Frobenius varieties

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- If S and T are numerical semigroups, then so is $S \cap T$
- If S is a numerical semigroup other than $\mathbb{N}$, then so is $S \cup \{\mathrm{F}(S)\}$

Frobenius varieties

Families of numerical semigroups closed under finite intersections and to the adjoin of the Frobenius number

J. C. Rosales, Families of numerical semigroups closed under finite intersections and for the Frobenius number, Houston J. Math.

Graph

A *graph* is a pair (V, E) with V a set (vertices) and E a subset of $\{(v, w) \in V \times V \mid v \neq w\}$ (edges)

Path

A *path* connecting vertices x and y of G is a sequence of distinct edges of the form $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ such that $v_0 = x$ and $v_n = y$

Trees

A graph G is a *tree* if there exists a vertex r of G (the root) such that for any other vertex x of G, there is a unique path connecting x with r

If (x, y) is an edge of a tree, we say that x is a *son* of y

Define

$$\mathcal{S} = \{S \mid S \text{ is a numerical semigroup}\}$$

$\mathbf{G}(\mathcal{S})$ the graph whose set of vertices is $\mathcal{S}$ and set of edges

$$\{(S,T) \in \mathcal{S}\times\mathcal{S} \mid T = S \cup \{\mathrm{F}(S)\}\}$$

Given a numerical semigroup S we define recurrently the sequence of numerical semigroups

- $S_0 = S$
- $S_{i+1} = \begin{cases} S_i \cup \{\mathrm{F}(S_i)\} \text{ if } S_i \neq \mathbb{N} \\ \mathbb{N} \text{ otherwise} \end{cases}$

$$S = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_{\mathrm{g}(S)} = \mathbb{N}$$

Chain of semigroups associated to S

$$\mathrm{C}(S) = \{S_0, S_1, \ldots, S_{\mathrm{g}(S)}\}$$

Theorem

$\mathrm{G}(\mathcal{S})$ is a tree rooted in $\mathbb{N}$. Moreover, the sons of $S \in \mathcal{S}$ are $S \setminus \{x_1\}, \ldots, S \setminus \{x_r\}$ where $x_1, \ldots, x_r$ are the minimal generators of S grater than $\mathrm{F}(S)$

This result allows us to construct recurrently the set of all numerical semigroups

$$
\begin{tikzcd}
 & & \mathbb{N} = \langle 1 \rangle & \\
 & & \langle 2,3 \rangle \arrow[u] & \\
 & \langle 3,4,5 \rangle \arrow[ur] & & \langle 2,5 \rangle \arrow[ul] \\
\langle 4,5,6 \rangle \arrow[ur] & \langle 3,5,7 \rangle \arrow[u] & \langle 3,4 \rangle \arrow[ul] & \langle 2,7 \rangle \arrow[u]
\end{tikzcd}
$$

- If (S,T) is an edge of $\mathbf{G}(\mathcal{S})$, then $\mathrm{F}(S) > \mathrm{F}(T)$ and $\mathrm{g}(T) = \mathrm{g}(S) - 1$

The preceding result allows us to recurrently apply the construction of $\mathbf{G}(\mathcal{S})$ to construct the set of all numerical semigroups with given Frobenius number or given gender

Our aim is to generalize the above results for any Frobenius variety

- $\mathcal{S}$ is a Frobenius variety
- If S is a numerical semigroup, then $\mathrm{C}(S)$ is a Frobenius variety
- For $A \subseteq \mathbb{N}$, the set

$$\mathcal{O}(A) = \{S \in \mathcal{S} \mid A \subseteq S\}$$

is a Frobenius variety. In particular, the set of oversemigroups of a numerical semigroup is a Frobenius variety

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Arf numerical semigroup

A numerical semigroup S has the *Arf property* if for any $x, y, z \in S$ with $x \geq y \geq z$, $x + y - z \in S$

