30. Two inaccessible points A and B are visible from D, but no other point can be found whence both are visible. Take some point C, whence A and D can be seen, and measure CD, 200 ft.; ADC, 89°; ACD, 50° 30′. Then take some point E, whence D and B are visible, and measure DE, 200; BDE, 54° 30′; BED, 88° 30′. At D measure ADB, 72° 30′. Compute the distance AB.

Ans. 345.4 ft.

31. The angle of elevation of an inaccessible tower situated on a horizontal plane is 63° 26′; at a point 500 ft. farther from the base of the tower the elevation of its top is 32° 14′. Find the height of the tower. *Ans.* 460.5 ft.

32. To compute the horizontal distance between two inaccessible points A and B, when no point can be found whence both can be seen. Take two points C and D, distant 200 yd., so that A can be seen from C, and B from D. From C measure CF, 200 yd. to F, whence A can be seen; and from D measure DE, 200 yd. to E, whence B can be seen. Measure AFC, 83°; ACD, 53° 30′; ACF, 54° 31′; BDE, 54° 30′; BDC, 156° 25′; DEB, 88° 30′. *Ans.* 345.3 yd.

33. A tower is situated on the bank of a river. From the opposite bank the angle of elevation of the tower is 60° 13′, and from a point 40 ft. more distant the elevation is 50° 19′. Find the breadth of the river. *Ans.* 88.9 ft.

34. A ship sailing north sees two lighthouses 8 mi. apart, in a line due west; after an hour's sailing one lighthouse bears S.W. and the other S.S.W. Find the ship's rate. *Ans.* 13.6 mi. per hour.

35. A column in the north temperate zone is east-southeast of an observer, and at noon the extremity of its shadow is northeast of him. The shadow is 80 ft. in length, and the elevation of the column at the observer's station is 45°. Find the height of the column. *Ans.* 61.23 ft.

36. At a distance of 40 ft. from the foot of a tower on an inclined plane the tower subtends an angle of 41° 19′; at a point 60 ft. farther away the angle subtended by the tower is 23° 45′. Find the height of the tower. *Ans.* 56.5 ft.

37. A tower makes an angle of 113° 12′ with the inclined plane on which it stands; and at a distance of 89 ft. from its base, measured down the plane, the angle subtended by the tower is 23° 27′. Find the height of the tower.

Ans. 51.6 ft.

38. From the top of a hill the angles of depression of two objects situated in the horizontal plane of the base of the hill are 45° and 30°; and the horizontal angle between the two objects is 30°. Show that the height of the hill is equal to the distance between the objects.

39. I observe the angular elevation of the summits of two spires which appear in a straight line to be α, and the angular depressions of their reflections in still water to be β and γ. If the height of my eye above the level of the water be c, then the horizontal distance between the spires is

$$\frac{2c\cos^2\alpha\sin(\beta-\gamma)}{\sin(\beta-\alpha)\sin(\gamma-\alpha)}.$$

40. The angular elevation of a tower due south at a place A is 30°, and at a place B, due west of A and at a distance a from it, the elevation is 18°. Show that the height of the tower is $\dfrac{a}{\sqrt{2\sqrt{5}+2}}$.

41. A boy standing c ft. behind and opposite the middle of a football goal sees that the angle of elevation of the nearer crossbar is A and the angle of elevation of the farther one is B. Show that the length of the field is $c(\tan A \cot B - 1)$.

42. A valley is crossed by a horizontal bridge whose length is l. The sides of the valley make angles A and B with the horizon. Show that the height of the bridge above the bottom of the valley is $\dfrac{l}{\cot A + \cot B}$.

43. A tower is situated on a horizontal plane at a distance a from the base of a hill whose inclination is α. A person on the hill, looking over the tower, can just see a pond, the distance of which from the tower is b. Show that, if the distance of the observer from the foot of the hill be c, the height of the tower is $\dfrac{bc \sin \alpha}{a + b + c \cos \alpha}$.

44. From a point on a hillside of constant inclination the angle of elevation of the top of an obelisk on its summit is observed to be α, and a ft. nearer to the top of the hill to be β; show that if h be the height of the obelisk, the inclination of the hill to the horizon will be

$$\cos^{-1}\left\{\frac{a}{h} \cdot \frac{\sin \alpha \sin \beta}{\sin (\beta - \alpha)}\right\}.$$

CHAPTER X

RECAPITULATION OF FORMULAS

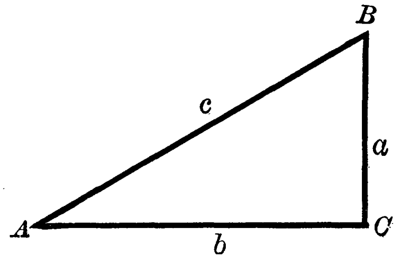
Plane Trigonometry

Right triangles, pp. 2–11.

(1) $\sin A = \frac{a}{c}\cdot$ (4) $\csc A = \frac{c}{a}\cdot$

(2) $\cos A = \frac{b}{c}\cdot$ (5) $\sec A = \frac{c}{b}\cdot$

(3) $\tan A = \frac{a}{b}\cdot$ (6) $\cot A = \frac{b}{a}\cdot$



(7) Side opposite an acute angle
= hypotenuse × sine of the angle.

(8) Side adjacent an acute angle
= hypotenuse × cosine of the angle.

(9) Side opposite an acute angle
= adjacent side × tangent of the angle.

Fundamental relations between the functions, p. 59.

(19) $\sin x = \frac{1}{\csc x}$, $\csc x = \frac{1}{\sin x}\cdot$

(20) $\cos x = \frac{1}{\sec x}$, $\sec x = \frac{1}{\cos x}\cdot$

(21) $\tan x = \frac{1}{\cot x}$, $\cot x = \frac{1}{\tan x}\cdot$

(22) $\tan x = \frac{\sin x}{\cos x}$, $\cot x = \frac{\cos x}{\sin x}\cdot$

(23) $\sin^2 x + \cos^2 x = 1.$

(24) $\sec^2 x = 1 + \tan^2 x.$ (25) $\csc^2 x = 1 + \cot^2 x.$

Functions of the sum and of the difference of two angles, pp. 63–69.

(40) $\sin(x + y) = \sin x \cos y + \cos x \sin y.$

(41) $\sin(x - y) = \sin x \cos y - \cos x \sin y.$

(42) $\cos(x + y) = \cos x \cos y - \sin x \sin y.$

(43) $\cos(x - y) = \cos x \cos y + \sin x \sin y.$

(44) $$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}.$$

(45) $$\tan(x-y)=\frac{\tan x-\tan y}{1+\tan x\tan y}.$$

(46) $$\cot(x+y)=\frac{\cot x\cot y-1}{\cot y+\cot x}.$$

(47) $$\cot(x-y)=\frac{\cot x\cot y+1}{\cot y-\cot x}.$$

Functions of twice an angle, p. 70.

(48) $$\sin 2x = 2\sin x\cos x.$$

(49) $$\cos 2x = \cos^2 x - \sin^2 x.$$

(50) $$\tan 2x = \frac{2\tan x}{1-\tan^2 x}.$$

Functions of an angle in terms of functions of half the angle, p. 72.

(51) $$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}.$$

(52) $$\cos x = \cos^2\frac{x}{2}-\sin^2\frac{x}{2}.$$

(53) $$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$$

Functions of half an angle, pp. 72–73.

(54) $$\sin\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{2}}.$$

(55) $$\cos\frac{x}{2}=\pm\sqrt{\frac{1+\cos x}{2}}.$$

(56) $$\tan\frac{x}{2}=\pm\sqrt{\frac{1-\cos x}{1+\cos x}}.$$

(57) $$\tan\frac{x}{2}=\frac{\sin x}{1+\cos x}.$$

(58) $$\tan\frac{x}{2}=\frac{1-\cos x}{\sin x}.$$

(59) $$\cot\frac{x}{2}=\pm\sqrt{\frac{1+\cos x}{1-\cos x}}.$$

(60) $$\cot\frac{x}{2}=\frac{1+\cos x}{\sin x}.$$

(61) $$\cot\frac{x}{2}=\frac{\sin x}{1-\cos x}.$$

Sums and differences of functions, p. 74.

(62) $$\sin A+\sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B).$$

(63) $$\sin A-\sin B = 2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).$$

(64) $$\cos A+\cos B = 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B).$$

(65) $$\cos A-\cos B = -2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B).$$

(66) $$\frac{\sin A+\sin B}{\sin A-\sin B}=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}.$$

Law of sines, p. 102.

(72) $$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

Law of cosines, p. 109.

(73) $$a^2 = b^2 + c^2 - 2\,bc\cos A.$$

Law of tangents, p. 112.

(79) $$\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}.$$

Functions of the half angles of a triangle in terms of the sides, pp. 113–115.

$$s = \tfrac{1}{2}(a+b+c).$$

(81) $$\sin\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

(82) $$\cos\tfrac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}.$$

(83) $$\tan\tfrac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

(84) $$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

(85) $$\tan\tfrac{1}{2}A = \frac{r}{s-a}.$$

(86) $$\tan\tfrac{1}{2}B = \frac{r}{s-b}.$$

(87) $$\tan\tfrac{1}{2}C = \frac{r}{s-c}.$$

Area of a triangle, p. 117.

(88) $$S = \tfrac{1}{2}\,bc\sin A.$$

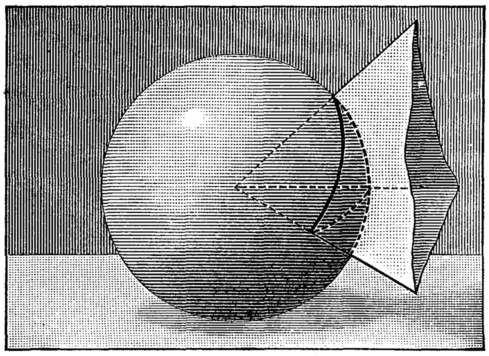
(89) $$S = \sqrt{s(s-a)(s-b)(s-c)}.$$

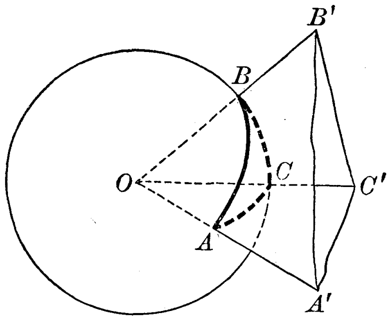
SPHERICAL TRIGONOMETRY

CHAPTER I

RIGHT SPHERICAL TRIANGLES

1. Correspondence between the face angles and the diedral angles of a triedral angle on the one hand, and the sides and angles of a spherical triangle on the other. Take any triedral angle $O\text{–}A'B'C'$ and let a sphere of any radius, as OA, be described about the vertex O as a center. The intersections of this sphere with the faces of the





triedral angle will be three arcs of great circles of the sphere, forming a spherical triangle, as ABC. The sides (arcs) AB, BC, CA of this triangle measure the face angles $A'OB'$, $B'OC'$, $C'OA'$ of the triedral angle. The angles ABC, BCA, CAB, are measured by the plane angles which also measure the diedral angles of the triedral angle; for, by Geometry, each is measured by the angle between two straight lines drawn, one in each face, perpendicular to the edge at the same point.

Spherical Trigonometry treats of the trigonometric relations between the six elements (three sides and three angles) of a spherical triangle; or, what amounts to the same thing, between the face and diedral angles of the triedral angle which intercepts it, as shown in the figure. Hence we have the

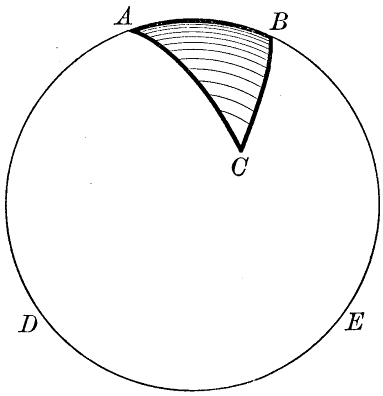
Theorem. *From any property of triedral angles an analogous property of spherical triangles can be inferred, and vice versa.*

It is evident that the face and diedral angles of the triedral angle are not altered in magnitude by varying the radius of the sphere; hence the relations between the sides and angles of a spherical triangle are independent of the length of the radius.

The sides of a spherical triangle, being arcs, are usually expressed in degrees.* The length of a side (arc) may be found in terms of any linear unit from the proportion

circumference of great circle : length of arc :: 360° : degrees in arc.

