Quotients of a numerical semigroup by a positive integer

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Notation

- $\mathbb{Z}$ denotes the set of integers
- $\mathbb{N}$ denotes the set of nonnegative integers
- $\langle n_1, \ldots, n_e \rangle = \{\lambda_1 n_1 + \cdots + \lambda_e n_e \mid \lambda_1, \ldots, \lambda_e \in \mathbb{N}\}$

For a numerical semigroup S

- $\mathrm{F}(S)$ the largest integer not in S, the Frobenius number of S
- $\mathrm{G}(S)$ the set of nonnegative integers not in S, the gaps of S
- $\mathrm{g}(S)$ the cardinality of $\mathrm{G}(S)$, the gender S or singularity degree of S

Quotient

Let S be a numerical semigroup and let p be a positive integer

$$\frac{S}{p} = \{x \in \mathbb{N} \mid px \in S\}$$

- This set is again a numerical semigroup
- $S \subseteq \dfrac{S}{p}$
- $\dfrac{S}{p} = \mathbb{N}$ iff $p \in S$

J. C. Rosales, J. M. Urbano, Proportionally modular diophantine inequalities and full semigroups, Semigroup Forum 72(2006), 362-374

Theorem

Let n_1, n_2 and p be positive integers with n_1 and n_2 relatively prime. Then $\frac{\langle n_1, n_2 \rangle}{p}$ is a proportionally modular numerical semigroup.
Every proportionally modular numerical semigroup is of this form

- We have an algorithm that allows us to determine whether or not a numerical semigroup is the quotient of an embedding dimension two numerical semigroup by a positive integer
- If $un_2 - vn_1 = 1$, then $\mathrm{S}\left(\left[\frac{n_1}{up}, \frac{n_2}{vp}\right]\right) = \dfrac{\langle n_1, n_2\rangle}{p}$. Thus by using Bézout sequences, one can compute a minimal generating system of $\dfrac{\langle n_1, n_2\rangle}{p}$.

Open problems

Find formulas for

1. the largest multiple of p not belonging to $\langle n_1, n_2\rangle$
2. the cardinality of the set of multiples of p not in $\langle n_1, n_2\rangle$
3. the least multiple of p in $\langle n_1, n_2\rangle$

A. M. Robles, J. C. Rosales, Equivalent proportionally modular Diophantine inequalities, Archiv Math. **90** (2008), 24-30

Theorem

Every proportionally modular numerical semigroup is of the form

$$\frac{\langle a, a+1 \rangle}{p}$$

with a and p positive integers

Open problem

$n_1 = a$ and $n_2 = a + 1$

A. Toms, Strongly perforated K_0-groups of simple C^*-algebra, Canad. Math. Bull. 46(2003), 457-472

Toms decomposition

A numerical semigroup S admits a Toms decomposition if there exist positive integers $q_1, \ldots, q_n$, $m_1, \ldots, m_n$ and L such that

1) $\gcd\{q_i, m_i\} = \gcd\{L, q_i\} = \gcd\{L, m_i\} = 1$ for all i

2) $S = \dfrac{1}{L} \bigcap_{i=1}^{n} \langle q_i, m_i \rangle$

J. C. Rosales, P. A. García-Sánchez, Numerical semigroups having a Toms decomposition, Canad. Math. Bull. 51 (2008), 134-139

Theorem

A numerical semigroup admits a Toms decomposition if and only if it is the intersection of finitely many proportionally modular numerical semigroups

M. Delgado, P. A. García-Sánchez, J. C. Rosales, J. M. Urbano-Blanco, Systems of proportionally modular Diophantine inequalities, Semigroup Forum

- Algorithm to detect whether or not a numerical semigroup admits a Toms decomposition

M. A. Moreno, J. Nicola, E. Pardo, H. Thomas, Numerical semigroups that cannot we written as an intersection a d-squashed semigroups, preprint

- There are numerical semigroups that are not the quotient of an embedding dimension three numerical semigroup

Open problem

Find a procedure to determine if a numerical semigroup is the quotient of an embedding dimension three numerical semigroup

J. C. Rosales, M. B. Branco, Irreducible numerical semigroups, Pacific J. Math. 209 (2003), 131-143

Irreducible numerical semigroup

A numerical semigroup is irreducible if it cannot be expressed as the intersection of numerical semigroups properly containing it

- Every numerical semigroup is a finite intersection of irreducible numerical semigroups

- A numerical semigroup is irreducible if and only if it is maximal in the set of numerical semigroups with its same Frobenius number

R. Fröberg, C. Gottlieb, R. Häggvist, On numerical semigroups, Semigroup Forum 35(1987), 63-83

- A numerical semigroup is irreducible if and only if it is either symmetric or pseudo-symmetric

J. C. Rosales, P. A. García-Sánchez, Every numerical semigroup is one half of symmetric numerical semigroup, Proc. Amer. Math. Soc. 136 (2008), 475-477

Theorem

Every numerical semigroup is one half of a symmetric numerical semigroup

J. C. Rosales, P. A. García-Sánchez, Every numerical semigroup is one half of infinitely many symmetric numerical semigroups, Comm. Algebra

