2 + 1$

Refinement of sequences: looking for minimal generators

Proper sequences

A Bézout sequence $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_p}{b_p}$ is *proper* if $a_{i+h}b_i - a_ib_{i+h} \geq 2$ for all $h \geq 2$ such that $i, i+h \in \{1, \ldots, p\}$

Remark

Every Bézout sequence can be refined to a proper Bézout sequence with the same ends

Examples

- $\frac{5}{3} < \frac{12}{7} < \frac{7}{4} < \frac{9}{5}$ is a non proper Bézout sequence
- $\frac{5}{3} < \frac{7}{4} < \frac{9}{5}$ is a proper Bézout sequence

Proper Bézout sequences are not the final solution

Proposition

If $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_p}{b_p}$ is a proper Bézout sequence, then $\max\{a_1, \ldots, a_p\} = \max\{a_1, a_p\}$

Corollary

If $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_p}{b_p}$ is a proper Bézout sequence, then there exists $h \in \{1, \ldots, p\}$ such that $a_1 \geq a_2 \geq \cdots \geq a_h \leq a_{h+1} \leq \cdots \leq a_p$

Example

$\frac{2}{1} < \frac{3}{1} < \frac{4}{1}$ is a proper Bézout sequence. However $\{2, 3, 4\}$ is not a minimal generating system for $\langle 2, 3, 4 \rangle$

Adjacent ends and minimal generators

Adjacent fractions

Two fractions $\frac{a_1}{b_1} < \frac{a_2}{b_2}$ are *adjacent* if $\frac{a_2}{b_2+1} < \frac{a_1}{b_1}$ and $b_1 = 1$, or $\frac{a_2}{b_2} < \frac{a_1}{b_1-1}$

Proposition

If $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_p}{b_p}$ is a proper Bézout sequence with adjacent ends, then $\{a_1, \ldots, a_p\}$ is the minimal generating system of $\mathrm{S}([\frac{a_1}{b_1}, \frac{a_p}{b_p}])$

Theorem

Let S be a proportionally modular numerical semigroup. Then there exists an arrangement $n_1, \ldots, n_p$ of its minimal generators and positive integers $b_1, \ldots, b_p$ such that $\frac{n_1}{b_1} < \frac{n_2}{b_2} < \cdots < \frac{n_p}{b_p}$ is a proper Bézout sequence with adjacent ends

Corollary

Let S be a proportionally modular numerical semigroup with minimal generating set $n_1 < n_2 < \cdots < n_p$ con $p \geq 3$. Then $\langle n_1, \ldots, n_{p-1} \rangle$ is also proportionally modular

This yields a characterization of proportionally modular numerical semigroups

Characterization

A numerical semigroup S is proportionally modular if and only if there exists a (convex) arrangement $n_1, \ldots, n_p$ of its minimal generators such that

1. $\mathrm{mcd}\{n_i, n_{i+1}\} = 1$ for all $i \in \{1, \ldots, p-1\}$
2. $n_{i-1} + n_{i+1} \equiv 0 \bmod n_i$ for all $i \in \{2, \ldots, p-1\}$

Examples

- $S = \langle 5, 7, 11 \rangle$ is not proportionally modular, since $5 + 11 \not\equiv 0 \bmod 7$ and $7 + 11 \not\equiv 0 \bmod 5$
- $S = \langle 6, 8, 11, 13 \rangle$ is not proportionally modular, $\mathrm{mcd}\{6, 8\} \neq 1$

Example

$S = \langle 8, 11, 21, 25, 39 \rangle$ is proportionally modular

	8	11		
21	8	11		
21	8	11	25	
21	8	11	25	39

A reformulation of the above characterization

Corollary

Let S be a proportionally modular numerical semigroup. Then there exists an arrangement $n_1, \ldots, n_p$ of its minimal generators so that
$S = \langle n_1, n_2\rangle \cup \langle n_2, n_3\rangle \cup \cdots \cup \langle n_{p-1}, n_p\rangle$

- $S = \langle 7, 8, 9, 10, 12\rangle$ is not proportionally modular
  $$\begin{cases} 7 & 8 \\ 7 & 8 & 9 \\ 7 & 8 & 9 & 10 \end{cases}$$
- $S = \langle 12, 7\rangle \cup \langle 7, 8\rangle \cup \langle 8, 9\rangle \cup \langle 9, 10\rangle$

Corollary

A numerical semigroup S is proportionally modular if and only if there exists a (convex) arrangement $n_1, \ldots, n_p$ of its minimal generators fulfilling that