A side or an angle of a spherical triangle may have any value from 0° to 360°, but any spherical triangle can always be made to depend on a spherical triangle having each element less than 180°.



Thus, a triangle such as $ADEBC$ (unshaded portion of hemisphere in figure), which has a side $ADEB$ greater than 180°, need not be considered, for its parts can be immediately found from the parts of the triangle ABC, each of whose sides is less than 180°. For arc $ADEB = 360° -$ arc AB, angle $CAD = 180° -$ angle CAB, etc. Only triangles whose elements are less than 180° are considered in this book.

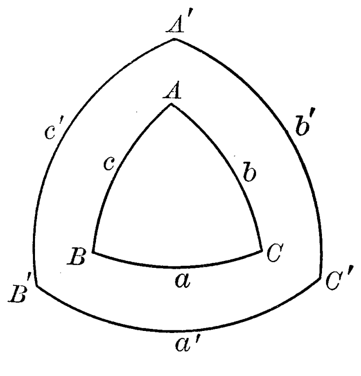
2. Properties of spherical triangles. The proofs of the following properties of spherical triangles may be found in any treatise on Spherical Geometry:

(*a*) Either side of a spherical triangle is less than the sum of the other two sides.

(*b*) If two sides of a spherical triangle are unequal, the angles opposite them are unequal, and the greater angle lies opposite the greater side, and conversely.

(*c*) The sum of the sides of a spherical triangle is less than 360°.†

(*d*) The sum of the angles of a spherical triangle is greater than 180° and less than 540°.‡



* One of the chief differences between Plane Trigonometry and Spherical Trigonometry is that in the former the *sides* of triangles are expressed in linear units, while in the latter *all* the parts are usually expressed in units of arc, i.e. degrees, etc.

† In a plane triangle the sum of the sides may have any magnitude.

‡ In a plane triangle the sum of the angles is always equal to 180°.

(*e*) If $A'B'C'$ is the polar triangle* of ABC, then, conversely, ABC is the polar triangle of $A'B'C'$.

(*f*) In two polar triangles each angle of one is the supplement of the side lying opposite to it in the other. Applying this to the last figure, we get

$$A = 180° - a', \qquad B = 180° - b', \qquad C = 180° - c',$$
$$A' = 180° - a, \qquad B' = 180° - b, \qquad C' = 180° - c.$$

A spherical triangle which has one or more right angles is called a *right spherical triangle.*

EXAMPLES

1. Find the sides of the polar triangles of the spherical triangles whose angles are as follows. Draw the figure in each case.

(a) $A = 70°$, $B = 80°$, $C = 100°$. *Ans.* $a' = 110°$, $b' = 100°$, $c' = 80°$.
(b) $A = 56°$, $B = 97°$, $C = 112°$.
(c) $A = 68° \, 14'$, $B = 52° \, 10'$, $C = 98° \, 44'$.
(d) $A = 115.6°$, $B = 89.9°$, $C = 74.2°$.

2. Find the angles of the polar triangles of the spherical triangles whose sides are as follows:

(a) $a = 94°$, $b = 52°$, $c = 100°$. *Ans.* $A' = 86°$, $B' = 128°$, $C' = 80°$.
(b) $a = 74° \, 42'$, $b = 95° \, 6'$, $c = 66° \, 25'$.
(c) $a = 106.4°$, $b = 64.3°$, $c = 51.7°$.

3. If a triangle has three right angles, show that the sides of the triangle are quadrants.

4. Show that if a triangle has two right angles, the sides opposite these angles are quadrants, and the third angle is measured by the opposite side.

5. Find the lengths of the sides of the triangles in Example 2 if the radius of the sphere is 4 ft.

3. Formulas relating to right spherical triangles. From the above Examples 3 and 4, it is evident that the only kind of right spherical triangle that requires further investigation is that which contains *only one* right angle.

In the figure shown on the next page let ABC be a right spherical triangle having only one right angle, the center of the sphere being at O. Let C be the right angle, and suppose first that each of the other elements is less than 90°, the radius of the sphere being unity.

* The *polar triangle* of any spherical triangle is constructed by describing arcs of great circles about the vertices of the original triangle as poles.

Pass an auxiliary plane through B perpendicular to OA, cutting OA at E and OC at D. Draw BE, BD, and DE. BE and DE are each perpendicular to OA;

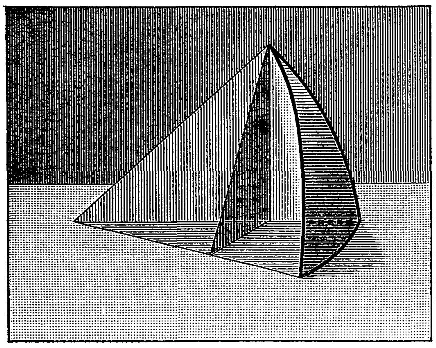
[If a straight line is ⊥ to a plane, it is ⊥ to every line in the plane.]

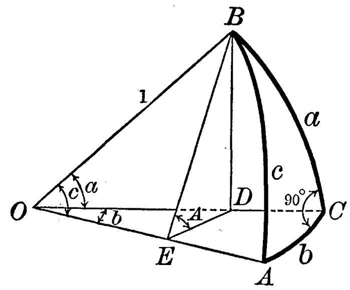
therefore angle $BED =$ angle A. The plane BDE is perpendicular to the plane AOC; [If a straight line is ⊥ to a plane, every plane passed through the line is ⊥ to the first plane.]

hence BD, which is the intersection of the planes BDE and BOC, is perpendicular to the plane AOC,

[If two intersecting planes are each ⊥ to a third plane, their intersection is also ⊥ to that plane.]

and therefore perpendicular to OC and DE.





In triangle EOD, remembering that angle $EOD = b$, we have

$$\frac{OE}{OD} = \cos b,$$

or, clearing of fractions,

$$(A) \qquad OE = OD \cdot \cos b.$$

But $\qquad OE = \cos c \ (= \cos EOB)$,

and $\qquad OD = \cos a \ (= \cos DOB)$.

Substituting in (A), we get

$$(1) \qquad \boldsymbol{\cos c = \cos a \cos b.}$$

In triangle BED, remembering that angle $BED =$ angle A, we have

$$\frac{BD}{BE} = \sin A,$$

or, clearing of fractions,

$$(B) \qquad BD = BE \cdot \sin A.$$

But $\qquad BD = \sin a \ (= \sin DOB)$,

and $\qquad BE = \sin c \ (= \sin EOB)$.

Substituting in (B), we get

(2) $$\sin a = \sin c \sin A.$$

Similarly, if we had passed the auxiliary plane through A perpendicular to OB,

(3) $$\sin b = \sin c \sin B.$$

Again, in the triangle BED,

(C) $$\cos A = \frac{DE}{BE}.$$

But $$DE = OD \sin b, \qquad \text{from } \sin b = \frac{DE}{OD}$$
$$OD = \cos a \; (= \cos DOB),$$
and $$BE = \sin c \; (= \sin EOB).$$

Substituting in (C),

(D) $$\cos A = \frac{OD \sin b}{\sin c} = \cos a \cdot \frac{\sin b}{\sin c}.$$

But from (3), $\dfrac{\sin b}{\sin c} = \sin B$. Therefore

(4) $$\cos A = \cos a \sin B.$$

Similarly, if we had passed the auxiliary plane through A perpendicular to OB,

(5) $$\cos B = \cos b \sin A.$$

The above five formulas are fundamental; that is, from them we may derive all other relations expressing any one part of a right spherical triangle in terms of two others. For example, to find a relation between A, b, c, proceed thus:

From (4), $$\cos A = \cos a \sin B$$
$$= \frac{\cos c}{\cos b} \cdot \frac{\sin b}{\sin c}$$

$\left[\text{Since } \cos a = \dfrac{\cos c}{\cos b} \text{ from (1), and } \sin B = \dfrac{\sin b}{\sin c} \text{ from (3).}\right]$

$$= \frac{\sin b}{\cos b} \cdot \frac{\cos c}{\sin c}.$$

(6) $$\therefore \cos A = \tan b \cot c.$$

Similarly, we may get

(7) $$\cos B = \tan a \cot c.$$

(8) $$\sin b = \tan a \cot A.$$

(9) $$\sin a = \tan b \cot B.$$

(10) $$\cos c = \cot A \cot B.$$

These ten formulas are sufficient for the solution of right spherical triangles. In deriving these formulas we assumed all the elements except the right angle to be less than 90°. But the formulas hold when this assumption is not made. For instance, let us suppose that a is greater that 90°. In this case the auxiliary plane BDE will cut CO and AO produced beyond the center O, and we have, in triangle EOD,

$$(E) \qquad \cos DOE \ (= \cos b) = \frac{OE}{OD}.$$

But $\qquad OE = \cos EOB = -\cos AOB = -\cos c,$

and $\qquad OD = \cos DOB = -\cos COB = -\cos a.$

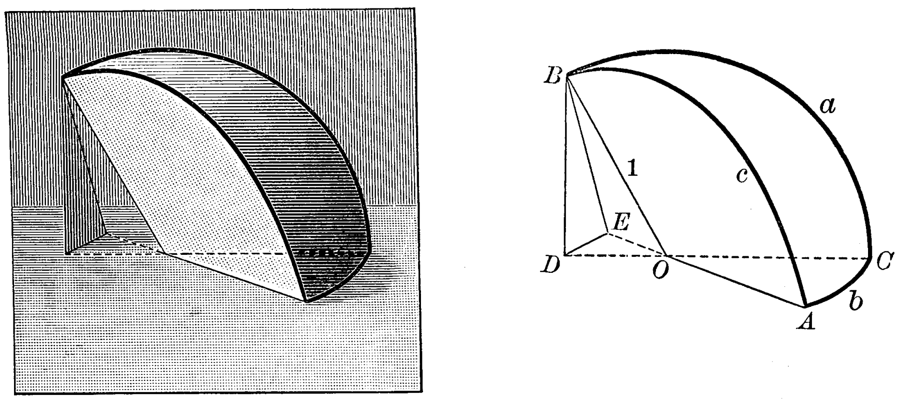
Substituting in (E), we get

$$\cos b = \frac{\cos c}{\cos a}, \text{ or } \cos c = \cos a \cos b,$$

which is the same as (**1**).

Likewise, the other formulas will hold true in this case. Similarly, they may be shown to hold true in all cases.

If the two sides including the right angle are either both less or



both greater than 90° (that is, $\cos a$ and $\cos b$ are either both positive or both negative), then the product

$$(F) \qquad \cos a \cos b$$

will always be positive, and therefore $\cos c$, from (**1**), will always be positive, that is, c will always be less than 90°. If, however, one of the sides including the right angle is less and the other is greater than 90°, the product (F), and therefore also $\cos c$, will be negative, and c will be greater than 90°.

Hence we have

Theorem I. *If the two sides including the right angle of a right spherical triangle are both less or both greater than 90°, the hypotenuse* *is less than 90°; if one side is less and the other is greater than 90°, the hypotenuse is greater than 90°.*

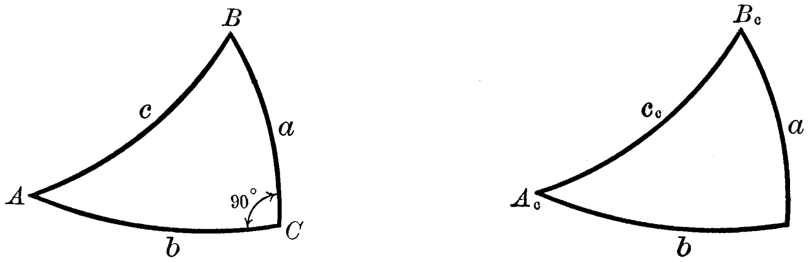
From (**4**) and (**5**), $\sin B = \dfrac{\cos A}{\cos a}$, and $\sin A = \dfrac{\cos B}{\cos b}$.

Since A and B are less than 180°, $\sin A$ and $\sin B$ must always be positive. But then $\cos A$ and $\cos a$ must have the same sign, that is, A and a are either both less than 90° or both greater than 90°. Similarly, for B and b. Hence we have

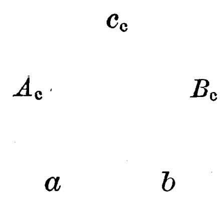
Theorem II. *In a right spherical triangle an oblique angle and the side opposite are either both less or both greater than 90°.*

4. Napier's rules of circular parts. The ten formulas derived in the last section express the relations between the three sides and the two oblique angles of a right spherical triangle. All these relations may be shown to follow from two very useful rules discovered by Baron Napier, the inventor of logarithms.