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- The family of Arf numerical semigroups is a Frobenius variety

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Saturated numerical semigroups

A numerical semigroup S is *saturated* if for any $s, s_1, \ldots, s_r \in S$ with $s_1, \ldots, s_r \leq s$, and $z_1, \ldots, z_r \in \mathbb{Z}$, $z_1 s_1 + \cdots + z_r s_r \geq 0$ implies that $s + z_1 s_1 + \cdots + z_r s_r \in S$

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Pattern

A *pattern* of length n is an expression of the form $a_1x_1 + \cdots + a_nx_n$, where $a_1, \ldots, a_n$ are nonzero integers

A numerical semigroup S *admits* the pattern P if for all $s_1, \ldots, s_n \in S$, $s_1 \geq s_2 \geq \cdots \geq s_n$ implies $a_1s_1 + \cdots + a_ns_n \in S$

Denote by $\mathcal{S}(P)$ the set of numerical semigroups admitting the pattern P

Arf

$\mathcal{S}(x_1 + x_2 - x_3)$ is the set of Arf numerical semigroups

Strongly admissible patterns

A pattern P is *admissible* if $\mathcal{S}(P)$ is not empty

If $P = a_1x_1 + \cdots + a_nx_n$, define

$$P' = \begin{cases} (a-1)x_1 + a_2x_2 + \cdots + a_nx_n & \text{if } a_1 > 1 \\ a_2x_2 + \cdots + a_nx_n & \text{otherwise} \end{cases}$$

P is *strongly admissible* if both P and P' are admissible

- If P is a strongly admissible pattern, then $\mathcal{S}(P)$ is a Frobenius variety

- The intersection of Frobenius varieties is a Frobenius variety

Hence we can construct new Frobenius varieties from the above examples

Arf and Toms decomposition

The set of Arf numerical semigroups having a Toms decomposition are a Frobenius variety

We can also talk about the Frobenius variety generated by a family X of numerical semigroups
We denote this variety by $\mathcal{F}(X)$

- $\mathcal{F}(X)$ is the smallest (with respect to set inclusion) Frobenius variety containing X

Given X a nonempty family of numerical semigroups, set

$$\mathrm{C}(X) = \bigcup_{S \in X} \mathrm{C}(S)$$

Theorem

$\mathcal{F}(X)$ is the set of all finite intersections of elements in $\mathrm{C}(X)$

- A Frobenius variety is finitely generated if and only if it is finite

In what follows $\mathcal{V}$ is a Frobenius variety

$\mathcal{V}$-monoid

A submonoid M of $\mathbb{N}$ is a $\mathcal{V}$-monoid if it can be expressed as an intersection of elements in $\mathcal{V}$

- The intersection of $\mathcal{V}$-monoids is a $\mathcal{V}$-monoid

Let $A \subseteq \mathbb{N}$

$\mathcal{V}$-generating system

The $\mathcal{V}$-*monoid generated* by A is the intersection of all $\mathcal{V}$-monoids containing A

This monoid is denoted by $\mathcal{V}(A)$

A is a $\mathcal{V}$-*generating system* of $\mathcal{V}(A)$, and it is *minimal* if none of its proper subsets $\mathcal{V}$-generates $\mathcal{V}(A)$

- If $x \in \mathcal{V}(A)$, then $x \in \mathcal{V}(\{a \in A \mid a \leq x\})$

Theorem

Every $\mathcal{V}$-monoid admits a unique minimal $\mathcal{V}$-generating system

- Let M be a $\mathcal{V}$-monoid and let $x \in M$. Then $M \setminus \{x\}$ is a $\mathcal{V}$-monoid if and only if x belongs to the minimal $\mathcal{V}$-generating system of M

The tree of a Frobenius variety

Define $\mathbf{G}(\mathcal{V})$ as the graph with set of vertices $\mathcal{V}$, and (S, T) is an edge if $T = S \cup \{\mathrm{F}(S)\}$