1. $\langle n_i, n_{i+1} \rangle$ is a numerical semigroup for all $i \in \{1, \ldots, p-1\}$
2. $\langle n_{i-1}, n_i, n_{i+1} \rangle = \langle n_{i-1}, n_i \rangle \cup \langle n_i, n_{i+1} \rangle$ for all $i \in \{2, \ldots, p-1\}$

Computing some invariants for the embedding dimension three case

- Given a numerical semigroup S and $n \in S \setminus \{0\}$, the *Apéry set* of n in S is
$$\mathrm{Ap}(S,n) = \{s \in S \mid s - n \notin S\}$$
- $\mathrm{Ap}(S,n) = \{0, w(1), \ldots, w(n-1)\}$ where $w(i)$ is the least element in S congruent with i modulo n
- $\#\mathrm{Ap}(S,n) = n$
- The Frobenius number of S is the largest integer not in S $\mathrm{F}(S) = \max(\mathrm{Ap}(S,n)) - n$
- The set of gaps of S is $\mathrm{H}(S) = \mathbb{N} \setminus S$ $\#\mathrm{H}(S) = \frac{1}{n}(w(1) + \cdots + w(n-1)) - \frac{n-1}{2}$

If S is a proportionally modular numerical semigroup with minimal generating set $\{n_1, n_2, n_3\}$, then we can assume that $\mathrm{mcd}\{n_1, n_2\} = \mathrm{mcd}\{n_2, n_3\} = 1$ and that $dn_2 = n_1 + n_3$ for some $d \in \mathbb{N} \setminus \{0, 1\}$

Proposition

$\mathrm{Ap}(S, n_2) = \{0, n_1, \ldots, \lfloor \frac{n_3}{d} \rfloor n_1, n_3, \ldots, (n_2 - \lfloor \frac{n_3}{d} \rfloor - 1) n_3\}$

$\mathrm{F}(S) = \max\{\lfloor \frac{n_3}{d} \rfloor n_1 - n_2, \lfloor \frac{n_1}{d} \rfloor n_3 - n_2\}$

$\#\mathrm{H}(S) = \frac{n_1(1+\lfloor \frac{n_3}{d} \rfloor)\lfloor \frac{n_3}{d} \rfloor + n_3(n_2 - \lfloor \frac{n_3}{d} \rfloor)(n_2 - \lfloor \frac{n_3}{d} \rfloor - 1) - n_2(n_2 - 1)}{2n_2}$

How to compute a Bézout sequence with given ends

Idea

- Every Bézout sequence can be obtained by gluing two Bézout sequences, the first with decreasing numerators, and the second with increasing numerators

$$\frac{131}{50} < \frac{76}{29} < \frac{21}{8} < \frac{8}{3} < \frac{11}{4} < \frac{25}{9} < \frac{39}{14} < \frac{131}{47}$$

$\frac{131}{50} < \frac{76}{29} < \frac{21}{8} < \frac{8}{3}$ sequence with decreasing numerators
$\frac{8}{3} < \frac{11}{4} < \frac{25}{9} < \frac{39}{14} < \frac{131}{47}$ sequence with increasing numerators

Decreasing numerators

Theorem

Let a_1 and b_1 be two positive integers with $a_1 \geq b_1 \geq 1$ and $\mathrm{mcd}\{a_1, b_1\} = 1$
As long as $b_i \neq 1$, set

$$a_{i+1} = b_i^{-1} \bmod a_i \text{ y } b_{i+1} = (-a_i)^{-1} \bmod b_i$$

Then there exists a positive integer p such that $a_p = \lceil \frac{a_1}{b_1} \rceil$, $b_p = 1$ and $\frac{a_1}{b_1} < \frac{a_2}{b_2} < \cdots < \frac{a_p}{b_p}$ is a Bézout sequence
Moreover, if $\frac{a_1}{b_1} < \frac{a'_2}{b'_2} < \cdots < \frac{a'_q}{b'_q}$ a Bézout sequence such that $a_1 \geq a'_2 \geq \cdots \geq a'_q$, then $q \leq p$, $a'_i = a_i$ and $b'_i = b_i$ for all $i \in \{2, \ldots, q\}$

Proposition

If $i \in \{2, \ldots, p-1\}$, then $a_{i+1} = (-a_{i-1}) \bmod a_i$ and $b_{i+1} = (-b_{i-1}) \bmod b_i$

Example

Let us construct the maximal Bézout sequence with decreasing numerators and left end $\frac{131}{50}$

$a_1 = 131$ and $b_1 = 50$
Then $a_2 = 50^{-1} \bmod 131 = 76$ and $b_2 = (-131)^{-1} \bmod 50 = 29$
By using that $a_{i+1} = (-a_{i-1}) \bmod a_i$ and $b_{i+1} = (-b_{i-1}) \bmod b_i$, we get

$$\frac{131}{50} < \frac{76}{29} < \frac{21}{8} < \frac{8}{3} < \frac{3}{1}$$

Increasing numerators

Theorem

Let a_1 and b_1 be two positive integers with $a_1 \geq b_1 \geq 1$ and $\mathrm{mcd}\{a_1, b_1\} = 1$