For this purpose the right angle (not entering the formulas) is not taken into account, and we replace the hypotenuse and the two



oblique angles by their respective *complements;* so that the five parts, called the *circular parts,* used in Napier's rules are a, b, A_c, c_c, B_c. The subscript c indicates that the complement is to be used. The first figure illustrates the ordinary method of representing a right spherical triangle. To emphasize the circular parts employed in Napier's rules, the same triangle might be represented as shown in the second figure. It is not necessary, however, to draw the triangle at all when using Napier's rules; in fact, it is found to be more convenient to simply write down the five parts in their proper order as on the circumference of a circle, as shown in the third figure (hence the name *circular parts*).



Any one of these parts may be called a *middle* part; then the two parts immediately adjacent to it are called *adjacent* parts, and the other two *opposite* parts. Thus, if a is taken as a middle part, A_c and b are the adjacent parts, while c_c and B_c are the opposite parts.

Napier's rules of circular parts.

Rule I. *The sine of any middle part is equal to the product of the tangents of the adjacent parts.*

Rule II. *The sine of any middle part is equal to the product of the cosines of the opposite parts.*

These rules are easily remembered if we associate the first one with the expression "**tan-adj.**" and the second one with "**cos-opp.**" *

Napier's rules may be easily verified by applying them in turn to each one of the five circular parts taken as a middle part, and comparing the results with (**1**) to (**10**).

For example, let c_c be taken as a middle part; then A_c and B_c are the adjacent parts, while a and b are the opposite parts.

	c_c	
A_c		B_c
a		b

Then, by Rule I, $\sin c_c = \tan A_c \tan B_c,$

or, $\cos c = \cot A \cot B;$

which agrees with (**10**), p. 197.

By Rule II, $\sin c_c = \cos a \cos b,$

or, $\cos c = \cos a \cos b;$

which agrees with (**1**), p. 196.

The student should verify Napier's rules in this manner by taking each one of the other four circular parts as the middle part.

Writers on Trigonometry differ as to the practical value of Napier's rules, but it is generally conceded that they are a great aid to the memory in applying formulas (**1**) to (**10**) to the solution of right spherical triangles, and we shall so employ them.

5. Solution of right spherical triangles. To solve a right spherical triangle, two elements (parts) must be given in addition to the right angle. For the sake of uniformity we shall continue to denote the right angle in a spherical triangle ABC by the letter C.

General directions for solving right spherical triangles.

	c_c			c_c			c_c			$\underline{c_c}$	
A_c		B_c	$\underline{A_c}$		B	$\underline{A_c}$		B_c	A_c		$\underline{B_c}$
b		a	$\underline{b}$		$\underline{a}$	$\underline{b}$		$\underline{a}$	$\underline{b}$		a

First step. *Write down the five circular parts as in first figure.*

Second step. *Underline the two given parts and the required unknown part. Thus, if A_c and a are given to find b, we underline all three as is shown in the second figure.*

* Or by noting that a is the first vowel in the words "tangent" and "adjacent," while o is the first vowel in the words "cosine" and "opposite."

Third step. *Pick out the middle part (in this case b) and cross the line under it as indicated in the third figure.*

Fourth step. *Use Rule I if the other two parts are adjacent to the middle part (as in case illustrated), or Rule II if they are opposite, and solve for the unknown part.*

Check: *Check with that rule which involves the three required parts.**

Careful attention must be paid to the algebraic signs of the functions when solving spherical triangles; the cosines, tangents, and cotangents of angles or arcs greater than 90° being negative. When computing with logarithms we shall write (n) after the logarithms when the functions are negative. If the number of negative factors is even, the result will be positive; if it is odd, the result will be negative and (n) should be written after the resulting logarithm. In order to be able to show our computations in compact form, we shall write down all the logarithms of the trigonometric functions just as they are given in our table; that is, when a logarithm has a negative characteristic we will not write down -10 after it.†

Ex. 1. Solve the right spherical triangle, having given $B = 33° 50'$, $a = 108°$.

Solution. Follow the above general directions.

To find A	*To find b*	*To find c*
c_c	c_c	$\underline{c_c}$
$\not{\underline{A_c}}$ $\underline{B_c}$	A_c $\underline{B_c}$	A_c $\not{\underline{B_c}}$
b $\underline{a}$	$\underline{b}$ $\not{\underline{a}}$	b $\underline{a}$
Using Rule II	Using Rule I	Using Rule I
$\sin A_c = \cos B_c \cos a$	$\sin a = \tan B_c \tan b$	$\sin B_c = \tan c_c \tan a$
$\cos A = \sin B \cos a$	$\tan b = \sin a \tan B$	$\cot c = \cos B \cot a$
$\log \sin B = 9.7457$	$\log \sin a = 9.9782$	$\log \cos B = 9.9194$
$\log \cos a = 9.4900\ (n)$	$\log \tan B = 9.8263$	$\log \cot a = 9.5118\ (n)$
$\log \cos A = 9.2357\ (n)$	$\log \tan b = 9.8045$	$\log \cot c = 9.4312\ (n)$
$\therefore 180° - A‡ = 80° 6'$	$\therefore b = 32° 31'$.	$\therefore 180° - c = 74° 54'$
and $A = 99° 54'$.		and $c = 105° 6'$.

The value of $\log \cos A$ found is the same as that found in our first computation. The student should observe that in checking our work in this example

* Thus, in above case, A_c and a are given; therefore we underline the three required parts and cross b as the middle part. Applying Rule II, c_c and B_c being opposite parts, we get $\sin b = \cos c_c \cos B_c$, or, $\sin b = \sin c \sin B$.

† For example, as in the table, we will write $\log \sin 24° = 9.6093$. To be exact, this should be written $\log \sin 24° = 9.6093 - 10$, or, $\log \sin 24° = \bar{1}.6093$.

‡ Since $\cos A$ is negative, we get the supplement of A from the table.

it was not necessary to look up any new logarithms. Hence the check in this case is only on the correctness of the logarithmic work.*

$\underline{c_c}$	*Check:* Using Rule I
A_c B_c	$\sin A_c = \tan b \tan c_c$
	$\cos A = \tan b \cot c$
$\underline{b}$ a	$\log \tan b = 9.8045$
	$\log \cot c = \underline{9.4312}$ (n)
	$\log \cos A = 9.2357$ (n)

In logarithmic computations the student should always write down an outline or skeleton of the computation before using his logarithmic table at all. In the last example this outline would be as follows:

$\log \sin B =$	$\log \sin a =$	$\log \cos B =$
$\log \cos a = \underline{\qquad}$ (n)	$\log \tan B = \underline{\qquad}$	$\log \cot a = \underline{\qquad}$ (n)
$\log \cos A =$ (n)	$\log \tan b =$	$\log \cot c =$ (n)
$\therefore\ 180° - A =$	$\therefore\ b =$	$\therefore\ 180° - c =$
and $A =$		and $c =$

It saves time to look up all the logarithms at once, and besides it reduces the liability of error to thus separate the theoretical part of the work from that which is purely mechanical. Students should be drilled in writing down forms like that given above before attempting to solve examples.

Ex. 2. Solve the right spherical triangle, having given $c = 70° 30'$, $A = 100°$.

Solution. Follow the general directions.

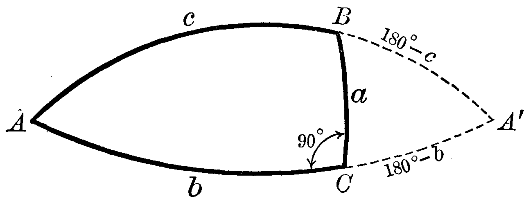
To find a	*To find b*	*To find B*
$\underline{c_c}$	$\underline{c_c}$	c_c
$\underline{A_c}$ B_c	A_c B_c	$\underline{A_c}$ $\underline{B_c}$
b a	$\underline{b}$ a	b a
Using Rule II	Using Rule I	Using Rule I
$\sin a = \cos c_c \cos A_c$	$\sin A_c = \tan b \tan c_c$	$\sin c_c = \tan A_c \tan B_c$
$\sin a = \sin c \sin A$	$\tan b = \cos A \tan c$	$\cot B = \cos c \tan A$
$\log \sin c = 9.9743$	$\log \cos A = 9.2397$ (n)	$\log \cos c = 9.5235$
$\log \sin A = \underline{9.9934}$	$\log \tan c = \underline{0.4509}$	$\log \tan A = \underline{0.7537}$ (n)
$\log \sin a = 9.9677$	$\log \tan b = 9.6906$ (n)	$\log \cot B = 0.2772$ (n)
$\therefore\ 180° - a† = 68° 10'$	$\therefore\ 180° - b = 26° 8'$	$\therefore\ 180° - B = 27° 51'$
and $a = 111° 50'$.	and $b = 153° 52'$.	and $B = 152° 9'$.

The work of verifying the results is left to the student.

* In order to be sure that the angles and sides have been correctly taken from the tables, in such an example as this, we should use them together with some of the given data in relations not already employed.

† Since a is determined from its sine, it is evident that it may have the value $68° 10'$ found from the table, or the supplementary value $111° 50'$. Since $A > 90°$, however, we know from Th. II, p. 199, that $a > 90°$; hence $a = 111° 50'$ is the only solution.

6. The ambiguous case. Two solutions. When the given parts of a right spherical triangle are an oblique angle and its opposite side, there are two triangles which satisfy the given conditions. For, in the triangle ABC, let $C = 90°$, and let A and CB $(= a)$ be the given parts. If we extend AB and AC to A',



it is evident that the triangle $A'BC$ also satisfies the given conditions, since $BCA' = 90°$, $A' = A$, and $BC = a$. The remaining parts in $A'BC$ are supplementary to the respective remaining parts in ABC. Thus

$$A'B = 180° - c, \qquad A'C = 180° - b, \qquad A'BC = 180° - ABC.$$

This ambiguity also appears in the solution of the triangle, as is illustrated in the following example:

Ex. 3. Solve the right spherical triangle, having given $A = 105°\ 59'$, $a = 128°\ 33'$.

Solution. We proceed as in the previous examples.

To find b	*To find B*	*To find c*
C_c	C_c	C_c
A_c B_c	A_c B_c	A_c B_c
b a	b a	b a
$\sin b = \tan a \tan A_c$	$\sin A_c = \cos a \cos B_c$	$\sin a = \cos A_c \cos c_c$
$\sin b = \tan a \cot A$	$\sin B = \dfrac{\cos A}{\cos a}$	$\sin c = \dfrac{\sin a}{\sin A}$
$\log \tan a = 0.0986\ (n)$	$\log \cos A = 9.4399\ (n)$	$\log \sin a = 9.8932$
$\log \cot A = 9.4570\ (n)$	$\log \cos a = 9.7946\ (n)$	$\log \sin A = 9.9828$
$\log \sin B = 9.5556$	$\log \sin B = 9.6453$	$\log \sin c = 9.9104$
$\therefore\ b = 21°\ 4'$, or,	$\therefore\ B = 26°\ 14'$, or,	$\therefore\ c' = 54°\ 27'$, or,
$180° - b = 158°\ 56' = b'$.*	$180° - B = 153°\ 46' = B'$.†	$180° - c' = 125°\ 33' = c$.‡

Hence the two solutions are:

1. $b = 21°\ 4'$,	$c = 125°\ 33'$,	$B = 26°\ 14'$ (triangle ABC);
2. $b' = 158°\ 56'$,	$c' = 54°\ 27'$,	$B' = 153°\ 46'$ (triangle $A'BC$).

It is not necessary to check both solutions. We leave this to the student.

* Since $\sin B$ is positive and B is not known, we cannot remove the ambiguity. Hence both the acute angle taken from the table and its supplement must be retained.

† The two values of B must be retained, since b has two values which are supplementary.

‡ Since $a > 90°$ and b has two values, one $>$ and the other $< 90°$, it follows from Th. I, p. 198, that c will have two values, the first one $< 90°$ and the second $> 90°$.