Theorem

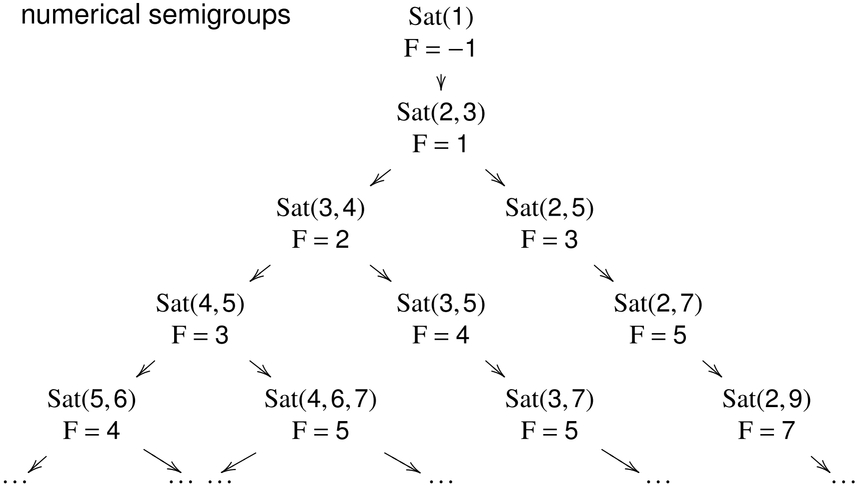
The graph $\mathbf{G}(\mathcal{V})$ is a tree rooted in $\mathbb{N}$. The sons of a vertex S are $S \setminus \{x_1\}, \ldots, S \setminus \{x_r\}$, where $x_1, \ldots, x_r$ are the elements in the minimal $\mathcal{V}$-generating system of S greater than $\mathrm{F}(S)$

The binary tree of Arf semigroups

$$
\begin{tikzcd}
 & & & \begin{array}{c}\mathbb{N} = \mathrm{Arf}(1)\\ \mathrm{F} = -1\end{array} \arrow[d] & \\
 & & & \begin{array}{c}\mathrm{Arf}(2,3)\\ \mathrm{F} = 1\end{array} \arrow[dl] \arrow[dr] & \\
 & & \begin{array}{c}\mathrm{Arf}(3,4)\\ \mathrm{F} = 2\end{array} \arrow[dl] \arrow[dr] & & \begin{array}{c}\mathrm{Arf}(2,5)\\ \mathrm{F} = 3\end{array} \arrow[d] \\
 & \begin{array}{c}\mathrm{Arf}(4,5)\\ \mathrm{F} = 3\end{array} \arrow[dl] \arrow[dr] & & \begin{array}{c}\mathrm{Arf}(3,5)\\ \mathrm{F} = 4\end{array} \arrow[d] & \begin{array}{c}\mathrm{Arf}(2,7)\\ \mathrm{F} = 5\end{array} \arrow[d] \\
\begin{array}{c}\mathrm{Arf}(5,6)\\ \mathrm{F} = 4\end{array} & & \begin{array}{c}\mathrm{Arf}(4,6,7)\\ \mathrm{F} = 5\end{array} & \begin{array}{c}\mathrm{Arf}(3,7)\\ \mathrm{F} = 5\end{array} & \begin{array}{c}\mathrm{Arf}(2,9)\\ \mathrm{F} = 7\end{array} \\
\vdots & & \vdots & \vdots & \vdots
\end{tikzcd}
$$

- If T is a son of S, then $T = S \setminus \{\mathrm{F}(S)+1\}$ or $T = S \setminus \{\mathrm{F}(S)+2\}$

The binary tree of saturated numerical semigroups



- If T is a son of S, then $T = S \setminus \{\mathrm{F}(S)+1\}$ or $T = S \setminus \{\mathrm{F}(S)+2\}$
- This tree has no leaves