Let S be a numerical semigroup

Pseudo-Frobenius number

The set of *pseudo-Frobenius* numbers of S is

$$\mathrm{PF}(S) = \{x \in \mathbb{Z} \setminus S \mid x + s \in S \text{ for all } x \in S \setminus \{0\}\}$$

The cardinality of $\mathrm{PF}(S)$ is the *type* of S, $\mathrm{t}(S)$

- S is symmetric if and only if $\mathrm{PF}(S) = \{\mathrm{F}(S)\}$ if and only $\mathrm{t}(S) = 1$
- S is pseudo-symmetric if and only if $\mathrm{PF}(S) = \left\{\mathrm{F}(S), \frac{\mathrm{F}(S)}{2}\right\}$

Let S be a numerical semigroup with $\mathrm{PF}(S) = \{f_1, \ldots, f_t\}$ and let $x \in S$

- $x \notin S$ if and only if $f_i - x \in S$ for some $i \in \{1, \ldots, t\}$

Theorem

Assume that $\{n_1, \ldots, n_p\}$ generates S. TFAE:

- T is a symmetric numerical semigroup with $S = \dfrac{T}{2}$
- $T = \langle 2n_1, \ldots, 2n_p, f - 2f_1, \ldots, f - 2f_t \rangle$ for some odd integer f such that $f - f_i - f_j \in S$ for all $i, j \in \{1, \ldots, t\}$

- Every numerical semigroup is one half of infinitely many symmetric numerical semigroups

J. C. Rosales, One half of a pseudo-symmetric numerical semigroup, Bull. London Math. Soc.

- If S is a numerical semigroup and $\mathrm{F}(S)$ is even, then $\mathrm{F}\left(\frac{S}{2}\right)=\frac{\mathrm{F}(S)}{2}$
- A numerical semigroup is not one half of infinitely many pseudo-symmetric numerical semigroups
- One half of a pseudo-symmetric numerical semigroup is always irreducible

Theorem

A numerical semigroup is irreducible if and only if if is one half of a pseudo-symmetric numerical semigroup

- Every numerical semigroup is one fourth of a pseudo-symmetric numerical semigroup

A. M. Robles-Pérez, J. C. Rosales, P. Vasco, The doubles of a numerical semigroup, preprint

Let S be a numerical semigroup

Doubles of S

$$\mathcal{D}(S) = \left\{ T \mid T \text{ is a numerical semigroup and } S = \frac{T}{2} \right\}$$

m*-upper sets of gaps*

A subset H of $\mathrm{G}(S)$ is an *m-upper subset* of $\mathrm{G}(S)$ if

1) $(m+H)\cap \mathrm{G}(S)$ is empty
2) $(m+H+H)\cap \mathrm{G}(S)$ is empty
3) if $h \in H$, then $\{g \in \mathrm{G}(S) \mid g-h \in S\} \subseteq H$

For S a numerical semigroup, m an odd integer in S and H an m-upper subset of $\mathrm{G}(S)$

$$\mathrm{S}(m,H) = (2S) \cup (m+2S) \cup (m+2H)$$

is a numerical semigroup

- $\mathrm{g}(\mathrm{S}(m,H)) = 2\mathrm{g}(S) + \frac{m-1}{2} - \#H$
- $\mathrm{F}(\mathrm{S}(m,H)) = \begin{cases} \max\{2\mathrm{F}(S), m-2\}, \text{if } H = \mathrm{G}(S), \\ \max\{2\mathrm{F}(S), 2\max(\mathrm{G}(S)\setminus H) + m\}, \text{otherwise} \end{cases}$

Theorem

$$\mathcal{D}(S) = \left\{ \mathrm{S}(m,H) \mid \begin{array}{l} m \text{ an odd integer in } S \\ H \text{ an } m-\text{upper subset of } \mathrm{G}(S) \end{array} \right\}$$

Moreover $\mathrm{S}(m_1,H_1) = \mathrm{S}(m_2,H_2)$ if and only if $(m_1,H_1) = (m_2,H_2)$

- If m is an odd integer in S greater than $\mathrm{F}(S)$, then

$$H = \{x \in \mathrm{G}(S) \mid \mathrm{F}(S) - x \in \mathrm{G}(S)\}$$

  is an m-upper subset of $\mathrm{G}(S)$ and $\mathrm{S}(m, H)$ is a symmetric numerical semigroup
- Every numerical semigroup is one half of infinitely many symmetric numerical semigroups

Balanced numerical semigroup

A numerical semigroup is *balanced* if it has as many odd gaps as even gaps

Let S be a numerical semigroup

- S is balanced if and only if $\mathrm{g}\left(\frac{S}{2}\right) = \frac{\mathrm{g}(S)}{2}$

Theorem

The set

$$\{T \in \mathcal{D}(S) \mid T \text{ is balanced}\}$$

is not empty and has finitely many elements

- Every numerical semigroup is one half of finitely many balanced numerical semigroups

Theorem

Every symmetric numerical semigroup is one half of a balanced pseudo-symmetric numerical semigroup

- Every numerical semigroup is one fourth of infinitely many balanced pseudo-symmetric numerical semigroups

Open problem

Let S be a numerical semigroup. Find a formula, depending on S, for

$$\min\{\mathrm{g}(T) \mid T \in \mathcal{D}(S)\}$$