EXAMPLES

Solve the following right spherical triangles:

No.	Given Parts		Required Parts		
1	$a = 132° 6'$	$b = 77° 51'$	$A = 131° 27'$	$B = 80° 55'$	$c = 98° 7'$
2	$a = 159°$	$c = 137° 20'$	$A = 148° 5'$	$B = 65° 23'$	$b = 38° 1'$
3	$A = 50° 20'$	$B = 122° 40'$	$a = 40° 42'$	$b = 134° 31'$	$c = 122° 7'$
4	$a = 160°$	$b = 38° 30'$	$A = 149° 41'$	$B = 66° 44'$	$c = 137° 20'$
5	$B = 80°$	$b = 67° 40'$	$A = 27° 12'$	$a = 25° 25'$	$c = 69° 54'$; or,
			$A' = 152° 48'$	$a' = 154° 35'$	$c = 110° 6'$
6	$B = 112°$	$c = 81° 50'$	$A = 109° 23'$	$a = 110° 58'$	$b = 113° 22'$
7	$a = 61°$	$B = 123° 40'$	$A = 66° 12'$	$b = 127° 17'$	$c = 107° 5'$
8	$a = 61° 40'$	$b = 144° 10'$	$A = 72° 29'$	$B = 140° 38'$	$c = 112° 38'$
9	$A = 99° 50'$	$a = 112°$	$B = 27° 7'$	$b = 25° 24'$	$c = 109° 46'$; or,
			$B' = 152° 53'$	$b' = 154° 36'$	$c' = 70° 14'$
10	$b = 15°$	$c = 152° 20'$	$A = 120° 44'$	$a = 156° 30'$	$B = 33° 53'$
11	$A = 62° 59'$	$B = 37° 4'$	$a = 41° 6'$	$b = 26° 25'$	$c = 47° 32'$
12	$A = 73° 7'$	$c = 114° 32'$	$a = 60° 31'$	$B = 143° 50'$	$b = 147° 32'$
13	$B = 144° 54'$	$b = 146° 32'$	$A = 78° 47'$	$a = 70° 10'$	$c = 106° 28'$; or,
			$A' = 101° 13'$	$a' = 109° 50'$	$c' = 73° 32'$
14	$B = 68° 18'$	$c = 47° 34'$	$A = 30° 32'$	$a = 22° 1'$	$b = 43° 18'$
15	$A = 161° 52'$	$b = 131° 8'$	$a = 166° 9'$	$B = 101° 49'$	$c = 50° 18'$
16	$a = 113° 25'$	$b = 110° 47'$	$A = 112° 3'$	$B = 109° 12'$	$c = 81° 54'$
17	$a = 137° 9'$	$B = 74° 51'$	$A = 135° 3'$	$b = 68° 17'$	$c = 105° 44'$
18	$A = 144° 54'$	$B = 101° 14'$	$a = 146° 33'$	$b = 109° 48'$	$c = 73° 35'$
19	$a = 69° 18'$	$c = 84° 27'$	$A = 70°$	$B = 75° 6'$	$b = 74° 7'$

20. For more examples take any two parts in the above triangles and solve for the other three.

7. Solution of isosceles and quadrantal triangles. Plane isosceles triangles were solved by dividing each one into two equal right triangles and then solving one of the right triangles. Similarly, we may solve an *isosceles spherical triangle* by dividing it into two symmetrical (equal) right spherical triangles by an arc drawn from the vertex perpendicular to the base, and then solving one of the right spherical triangles.

A *quadrantal triangle* is a spherical triangle one side of which is a quadrant (= 90°). By (*f*), p. 195, the polar triangle of a quadrantal triangle is a right triangle. Therefore, to solve a quadrantal triangle we have only to solve its polar triangle and take the *supplements* of the parts obtained by the calculation.

Ex. 1. Solve the triangle, having given $c = 90°$, $a = 67° 38'$, $b = 48° 50'$.

Solution. This is a quadrantal triangle since one side $c = 90°$. We then find the corresponding elements of its polar triangle by (*f*), p. 195. They are $C' = 90°$, $A' = 112° 22'$, $B' = 131° 10'$. We solve this right triangle in the usual way.

Construct the polar (right) triangle.
Given $A' = 112° 22'$, $B' = 131° 10'$:

To find a'

c'_c

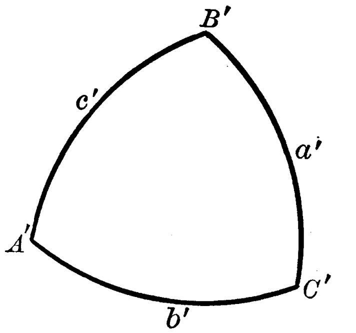
A'_c B'_c

b' a'

$$\sin A'_c = \cos a' \cos B'_c$$

$$\cos a' = \frac{\cos A'}{\sin B'}.$$

$\log \cos A' = 9.5804\ (n)$
$\log \sin B' = 9.8767$
$\log \cos a' = 9.7037\ (n)$
$180° - a' = 59° 38'.$
$a' = 120° 22'.$



Similarly, we get

$b' = 135° 23'$, $c' = 68° 55'$.

Hence in the given quadrantal triangle we have

$$A = 180° - a' = 59° 38',$$
$$B = 180° - b' = 44° 37',$$
$$C = 180° - c' = 111° 5'.$$

EXAMPLES

Solve the following quadrantal triangles:

No.	Given Parts			Required Parts		
1	$A = 139°$	$b = 143°$	$c = 90°$	$a = 117° 1'$	$B = 153° 42'$	$C = 132° 34'$
2	$A = 45° 30'$	$B = 139° 20'$	$c = 90°$	$a = 57° 22'$	$b = 129° 42'$	$C = 57° 53'$
3	$a = 30° 20'$	$C = 42° 40'$	$c = 90°$	$A = 20° 1'$	$B = 141° 30'$	$b = 113° 17'$
4	$B = 70° 12'$	$C = 106° 25'$	$c = 90°$	$A = 33° 28'$	$a = 35° 4'$	$b = 78° 47'$
5	$A = 105° 53'$	$a = 104° 54'$	$c = 90°$	$B = 69° 16'$	$b = 70°$	$C = 84° 30'$; or
				$B = 110° 44'$	$b = 110°$	$C = 95° 30'$

Solve the following isosceles spherical triangles:

No.	Given Parts			Required Parts		
6	$a = 54° 20'$	$c = 72° 54'$	$A = B$	$b = 54° 20'$	$A = B = 57° 59'$	$C = 93° 59'$
7	$a = 54° 30'$	$C = 71°$	$A = B$	$b = 54° 30'$	$A = B = 90°$	$c = 180°$
8	$a = 66° 29'$	$A = B = 50° 17'$		$b = 66° 29'$	$c = 111° 30'$	$C = 128° 42'$
9	$c = 156° 40'$	$C = 187° 46'$	$A = B$	$a = b = 3° 58'$ or $176° 2'$		
				$A = B = 89° 12'$ or $90° 48'$		

Prove the following relations between the elements of a right spherical triangle ($C = 90°$):

10. $\cos^2 A \sin^2 c = \sin(c + a)\sin(c - a)$.
11. $\tan a \cos c = \sin b \cot B$.
12. $\sin^2 A = \cos^2 B + \sin^2 a \sin^2 B$.
13. $\sin(b + c) = 2 \cos^2 \frac{1}{2} A \cos b \sin c$.
14. $\sin(c - b) = 2 \sin^2 \frac{1}{2} A \cos b \sin c$.

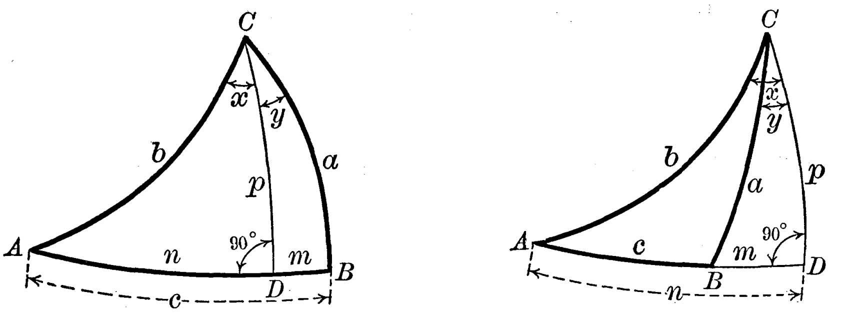
CHAPTER II

OBLIQUE SPHERICAL TRIANGLES

8. Fundamental formulas. In this chapter some relations between the sides and angles of any spherical triangle (whether right angled or oblique) will be derived.

9. Law of sines. *In a spherical triangle the sines of the sides are proportional to the sines of the opposite angles.*

Proof. Let ABC be any spherical triangle, and draw the arc CD perpendicular to AB. There will be two cases according as CD falls



upon AB (first figure) or upon AB produced (second figure). For the sake of brevity let $CD = p$, $AD = n$, $BD = m$, angle $ACD = x$, angle $BCD = y$.

In the right triangle ADC (either figure)

$$(A) \qquad \sin p = \sin b \sin A. \qquad \text{Rule II, p. 200}$$

In the right triangle BCD (first figure)

$$(B) \qquad \sin p = \sin a \sin B. \qquad \text{Rule II, p. 200}$$

This also holds true in the second figure, for

$$\sin DBC = \sin(180° - B) = \sin B.$$

Equating the values of $\sin p$ from (A) and (B),

$$\sin a \sin B = \sin b \sin A,$$

or, dividing through by $\sin A \sin B$,

$$(C) \qquad \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B}.$$

In like manner, by drawing perpendiculars from A and B, we get

$$(D) \qquad \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}, \text{ and}$$

$$(E) \qquad \frac{\sin c}{\sin C} = \frac{\sin a}{\sin A}, \text{ respectively.}$$

Writing (C), (D), (E) as a single statement, we get the **law of sines,**

$$(11) \qquad \frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.*$$

10. Law of cosines. *In a spherical triangle the cosine of any side is equal to the product of the cosines of the other two sides plus the product of the sines of these two sides and the cosine of their included angle.*

Proof. Using the same figures as in the last section, we have in the right triangle BDC,

$$\cos a = \cos p \cos m \qquad \text{Rule II, p. 200}$$
$$= \cos p \cos (c - n)$$
$$= \cos p \{\cos c \cos n + \sin c \sin n\}$$
$$(A) \qquad = \cos p \cos c \cos n + \cos p \sin c \sin n.$$

In the right triangle ADC,

$$(B) \qquad \cos p \cos n = \cos b.$$

Whence $$\cos p = \frac{\cos b}{\cos n},$$

and, multiplying both sides by $\sin n$,

$$(C) \qquad \cos p \sin n = \frac{\cos b}{\cos n} \cdot \sin n = \cos b \tan n.$$

But $$\cos A = \tan n \cot b, \text{ or,} \qquad \text{Rule I, p. 200}$$

$$(D) \qquad \tan n = \tan b \cos A.$$

Substituting value of $\tan n$ from (D) in (C), we have

$$(E) \qquad \cos p \sin n = \cos b \tan b \cos A = \sin b \cos A.$$

Substituting the value of $\cos p \cos n$ from (B) and the value of $\cos p \sin n$ from (E) in (A), we get the law of cosines,

$$(F) \qquad \cos a = \cos b \cos c + \sin b \sin c \cos A.$$

* Compare with the law of sines in Granville's *Plane Trigonometry*, p. 102.

Similarly, for the sides b and c we may obtain

$$(G) \qquad \cos b = \cos c \cos a + \sin c \sin a \cos B,$$

$$(H) \qquad \cos c = \cos a \cos b + \sin a \sin b \cos C.$$

11. Principle of Duality. Given any relation involving one or more of the sides a, b, c, and the angles A, B, C of any general spherical triangle. Now the polar triangle (whose sides are denoted by a', b', c', and angles by A', B', C') is also in this case a general spherical triangle, and the given relation must hold true for it also; that is, the given relation applies to the polar triangle if accents are placed upon the letters representing the sides and angles. Thus (F), (G), (H) of the last section give us the following law of cosines for the polar triangle:

$$(A) \qquad \cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

$$(B) \qquad \cos b' = \cos c' \cos a' + \sin c' \sin a' \cos B'.$$

$$(C) \qquad \cos c' = \cos a' \cos b' + \sin a' \sin b' \cos C'.$$

But by (f), p. 195,

$$a' = 180° - A, \qquad b' = 180° - B, \qquad c' = 180° - C,$$

$$A' = 180° - a, \qquad B' = 180° - b, \qquad C' = 180° - c.$$

Making these substitutions in (A), (B), (C), which refer to the polar triangle, we get

$$(D) \quad \cos(180° - A) = \cos(180° - B)\cos(180° - C) + \sin(180° - B)\sin(180° - C)\cos(180° - a),$$

$$(E) \quad \cos(180° - B) = \cos(180° - C)\cos(180° - A) + \sin(180° - C)\sin(180° - A)\cos(180° - b),$$

$$(F) \quad \cos(180° - C) = \cos(180° - A)\cos(180° - B) + \sin(180° - A)\sin(180° - B)\cos(180° - c),$$

which involve the sides and angles of the original triangle.