As long as $a_i \neq 1$, set a_{i+1} and b_{i+1} as follows

1) If $b_i \neq 1$, $a_{i+1} = (-b_i)^{-1} \bmod a_i$ and $b_{i+1} = a_i^{-1} \bmod b_i$

2) If $b_i = 1$, then $a_{i+1} = a_i - 1$ and $b_{i+1} = 1$

Then there exists a positive integer p such that $a_p = 1$, $b_p = 1$ and $\frac{a_p}{b_p} < \cdots < \frac{a_2}{b_2} < \frac{a_1}{b_1}$ is a Bézout sequence

Moreover, if $\frac{a'_s}{b'_s} < \cdots < \frac{a'_2}{b'_2} < \frac{a_1}{b_1}$ is a Bézout sequence such that $\frac{a'_s}{b'_s} \geq 1$ and $a'_s \leq \cdots \leq a'_2 \leq a_1$, then $s \leq p$, $a'_i = a_i$ and $b'_i = b_i$ for all $i \in \{2, \ldots, s\}$

Proposition

If $i \geq 2$ and $b_i \neq 1$, then $a_{i+1} = (-a_{i-1}) \bmod a_i$ and $b_{i+1} = (-b_{i-1}) \bmod b_i$

Example

Let us construct the maximal Bézout sequence with fractions greater than or equal to one, with increasing numerators and with right end $\frac{131}{47}$

$a_1 = 131$ and $b_1 = 47$
Then $a_2 = (-47)^{-1} \bmod 131 = 39$ and $b_2 = 131^{-1} \bmod 47 = 14$
By using that if $b_i \neq 1$, then $a_{i+1} = (-a_{i-1}) \bmod a_i$ and $b_{i+1} = (-b_{i-1}) \bmod b_i$, we obtain

$$\frac{131}{47} > \frac{39}{14} > \frac{25}{9} > \frac{11}{4} > \frac{8}{3} > \frac{5}{2} > \frac{2}{1} > \frac{1}{1}$$

Example

Let us construct the Bézout sequence with ends $\frac{131}{50}$ and $\frac{131}{47}$

1) We compute the maximal Bézout sequence with decreasing numerators and left end $\frac{131}{50}$

$$\frac{131}{50} < \frac{76}{29} < \frac{21}{8} < \frac{8}{3} < \frac{3}{1}$$

2) Then we keep constructing the maximal Bézout sequence with increasing denominators and right end $\frac{131}{47}$ until we find an element in the above sequence

$$\frac{131}{47} > \frac{39}{14} > \frac{25}{9} > \frac{11}{4} > \frac{8}{3}$$

3) The desired sequence is

$$\frac{131}{50} < \frac{76}{29} < \frac{21}{8} < \frac{8}{3} < \frac{11}{4} < \frac{25}{9} < \frac{39}{14} < \frac{131}{47}$$

Theorem

Let a, b, c and d be positive integers such that $\mathrm{mcd}\{a,b\} = \mathrm{mcd}\{c,d\} = 1$ and $\frac{a}{b} < \frac{c}{d}$
Then there exists a unique proper Bézout sequence with ends $\frac{a}{b}$ and $\frac{c}{d}$

Lemma

Let a_1, b_1, a_2 and b_2 be positive integers
Then $\frac{a_1}{b_1} < \frac{a_2}{b_2}$ is a Bézout sequence if and only if $\frac{a_1+b_1}{b_1} < \frac{a_2+b_2}{b_2}$ is a Bézout sequence

Example

Let us construct the unique proper Bézout sequence with ends $\frac{8}{25}$ and $\frac{20}{7}$

1) We construct the proper Bézout sequence with ends $\frac{8+25}{25} = \frac{33}{25}$ and $\frac{20+7}{7} = \frac{27}{7}$

$$\frac{33}{25} < \frac{4}{3} < \frac{3}{2} < \frac{2}{1} < \frac{3}{1} < \frac{7}{2} < \frac{11}{3} < \frac{15}{4} < \frac{19}{5} < \frac{23}{6} < \frac{27}{7}$$

2) Subtracting 1 to all the terms, we obtain the desired sequence

$$\frac{8}{25} < \frac{1}{3} < \frac{1}{2} < \frac{1}{1} < \frac{2}{1} < \frac{5}{2} < \frac{8}{3} < \frac{11}{4} < \frac{14}{5} < \frac{17}{6} < \frac{20}{7}$$