The result of the preceding discussion may then be stated in the following form:

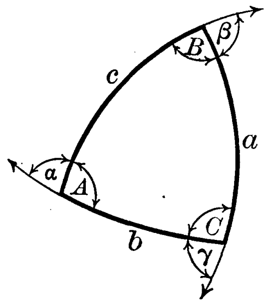
Theorem. *In any relation between the parts of a general spherical triangle, each part may be replaced by the supplement of the opposite part, and the relation thus obtained will hold true.*

The Principle of Duality follows when the above theorem is applied to a relation involving one or more of the sides and the supplements of the angles (instead of the angles themselves).

Let the supplements of the angles of the triangle be denoted by α, β, γ*; that is,

$$\alpha = 180° - A, \quad \beta = 180° - B, \quad \gamma = 180° - C,$$

or, $A = 180° - \alpha, \quad B = 180° - \beta, \quad C = 180° - \gamma.$



When we apply the above theorem to a relation between the sides and supplements of the angles of a triangle, we, in fact,

replace a by α $(= 180° - A)$,
replace b by β $(= 180° - B)$,
replace c by γ $(= 180° - C)$,
replace α $(= 180° - A)$ by $180° - (180° - a) = a$,
replace β $(= 180° - B)$ by $180° - (180° - b) = b$,
replace γ $(= 180° - C)$ by $180° - (180° - c) = c$,

or, what amounts to the same thing, *interchange the Greek and Roman letters.* For instance, substitute

$$A = 180° - \alpha, \qquad B = 180° - \beta, \qquad C = 180° - \gamma$$

in (F), (G), (H) of the last section. This gives the law of cosines for the sides in the new form

(12) $$\cos a = \cos b \cos c - \sin b \sin c \cos \alpha,$$

(13) $$\cos b = \cos c \cos a - \sin c \sin a \cos \beta,$$

(14) $$\cos c = \cos a \cos b - \sin a \sin b \cos \gamma.$$

[Since $\cos A = \cos(180° - \alpha) = -\cos\alpha$, etc.]

If we now apply the above theorem to these formulas, we get the law of cosines for the angles, namely,

(15) $$\cos \alpha = \cos \beta \cos \gamma - \sin \beta \sin \gamma \cos a,$$

(16) $$\cos \beta = \cos \gamma \cos \alpha - \sin \gamma \sin \alpha \cos b,$$

(17) $$\cos \gamma = \cos \alpha \cos \beta - \sin \alpha \sin \beta \cos c,$$

* α, β, γ are then the exterior angles of the triangle, as shown in the figure.

that is, we have derived three new relations between the sides and supplements of the angles of the triangle.* We may now state the

Principle of Duality. *If the sides of a general spherical triangle are denoted by the Roman letters a, b, c, and the supplements of the corresponding opposite angles by the Greek letters α, β, γ, then, from any given formula involving any of these six parts, we may write down a* **dual** *formula by simply interchanging the corresponding Greek and Roman letters.*

The immediate consequence of this principle is that formulas in Spherical Trigonometry occur in *pairs*, either one of a pair being the *dual* of the other.

Thus (**12**) and (**15**) are dual formulas; also (**13**) and (**16**), or (**14**) and (**17**).

If we substitute

$$A = 180^\circ - \alpha, \qquad B = 180^\circ - \beta, \qquad C = 180^\circ - \gamma$$

in the law of sines (p. 207), we get

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.$$

[Since $\sin A = \sin(180^\circ - \alpha) = \sin \alpha$, etc.]

Applying the Principle of Duality to this relation, we get

$$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c},$$

which is essentially the same as the previous form.

The forms of the law of cosines that we have derived involve algebraic sums. As these are not convenient for logarithmic calculations, we will reduce them to the form of products.

12. Trigonometric functions of half the supplements of the angles of a spherical triangle in terms of its sides. Denote half the sum of the sides of a triangle (i.e. half the perimeter) by s. Then

$$(A) \qquad 2s = a + b + c,$$

or,

$$s = \tfrac{1}{2}(a + b + c).$$

* If we had employed the interior angles of the triangle in our formulas (as has been the almost universal practice of writers on Spherical Trigonometry), the two sets of cosine formulas would not have been of the same form. That the method used here has many advantages will become more and more apparent as the reading of the text is continued. Not only are the resulting formulas much easier to memorize, but much labor is saved in that, when we have derived one set of formulas for the angles (or sides), the corresponding set of formulas for the sides (or angles) may be written down at once by mere inspection by applying this Principle of Duality. The great advantage of using this Principle of Duality was first pointed out by Möbius (1790–1868).

Subtracting $2c$ from both sides of (A),

$$2s - 2c = a + b + c - 2c, \text{ or,}$$

(B) $$s - c = \tfrac{1}{2}(a + b - c).$$

Similarly,

(C) $$s - b = \tfrac{1}{2}(a - b + c), \text{ and}$$

(D) $$s - a = \tfrac{1}{2}(-a + b + c) = \tfrac{1}{2}(b + c - a).$$

From Plane Trigonometry,

(E) $$2\sin^2\tfrac{1}{2}\alpha = 1 - \cos\alpha,$$

(F) $$2\cos^2\tfrac{1}{2}\alpha = 1 + \cos\alpha.$$

But from (**12**), p. 209, solving for $\cos\alpha$,

$$\cos\alpha = \frac{\cos b \cos c - \cos a}{\sin b \sin c};$$

hence (E) becomes

$$\begin{aligned} 2\sin^2\tfrac{1}{2}\alpha &= 1 - \frac{\cos b \cos c - \cos a}{\sin b \sin c} \\ &= \frac{\sin b \sin c - \cos b \cos c + \cos a}{\sin b \sin c} \\ &= \frac{\cos a - (\cos b \cos c - \sin b \sin c)}{\sin b \sin c} \\ &= \frac{\cos a - \cos(b + c)}{\sin b \sin c} \\ &= \frac{-2\sin\frac{1}{2}(a+b+c)\sin\frac{1}{2}(a-b-c)}{\sin b \sin c},^{*} \text{ or,} \end{aligned}$$

(G) $$2\sin^2\tfrac{1}{2}\alpha = \frac{2\sin\frac{1}{2}(a + b + c)\sin\frac{1}{2}(b + c - a)}{\sin b \sin c}.$$

[Since $\sin\frac{1}{2}(a - b - c) = -\sin\frac{1}{2}(-a + b + c) = -\sin\frac{1}{2}(b + c - a)$.]

Substituting from (A) and (D) in (G), we get

$$\sin^2\tfrac{1}{2}\alpha = \frac{\sin s \sin(s - a)}{\sin b \sin c}, \text{ or,}$$

(18) $$\boldsymbol{\sin\tfrac{1}{2}\alpha = \sqrt{\frac{\sin s \sin(s - a)}{\sin b \sin c}}}.$$

* Let

$$\begin{aligned} A &= a \\ B &= b + c \\ A + B &= a + b + c \\ \tfrac{1}{2}(A + B) &= \tfrac{1}{2}(a + b + c). \end{aligned} \qquad \begin{aligned} A &= a \\ B &= b + c \\ A - B &= a - b - c \\ \tfrac{1}{2}(A - B) &= \tfrac{1}{2}(a - b - c). \end{aligned}$$

Hence, substituting in (**65**), p. 74, Granville's *Plane Trigonometry*, namely,

$$\cos A - \cos B = -2\sin\tfrac{1}{2}(A + B)\sin\tfrac{1}{2}(A - B),$$

we get $$\cos a - \cos(b + c) = -2\sin\tfrac{1}{2}(a + b + c)\sin\tfrac{1}{2}(a - b - c).$$

Similarly, (F) becomes

$$2\cos^2\tfrac{1}{2}a = 1 + \frac{\cos b\cos c - \cos a}{\sin b\sin c}$$

$$= \frac{\sin b\sin c + \cos b\cos c - \cos a}{\sin b\sin c}$$

$$= \frac{\cos(b-c) - \cos a}{\sin b\sin c}$$

$$= \frac{-2\sin\tfrac{1}{2}(a+b-c)\sin\tfrac{1}{2}(b-c-a)}{\sin b\sin c},\text{* or,}$$

$$(H)\qquad 2\cos^2\tfrac{1}{2}a = \frac{2\sin\tfrac{1}{2}(a+b-c)\sin\tfrac{1}{2}(a-b+c)}{\sin b\sin c}.$$

[Since $\sin\frac{1}{2}(b-c-a) = -\sin\frac{1}{2}(-b+c+a) = -\sin\frac{1}{2}(a-b+c)$.]

Substituting from (B) and (C) in (H), we get

$$\cos^2\tfrac{1}{2}a = \frac{\sin(s-c)\sin(s-b)}{\sin b\sin c},\text{ or,}$$

$$\textbf{(19)}\qquad \cos\tfrac{1}{2}a = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}.$$

Since $\tan\frac{1}{2}a = \dfrac{\sin\frac{1}{2}a}{\cos\frac{1}{2}a}$, we get from this, by substitution from **(18)** and **(19)**,

$$\textbf{(20)}\qquad \tan\tfrac{1}{2}a = \sqrt{\frac{\sin s\sin(s-a)}{\sin(s-b)\sin(s-c)}}.\dagger$$

* Let

$$\begin{aligned} A &= b-c \\ B &= a \\ A+B &= a+b-c \\ \tfrac{1}{2}(A+B) &= \tfrac{1}{2}(a+b-c). \end{aligned} \qquad\qquad \begin{aligned} A &= b-c \\ B &= a \\ A-B &= b-c-a \\ \tfrac{1}{2}(A-B) &= \tfrac{1}{2}(b-c-a). \end{aligned}$$

Hence, substituting in formula (**65**), found on p. 74, Granville's *Plane Trigonometry*, namely,

$$\cos A - \cos B = -2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B),$$

we get

$$\cos(b-c) - \cos a = -2\sin\tfrac{1}{2}(a+b-c)\sin\tfrac{1}{2}(b-c-a).$$

† In memorizing these formulas it will be found an aid to the memory to note the fact that under each radical

(*a*) only the sine function occurs.

(*b*) The denominators of the sine and cosine formulas involve those two sides of the triangle which are not opposite to the angle sought.

(*c*) When reading the numerator and denominator of the fraction in the tangent formula, s comes first and then the differences

$$s-a,\quad s-b,\quad s-c,$$

in cyclical order; s and the first difference occurring also in the numerator of the corresponding sine formula, while the last two differences occur in the numerator of the corresponding cosine formula.

In like manner, we may get

(21) $$\sin\tfrac{1}{2}\beta = \sqrt{\frac{\sin s \sin(s-b)}{\sin c \sin a}},$$

(22) $$\cos\tfrac{1}{2}\beta = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin c \sin a}},$$

(23) $$\tan\tfrac{1}{2}\beta = \sqrt{\frac{\sin s \sin(s-b)}{\sin(s-c)\sin(s-a)}}.$$

Also

(24) $$\sin\tfrac{1}{2}\gamma = \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}},$$

(25) $$\cos\tfrac{1}{2}\gamma = \sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin a \sin b}},$$

(26) $$\tan\tfrac{1}{2}\gamma = \sqrt{\frac{\sin s \sin(s-c)}{\sin(s-a)\sin(s-b)}}.$$

In solving triangles it is sometimes more convenient to use other forms of (**20**), (**23**), (**26**). Thus, in the right-hand member of (**20**), multiply both the numerator and denominator of the fraction under the radical by $\sin(s-a)$. This gives

$$\tan\tfrac{1}{2}\alpha = \sqrt{\frac{\sin s \sin^2(s-a)}{\sin(s-a)\sin(s-b)\sin(s-c)}}$$

$$= \sin(s-a)\sqrt{\frac{\sin s}{\sin(s-a)\sin(s-b)\sin(s-c)}}.$$

Let $$\tan\tfrac{1}{2}d^{*} = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

then $$\tan\tfrac{1}{2}\alpha = \frac{\sin(s-a)}{\tan\tfrac{1}{2}d}.$$

Similarly, for $\tan\frac{1}{2}\beta$ and $\tan\frac{1}{2}\gamma$. Hence

(27) $$\tan\tfrac{1}{2}d = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

(28) $$\tan\tfrac{1}{2}\alpha = \frac{\sin(s-a)}{\tan\tfrac{1}{2}d},$$

(29) $$\tan\tfrac{1}{2}\beta = \frac{\sin(s-b)}{\tan\tfrac{1}{2}d},$$

(30) $$\tan\tfrac{1}{2}\gamma = \frac{\sin(s-c)}{\tan\tfrac{1}{2}d}.$$

* It may be shown that $d =$ diameter of the circle inscribed in the spherical triangle.

13. Trigonometric functions of the half sides of a spherical triangle in terms of the supplements of the angles. By making use of the Principle of Duality on p. 208, we get at once from formulas (**18**) to (**30**), by replacing the supplement of an angle by the opposite side and each side by the supplement of the opposite angle, the following formulas:

$$\sin\tfrac{1}{2}a = \sqrt{\frac{\sin\sigma\sin(\sigma-\alpha)}{\sin\beta\sin\gamma}}, \quad (31)$$

$$\cos\tfrac{1}{2}a = \sqrt{\frac{\sin(\sigma-\beta)\sin(\sigma-\gamma)}{\sin\beta\sin\gamma}}, \quad (32)$$

$$\tan\tfrac{1}{2}a = \sqrt{\frac{\sin\sigma\sin(\sigma-\alpha)}{\sin(\sigma-\beta)\sin(\sigma-\gamma)}}, \quad (33)$$

$$\sin\tfrac{1}{2}b = \sqrt{\frac{\sin\sigma\sin(\sigma-\beta)}{\sin\gamma\sin\alpha}}, \quad (34)$$

$$\cos\tfrac{1}{2}b = \sqrt{\frac{\sin(\sigma-\gamma)\sin(\sigma-\alpha)}{\sin\gamma\sin\alpha}}, \quad (35)$$

$$\tan\tfrac{1}{2}b = \sqrt{\frac{\sin\sigma\sin(\sigma-\beta)}{\sin(\sigma-\gamma)\sin(\sigma-\alpha)}}, \quad (36)$$

$$\sin\tfrac{1}{2}c = \sqrt{\frac{\sin\sigma\sin(\sigma-\gamma)}{\sin\alpha\sin\beta}}, \quad (37)$$

$$\cos\tfrac{1}{2}c = \sqrt{\frac{\sin(\sigma-\alpha)\sin(\sigma-\beta)}{\sin\alpha\sin\beta}}, \quad (38)$$

$$\tan\tfrac{1}{2}c = \sqrt{\frac{\sin\sigma\sin(\sigma-\gamma)}{\sin(\sigma-\alpha)\sin(\sigma-\beta)}}, \quad (39)$$

$$\tan\tfrac{1}{2}\delta^* = \sqrt{\frac{\sin(\sigma-\alpha)\sin(\sigma-\beta)\sin(\sigma-\gamma)}{\sin\sigma}}, \quad (40)$$

$$\tan\tfrac{1}{2}a = \frac{\sin(\sigma-\alpha)}{\tan\frac{1}{2}\delta}, \quad (41)$$

$$\tan\tfrac{1}{2}b = \frac{\sin(\sigma-\beta)}{\tan\frac{1}{2}\delta}, \quad (42)$$

$$\tan\tfrac{1}{2}c = \frac{\sin(\sigma-\gamma)}{\tan\frac{1}{2}\delta}, \quad (43)$$

where

$$\begin{aligned}\sigma &= \tfrac{1}{2}(\alpha+\beta+\lambda)\\ &= \tfrac{1}{2}(180^\circ - A + 180^\circ - B + 180^\circ - C)\\ &= 270^\circ - \tfrac{1}{2}(A+B+C).\end{aligned}$$

What we have done amounts to interchanging the corresponding Greek and Roman letters.

* It may be shown that δ is the supplement of the diameter of the circumscribed circle.

14. Napier's analogies. Dividing (**20**) by (**23**), we get

$$\frac{\tan\frac{1}{2}\alpha}{\tan\frac{1}{2}\beta}=\sqrt{\frac{\sin s\sin(s-a)}{\sin(s-b)\sin(s-c)}}\div\sqrt{\frac{\sin s\sin(s-b)}{\sin(s-c)\sin(s-a)}},$$

or,

$$\frac{\frac{\sin\frac{1}{2}\alpha}{\cos\frac{1}{2}\alpha}}{\frac{\sin\frac{1}{2}\beta}{\cos\frac{1}{2}\beta}}=\sqrt{\frac{\frac{\sin s\sin(s-a)}{\sin(s-b)\sin(s-c)}}{\frac{\sin s\sin(s-b)}{\sin(s-c)\sin(s-a)}}}.$$

Hence $$\frac{\sin\frac{1}{2}\alpha\cos\frac{1}{2}\beta}{\cos\frac{1}{2}\alpha\sin\frac{1}{2}\beta}=\frac{\sin(s-a)}{\sin(s-b)}.$$

By composition and division, in proportion,

$$\frac{\sin\frac{1}{2}\alpha\cos\frac{1}{2}\beta+\cos\frac{1}{2}\alpha\sin\frac{1}{2}\beta}{\sin\frac{1}{2}\alpha\cos\frac{1}{2}\beta-\cos\frac{1}{2}\alpha\sin\frac{1}{2}\beta}=\frac{\sin(s-a)+\sin(s-b)}{\sin(s-a)-\sin(s-b)}.$$

From (**40**), (**41**), p. 63, and (**66**), p. 74, Granville's *Plane Trigonometry*, the left-hand member equals

$$\frac{\sin(\frac{1}{2}\alpha+\frac{1}{2}\beta)}{\sin(\frac{1}{2}\alpha-\frac{1}{2}\beta)},$$

and the right-hand member

$$\frac{\sin(s-a)+\sin(s-b)}{\sin(s-a)-\sin(s-b)}=\frac{\tan\frac{1}{2}[s-a+(s-b)]}{\tan\frac{1}{2}[s-a-(s-b)]}=\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(b-a)}.^*$$

Equating these results, we get, noting that $\tan\frac{1}{2}(b-a)=-\tan\frac{1}{2}(a-b)$,

$$\frac{\sin\frac{1}{2}(\alpha+\beta)}{\sin\frac{1}{2}(\alpha-\beta)}=-\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a-b)},\text{ or,}$$

(**44**) $$\tan\tfrac{1}{2}(a-b)=-\frac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)}\tan\tfrac{1}{2}c.$$

In the same manner we may get the two similar formulas for $\tan\frac{1}{2}(b-c)$ and $\tan\frac{1}{2}(c-a)$.

Multiplying (**20**) and (**23**), we get

$$\tan\tfrac{1}{2}\alpha\tan\tfrac{1}{2}\beta=\sqrt{\frac{\sin s\sin(s-a)}{\sin(s-b)\sin(s-c)}}\sqrt{\frac{\sin s\sin(s-b)}{\sin(s-c)\sin(s-a)}},$$

or, $$\frac{\sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta}{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta}=\frac{\sin s}{\sin(s-c)}.$$

By composition and division, in proportion,

$$\frac{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta-\sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta}{\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta+\sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta}=\frac{\sin(s-c)-\sin s}{\sin(s-c)+\sin s}.$$

* For $s-a+s-b=2s-a-b=a+b+c-a-b=c$, and $s-a-s+b=b-a$.

From (**42**), (**43**), p. 63, and (**66**), p. 74, Granville's *Plane Trigonometry*, the left-hand member equals

$$\frac{\cos\left(\frac{1}{2}\alpha + \frac{1}{2}\beta\right)}{\cos\left(\frac{1}{2}\alpha - \frac{1}{2}\beta\right)};$$

and the right-hand member

$$\frac{\sin(s-c) - \sin s}{\sin(s-c) + \sin s} = \frac{\tan\frac{1}{2}(s-c-s)}{\tan\frac{1}{2}(s-c+s)} = \frac{\tan\frac{1}{2}(-c)}{\tan\frac{1}{2}(a+b)}.^*$$

Equating these results, we get, noting that $\tan\frac{1}{2}(-c) = -\tan\frac{1}{2}c$,

$$\frac{\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)} = -\frac{\tan\frac{1}{2}c}{\tan\frac{1}{2}(a+b)}, \text{ or,}$$

$$\textbf{(45)} \qquad \tan\tfrac{1}{2}(a+b) = -\frac{\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)}\tan\tfrac{1}{2}c.$$

In the same manner we may get the two similar formulas for $\tan\frac{1}{2}(b+c)$ and $\tan\frac{1}{2}(c+a)$.

By making use of the Principle of Duality on p. 208, we get at once from formulas (**44**) and (**45**),

$$\textbf{(46)} \qquad \tan\tfrac{1}{2}(\alpha-\beta) = -\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\tan\tfrac{1}{2}\gamma,$$

$$\textbf{(47)} \qquad \tan\tfrac{1}{2}(\alpha+\beta) = -\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\tan\tfrac{1}{2}\gamma.$$

By changing the letters in cyclic order we may at once write down the corresponding formulas for $\tan\frac{1}{2}(\beta-\gamma)$, $\tan\frac{1}{2}(\gamma-\alpha)$, $\tan\frac{1}{2}(\beta+\gamma)$, and $\tan\frac{1}{2}(\gamma+\alpha)$.

The relations derived in this section are known as *Napier's analogies*.

Since $\cos\frac{1}{2}(a-b)$ and $\tan\frac{1}{2}\gamma = \tan\frac{1}{2}(180° - C) = \tan(90° - \frac{1}{2}C) = \cot\frac{1}{2}C$ are always positive, it follows from (**47**) that $\cos\frac{1}{2}(a+b)$ and $\tan\frac{1}{2}(\alpha+\beta)$ always have opposite signs; or, since $\tan\frac{1}{2}(\alpha+\beta) = \tan\frac{1}{2}(180° - A + 180° - B) = \tan\frac{1}{2}[360° - (A+B)] = \tan[180° - \frac{1}{2}(A+B)] = -\tan\frac{1}{2}(A+B)$, we may say that $\cos\frac{1}{2}(a+b)$ and $\tan\frac{1}{2}(A+B)$ always have the same sign. Hence we have the

Theorem. *In a spherical triangle the sum of any two sides is less than, greater than, or equal to 180°, according as the sum of the corresponding opposite angles is less than, greater than, or equal to 180°.*

15. Solution of oblique spherical triangles. We shall now take up the numerical solution of oblique spherical triangles. There are three cases to consider with two subdivisions under each case.

* For $\qquad s - c - s = -c,$
and $\qquad s - c + s = 2s - c = a + b + c - c = a + b.$

CASE I. (*a*) *Given the three sides.*
(*b*) *Given the three angles.*
CASE II. (*a*) *Given two sides and their included angle.*
(*b*) *Given two angles and their included side.*
CASE III. (*a*) *Given two sides and the angle opposite one of them.*
(*b*) *Given two angles and the side opposite one of them.*

16. Case I. (*a*) **Given the three sides.** *Use formulas from* *p. 213*, *namely,*

(27) $$\tan \tfrac{1}{2} d = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

(28) $$\tan \tfrac{1}{2} \alpha = \frac{\sin(s-a)}{\tan \frac{1}{2} d},$$

(29) $$\tan \tfrac{1}{2} \beta = \frac{\sin(s-b)}{\tan \frac{1}{2} d},$$

(30) $$\tan \tfrac{1}{2} \gamma = \frac{\sin(s-c)}{\tan \frac{1}{2} d},$$

to find α, β, γ, *and therefore* A, B, C, *and check by the law of sines,*

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

Ex. 1. Given $a = 60°$, $b = 137° 20'$, $c = 116°$; find A, B, C.

Solution.

$a = 60°$
$b = 137° 20'$
$c = 116°$
$2s = 313° 20'$
$s = 156° 40'$.
$s - a = 96° 40'$.
$s - b = 19° 20'$.
$s - c = 40° 40'$.

To find $\log \tan \frac{1}{2} d$ *use* (27)

$\log \sin(s-a) = 9.9971$
$\log \sin(s-b) = 9.5199$
$\log \sin(s-c) = 9.8140$
29.3310
$\log \sin s = 9.5978$
$2 \,|\, 19.7332$
$\log \tan \frac{1}{2} d = 9.8666$

To find A use (28)

$\log \sin(s-a) = 9.9971$
$\log \tan \frac{1}{2} d = 9.8666$
$\log \tan \frac{1}{2} \alpha = 0.1305$
$\frac{1}{2}\alpha = 53° 29'$.
$\alpha = 106° 58'$.
$\therefore A = 180° - 106° 58' = 73° 2'$.

To find B use (29)

$\log \sin(s-b) = 9.5199$
$\log \tan \frac{1}{2} d = 9.8666$
$\log \tan \frac{1}{2} \beta = 9.6533$
$\frac{1}{2}\beta = 24° 14'$.
$\beta = 48° 28'$.
$\therefore B = 180° - 48° 28' = 131° 32'$.

To find C use (30)

$\log \sin(s-c) = 9.8140$
$\log \tan \frac{1}{2} d = 9.8666$
$\log \tan \frac{1}{2} \gamma = 9.9474$
$\frac{1}{2}\gamma = 41° 32'$.
$\gamma = 83° 4'$.
$\therefore C = 180° - 83° 4' = 96° 56'$.

Check:

$\log \sin a = 9.9375$	$\log \sin b = 9.8311$	$\log \sin c = 9.9537$
$\log \sin A = 9.9807$	$\log \sin B = 9.8743$	$\log \sin C = 9.9969$
9.9568	9.9568	9.9568

This checks up closer than is to be expected in general. There may be a variation of at most two units in the last figure when the work is accurate.

EXAMPLES

Solve the following oblique spherical triangles :

No.	Given Parts			Required Parts		
1	$a=38°$	$b=51°$	$c=42°$	$A=51°\,58'$	$B=83°\,54'$	$C=58°\,53'$
2	$a=101°$	$b=49°$	$c=60°$	$A=142°\,32'$	$B=27°\,52'$	$C=32°\,28'$
3	$a=61°$	$b=39°$	$c=92°$	$A=35°\,32'$	$B=24°\,42'$	$C=138°\,24'$
4	$a=62°\,20'$	$b=54°\,10'$	$c=97°\,50'$	$A=47°\,22'$	$B=42°\,20'$	$C=124°\,38'$
5	$a=58°$	$b=80°$	$c=96°$	$A=55°\,58'$	$B=74°\,14'$	$C=103°\,36'$
6	$a=46°\,30'$	$b=62°\,40'$	$c=83°\,20'$	$A=43°\,58'$	$B=58°\,14'$	$C=108°\,6'$
7	$a=71°\,15'$	$b=39°\,10'$	$c=40°\,35'$	$A=130°\,36'$	$B=30°\,26'$	$C=31°\,26'$
8	$a=47°\,30'$	$b=55°\,40'$	$c=60°\,10'$	$A=56°\,32'$	$B=69°\,7'$	$C=78°\,58'$
9	$a=43°\,30'$	$b=72°\,24'$	$c=87°\,50'$	$A=41°\,27'$	$B=66°\,26'$	$C=106°\,3'$
10	$a=110°\,40'$	$b=45°\,10'$	$c=73°\,30'$	$A=144°\,27'$	$B=26°\,9'$	$C=36°\,35'$

17. Case I. (*b*) Given the three angles. *Use formulas from p. 214, namely,**

(40) $$\tan\tfrac{1}{2}\delta=\sqrt{\frac{\sin(\sigma-\alpha)\sin(\sigma-\beta)\sin(\sigma-\gamma)}{\sin\sigma}},$$

(41) $$\tan\tfrac{1}{2}a=\frac{\sin(\sigma-\alpha)}{\tan\frac{1}{2}\delta},$$

(42) $$\tan\tfrac{1}{2}b=\frac{\sin(\sigma-\beta)}{\tan\frac{1}{2}\delta},$$

(43) $$\tan\tfrac{1}{2}c=\frac{\sin(\sigma-\gamma)}{\tan\frac{1}{2}\delta},$$

to find a, b, c; and check by the law of sines,

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}.$$

Ex. 1. Given $A=70°$, $B=131°\,10'$, $C=94°\,50'$; find a, b, c.

Solution. Here we use the supplements of the angles.

$$\begin{aligned}\alpha&=180°-A=110°\\ \beta&=180°-B=\ \ 48°\,50'\\ \gamma&=180°-C=\ \ 85°\,10'\\ 2\sigma&=244°\\ \sigma&=122°.\\ \sigma-\alpha&=\ \ 12°.\\ \sigma-\beta&=\ \ 73°\,10'.\\ \sigma-\gamma&=\ \ 36°\,50'.\end{aligned}$$

To find log tan $\frac{1}{2}\delta$ use **(40)**

$$\begin{aligned}\log\sin(\sigma-\alpha)&=\ \ 9.3179\\ \log\sin(\sigma-\beta)&=\ \ 9.9810\\ \log\sin(\sigma-\gamma)&=\ \ 9.7778\\ &\ \ \ 29.0767\\ \log\sin\sigma&=\ \ 9.9284\\ &2\,|\,19.1483\\ \log\tan\tfrac{1}{2}\delta&=\ \ 9.5742\end{aligned}$$

* These formulas may be written down at once from those used in Case I, (*a*), p. 217, by simply interchanging the corresponding Greek and Roman letters.

To find a use (**41**)	*To find b use* (**42**)	*To find c use* (**43**)
$\log\sin(\sigma-\alpha) = 9.3179$	$\log\sin(\sigma-\beta) = 9.9810$	$\log\sin(\sigma-\gamma) = 9.7778$
$\log\tan\frac{1}{2}\delta = 9.5742$	$\log\tan\frac{1}{2}\delta = 9.5742$	$\log\tan\frac{1}{2}\delta = 9.5742$
$\log\tan\frac{1}{2}a = 9.7437$	$\log\tan\frac{1}{2}b = 0.4068$	$\log\tan\frac{1}{2}c = 0.2036$
$\frac{1}{2}a = 29°.$	$\frac{1}{2}b = 68°\,36'.$	$\frac{1}{2}c = 57°\,58'.$
$\therefore\ a = 58°.$	$b = 137°\,12'.$	$c = 115°\,56'.$

Check: $\log\sin a = 9.9284$ $\quad\log\sin b = 9.8321$ $\quad\log\sin c = 9.9539$

$\log\sin A = 9.9730$ $\quad\log\sin B = 9.8767$ $\quad\log\sin C = 9.9985$

9.9554 $\quad 9.9554$ $\quad 9.9554$

EXAMPLES

Solve the following oblique spherical triangles:

No.	Given Parts			Required Parts		
1	$A=75°$	$B=82°$	$C=61°$	$a=67°\,52'$	$b=71°\,44'$	$c=57°$
2	$A=120°$	$B=130°$	$C=80°$	$a=144°\,10'$	$b=148°\,49'$	$c=41°\,44'$
3	$A=91°\,10'$	$B=85°\,40'$	$C=72°\,30'$	$a=89°\,51'$	$b=85°\,49'$	$c=72°\,32'$
4	$A=138°\,16'$	$B=31°\,11'$	$C=35°\,53'$	$a=100°\,5'$	$b=49°\,59'$	$c=60°\,6'$
5	$A=78°\,40'$	$B=63°\,50'$	$C=46°\,20'$	$a=39°\,30'$	$b=35°\,36'$	$c=27°\,59'$
6	$A=121°$	$B=102°$	$C=68°$	$a=130°\,50'$	$b=120°\,18'$	$c=54°\,56'$
7	$A=130°$	$B=110°$	$C=80°$	$a=139°\,21'$	$b=126°\,58'$	$c=56°\,52'$
8	$A=28°$	$B=92°$	$C=85°\,26'$	$a=27°\,56'$	$b=85°\,40'$	$c=84°\,2'$
9	$A=59°\,18'$	$B=108°$	$C=76°\,22'$	$a=61°\,44'$	$b=103°\,4'$	$c=84°\,32'$
10	$A=100°$	$B=100°$	$C=50°$	$a=112°\,14'$	$b=112°\,14'$	$c=46°\,4'$

18. Case II. (*a*) **Given two sides and their included angle, as *a*, *b*, *C*.** *Use formulas on p. 216, namely,*

$$\textbf{(46)}\qquad \tan\tfrac{1}{2}(\alpha-\beta) = -\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\tan\tfrac{1}{2}\gamma,$$

$$\textbf{(47)}\qquad \tan\tfrac{1}{2}(\alpha+\beta) = -\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\tan\tfrac{1}{2}\gamma,$$

to find α and β and therefore A and B; and from p. 215 use (**44**) *solved for* $\tan\frac{1}{2}c$, *namely,*

$$\textbf{(44)}\qquad \tan\tfrac{1}{2}c = -\frac{\sin\frac{1}{2}(\alpha+\beta)\tan\frac{1}{2}(a-b)}{\sin\frac{1}{2}(\alpha-\beta)},$$

to find c. Check by the law of sines.

Ex. 1. Given $a = 64°\,24'$, $b = 42°\,30'$, $C = 58°\,40'$; find A, B, c.

Solution. $\gamma = 180° - C = 121°\,20'$. $\therefore\ \frac{1}{2}\gamma = 60°\,40'$.

$$\begin{aligned} a &= 64°\,24' \\ b &= 42°\,30' \\ a+b &= 106°\,54' \\ \therefore\ \tfrac{1}{2}(a+b) &= 53°\,27'. \end{aligned}\qquad\qquad \begin{aligned} a &= 64°\,24' \\ b &= 42°\,30' \\ a-b &= 21°\,54' \\ \therefore\ \tfrac{1}{2}(a-b) &= 10°\,57'. \end{aligned}$$

To find $\frac{1}{2}(\alpha - \beta)$ *use* **(46)**

$$\begin{aligned}
\log\sin\tfrac{1}{2}(a-b) &= 9.2786\\
\log\tan\tfrac{1}{2}\gamma &= \underline{0.2503}\\
&\ 9.5289\\
\log\sin\tfrac{1}{2}(a+b) &= \underline{9.9049}\\
\log\tan\tfrac{1}{2}(\alpha-\beta) &= 9.6240\ (n)\\
\therefore\ \tfrac{1}{2}(\alpha-\beta) &= -\,22^\circ\,49'.^*
\end{aligned}$$

To find $\frac{1}{2}(\alpha + \beta)$ *use* **(47)**

$$\begin{aligned}
\log\cos\tfrac{1}{2}(a-b) &= 9.9920\\
\log\tan\tfrac{1}{2}\gamma &= \underline{0.2503}\\
&\ 10.2423\\
\log\cos\tfrac{1}{2}(a+b) &= \underline{9.7749}\\
\log\tan\tfrac{1}{2}(\alpha+\beta) &= 0.4674\ (n)\\
180^\circ - \tfrac{1}{2}(\alpha+\beta) &= 71^\circ\,11'.^\dagger\\
\therefore\ \tfrac{1}{2}(\alpha+\beta) &= 108^\circ\,49'.
\end{aligned}$$

To find A and B

$$\begin{aligned}
\tfrac{1}{2}(\alpha+\beta) &= 108^\circ\,49'\\
\tfrac{1}{2}(\alpha-\beta) &= -\,\underline{22^\circ\,49'}\\
\text{Adding,}\qquad \alpha &= 86^\circ\\
\text{Subtracting,}\qquad \beta &= 131^\circ\,38'.\\
\therefore\ A &= 180^\circ - \alpha = 94^\circ.\\
B &= 180^\circ - \beta = 48^\circ\,22'.
\end{aligned}$$

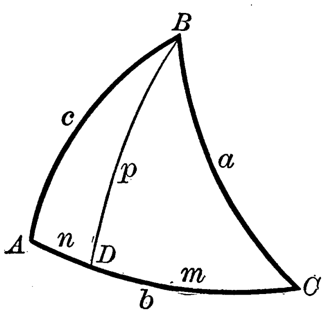
To find c use **(44)**

$$\begin{aligned}
\log\sin\tfrac{1}{2}(\alpha+\beta) &= 9.9761\\
\log\tan\tfrac{1}{2}(a-b) &= \underline{9.2867}\\
&\ 19.2628\\
\log\sin\tfrac{1}{2}(\alpha-\beta) &= \underline{9.5886}\ (n)\\
\log\tan\tfrac{1}{2}c &= 9.6742\ ^\ddagger\\
\tfrac{1}{2}c &= 25^\circ\,17'.\\
\therefore\ c &= 50^\circ\,34'.
\end{aligned}$$

Check:

$\log\sin a = 9.9551$	$\log\sin b = 9.8297$	$\log\sin c = 9.8878$
$\log\sin A = \underline{9.9989}$	$\log\sin B = \underline{9.8735}$	$\log\sin C = \underline{9.9315}$
9.9562	9.9562	9.9563

If c only is wanted, we may find it from the law of cosines, **(14)**, p. 209, without previously determining A and B. But this formula is not well adapted to logarithmic calculations. Another method is illustrated below, which depends on the solution of right spherical triangles, and hence requires only those formulas which follow from applying *Napier's rules of circular parts*, p. 200.



Through B draw an arc of a great circle perpendicular to AC, intersecting AC at D. Let

$$BD = p, \qquad CD = m, \qquad AD = n.$$

Applying Rule I, p. 200, to the right spherical triangle BCD, we have

$$\cos C = \tan m \cot a, \text{ or,}$$

$$(A)\qquad \tan m = \tan a \cos C.$$

Applying Rule II, p. 200, to BCD,

$$\cos a = \cos m \cos p, \text{ or,}$$

$$(B)\qquad \cos p = \cos a \sec m.$$

* Since $\tan\frac{1}{2}(\alpha-\beta)$ is negative, $\frac{1}{2}(\alpha-\beta)$ may be an angle in the second or fourth quadrant. But $a > b$, therefore $A > B$ and $\alpha < \beta$, since α and β are the supplements of A and B. Hence $\frac{1}{2}(\alpha-\beta)$ must be a negative angle numerically less than 90°.

† Here $\frac{1}{2}(\alpha+\beta)$ must be a positive angle less than 180°. Since $\tan\frac{1}{2}(\alpha+\beta)$ is negative, $\frac{1}{2}(\alpha+\beta)$ must lie in the second quadrant, and we get its supplement from the table.

‡ $\tan\frac{1}{2}c$ is positive, since $\sin\frac{1}{2}(\alpha-\beta)$ is negative and there is a minus sign before the fraction.

Applying the same rule to ABD,

$$\cos c = \cos n \cos p, \text{ or,}$$

$$(C) \qquad \cos p = \cos c \sec n.$$

Equating (B) and (C),

$$\cos c \sec n = \cos a \sec m, \text{ or,}$$

$$\cos c = \cos a \sec m \cos n.$$

But $n = b - m$; therefore

$$(D) \qquad \cos c = \cos a \sec m \cos (b - m).$$

Now c may be computed from (A) and (D), namely,

$$\textbf{(48)} \qquad \tan m = \tan a \cos C,$$

$$\textbf{(49)} \qquad \cos c = \frac{\cos a \cos (b - m)}{\cos m}.$$

Ex. 2. Given $a = 98°$, $b = 80°$, $C = 110°$; find c.

Solution. Apply the method just explained.

To find $b - m$ use **(48)**

$\log \tan a = 0.8522$ (n)
$\log \cos C = 9.5341$ (n)
$\log \tan m = 0.3863$
$m = 67° 40'$.
$\therefore b - m = 12° 20'$.

To find c use **(49)**

$\log \cos a = 9.1436$ (n)
$\log \cos (b - m) = 9.9899$
19.1335
$\log \cos m = 9.5798$
$\log \cos c = 9.5537$ (n)
$180° - c = 69° 2'$.
$c = 110° 58'$.

EXAMPLES

Solve the following oblique spherical triangles

No.	Given Parts			Required Parts		
1	$a = 137° 20'$	$c = 116°$	$A = 70°$	$B = 131° 17'$	$C = 94° 48'$	$a = 57° 57'$
2	$a = 72°$	$b = 47°$	$C = 33°$	$A = 121° 33'$	$B = 40° 57'$	$c = 37° 26'$
3	$a = 98°$	$c = 60°$	$B = 110°$	$A = 87°$	$C = 60° 51'$	$b = 111° 17'$
4	$b = 120° 20'$	$c = 70° 40'$	$A = 50°$	$B = 134° 57'$	$C = 50° 41'$	$a = 69° 9'$
5	$a = 125° 10'$	$b = 153° 50'$	$C = 140° 20'$	$A = 147° 29'$	$B = 163° 9'$	$c = 76° 8'$
6	$a = 93° 20'$	$b = 56° 30'$	$C = 74° 40'$	$A = 101° 24'$	$B = 54° 58'$	$c = 79° 10'$
7	$b = 76° 30'$	$c = 47° 20'$	$A = 92° 30'$	$B = 78° 21'$	$C = 47° 47'$	$a = 82° 42'$
8	$c = 40° 20'$	$a = 100° 30'$	$B = 46° 40'$	$A = 131° 29'$	$C = 29° 33'$	$b = 72° 40'$
9	$b = 76° 36'$	$c = 110° 26'$	$A = 46° 50'$	$B = 57° 43'$	$C = 125° 28'$	$a = 57° 13'$
10	$a = 84° 23'$	$b = 124° 48'$	$C = 62°$	$A = 68° 27'$	$B = 129° 51'$	$c = 70° 52'$

19. Case II. (*b*) **Given two angles and their included side, as** A, B, c. *Use formulas* on pp. 215, 216, namely,*

$$\text{(44)} \qquad \tan \tfrac{1}{2}(a-b) = -\frac{\sin \frac{1}{2}(\alpha-\beta)}{\sin \frac{1}{2}(\alpha+\beta)} \tan \tfrac{1}{2} c,$$

$$\text{(45)} \qquad \tan \tfrac{1}{2}(a+b) = -\frac{\cos \frac{1}{2}(\alpha-\beta)}{\cos \frac{1}{2}(\alpha+\beta)} \tan \tfrac{1}{2} c,$$

to find a and b; and from p. 216, use (**46**) *solved for* $\tan \frac{1}{2}\gamma$, *namely,*

$$\text{(46)} \qquad \tan \tfrac{1}{2}\gamma = -\frac{\sin \frac{1}{2}(a+b) \tan \frac{1}{2}(\alpha-\beta)}{\sin \frac{1}{2}(a-b)},$$

to find γ *and therefore C. Check by the law of sines.*

Ex. 1. Given $c = 116°$, $A = 70°$, $B = 131° 20'$; find a, b, C.

Solution. $\alpha = 180° - A = 110°$, and $\beta = 180° - B = 48° 40'$.

$\alpha = 110°$	$\alpha = 110°$	
$\beta = 48° 40'$	$\beta = 48° 40'$	
$\alpha + \beta = 158° 40'$	$\alpha - \beta = 61° 20'$	$c = 116°$.
$\therefore \frac{1}{2}(\alpha+\beta) = 79° 20'$.	$\therefore \frac{1}{2}(\alpha-\beta) = 30° 40'$.	$\therefore \frac{1}{2}c = 58°$.

To find $\frac{1}{2}(a-b)$ *use* (**44**)

$\log \sin \frac{1}{2}(\alpha-\beta) = 9.7076$
$\log \tan \frac{1}{2} c = 0.2042$
9.9118
$\log \sin \frac{1}{2}(\alpha+\beta) = 9.9924$
$\log \tan \frac{1}{2}(a-b) = 9.9194\ (n)$
$\therefore \frac{1}{2}(a-b) = -39° 43'$.†

To find $\frac{1}{2}(a+b)$ *use* (**45**)

$\log \cos \frac{1}{2}(\alpha-\beta) = 9.9346$
$\log \tan \frac{1}{2} c = 0.2042$
10.1388
$\log \cos \frac{1}{2}(\alpha+\beta) = 9.2674$
$\log \tan \frac{1}{2}(a+b) = 0.8714\ (n)$
$180° - \frac{1}{2}(a+b) = 82° 21'$.
$\therefore \frac{1}{2}(a+b) = 97° 39'$.

To find a and b

$\frac{1}{2}(a+b) = 97° 39'$
$\frac{1}{2}(a-b) = -39° 43'$
Adding, $a = 57° 56'$
Subtracting, $b = 137° 22'$.

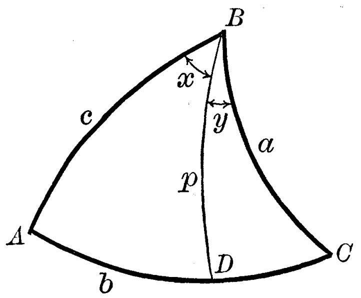
To find C use (**46**)

$\log \sin \frac{1}{2}(a+b) = 9.9961$
$\log \tan \frac{1}{2}(\alpha-\beta) = 9.7730$
19.7691
$\log \sin \frac{1}{2}(a-b) = 9.8055\ (n)$
$\log \tan \frac{1}{2}\gamma = 9.9636$
$\frac{1}{2}\gamma = 42° 36'$.
$\gamma = 85° 12'$.
$\therefore C = 180° - \gamma = 94° 48'$.

Check:		
$\log \sin a = 9.9281$	$\log \sin b = 9.8308$	$\log \sin c = 9.9537$
$\log \sin A = 9.9730$	$\log \sin B = 9.8756$	$\log \sin C = 9.9985$
9.9551	9.9552	9.9552

* Same as those used in Case II, (*a*), p. 219, with Greek and Roman letters interchanged.
† Since $A < B$ it follows that $a < b$, and $\frac{1}{2}(a-b)$ is negative.

If C only is wanted, we can calculate it without previously determining a and b, by dividing the given triangle into two right spherical triangles, as was illustrated on p. 220.



Through B draw an arc of a great circle perpendicular to AC, intersecting AC at D. Let $BD = p$, angle $ABD = x$, angle $CBD = y$. Applying Rule I of Napier's rules, p. 200, to the right spherical triangle ABD, we have

$$\cos c = \cot x \cot A, \text{ or,}$$

$$(A) \qquad \cot x = \tan A \cos c.$$

Applying Rule II, p. 200, to ABD, we have

$$\cos A = \cos p \sin x, \text{ or,}$$

$$(B) \qquad \cos p = \cos A \csc x.$$

Applying the same rule to CBD,

$$\cos C = \cos p \sin y, \text{ or,}$$

$$(C) \qquad \cos p = \cos C \csc y.$$

Equating (B) and (C),

$$\cos C \csc y = \cos A \csc x, \text{ or,}$$

$$\cos C = \cos A \csc x \sin y.$$

But $y = B - x$; therefore

$$(D) \qquad \cos C = \cos A \csc x \sin (B - x).$$

Now C may be computed from (A) and (D), namely,

$$\textbf{(50)} \qquad \boldsymbol{\cot x = \tan A \cos c.}$$

$$\textbf{(51)} \qquad \boldsymbol{\cos C = \frac{\cos A \sin (B - x)}{\sin x}}.$$

Ex. 2. Given $A = 35° 46'$, $B = 115° 9'$, $c = 51° 2'$; find C.

Solution. Apply the method just explained.

To find B – x use **(50)**	*To find C use* **(51)**
$\log \tan A = 9.8575$	$\log \cos A = 9.9093$
$\log \cos c = 9.7986$	$\log \sin (B - x) = 9.8811$
$\log \cot x = 9.6561$	19.7904
$x = 65° 38'$.	$\log \sin x = 9.9595$
$\therefore\ B - x = 49° 31'$.	$\log \cos C = 9.8309$
	$C = 47° 21'$.

EXAMPLES

Solve the following oblique spherical triangles:

No.	Given Parts			Required Parts		
1	$A = 67° 30'$	$B = 45° 50'$	$c = 74° 20'$	$a = 63° 15'$	$b = 43° 54'$	$C = 95° 1'$
2	$B = 98° 30'$	$C = 67° 20'$	$a = 60° 40'$	$b = 86° 40'$	$c = 68° 40'$	$A = 59° 44'$
3	$C = 110°$	$A = 94°$	$b = 44°$	$a = 114° 10'$	$c = 120° 46'$	$B = 130° 34'$
4	$C = 70° 20'$	$B = 43° 50'$	$a = 50° 46'$	$b = 32° 59'$	$c = 47° 45'$	$A = 80° 14'$
5	$A = 78°$	$B = 41°$	$c = 108°$	$a = 95° 38'$	$b = 41° 52'$	$C = 110° 49'$
6	$B = 135°$	$C = 50°$	$a = 70° 20'$	$b = 120° 17'$	$c = 69° 20'$	$A = 50° 26'$
7	$A = 31° 40'$	$C = 122° 20'$	$b = 40° 40'$	$a = 34° 3'$	$c = 64° 19'$	$B = 37° 40'$
8	$A = 108° 12'$	$B = 145° 46'$	$c = 126° 32'$	$a = 69° 4'$	$b = 146° 26'$	$C = 125° 12'$
9	$A = 130° 36'$	$B = 30° 26'$	$c = 40° 35'$	$a = 71° 15'$	$b = 39° 10'$	$C = 31° 26'$
10	$A = 51° 58'$	$B = 83° 54'$	$c = 42°$	$a = 38°$	$b = 51°$	$C = 58° 53'$

20. Case III. **(*a*) Given two sides and the angle opposite one of them, as *a*, *b*, *B* (ambiguous case *).**

From the law of sines, p. 207, we get

$$\text{(11)} \qquad \sin A = \frac{\sin a \sin B}{\sin b},$$

which gives A†. *To find C we use, from p. 216, formula* (**46**), *solved for* $\tan \frac{1}{2}\gamma$, *namely,*

$$\text{(46)} \qquad \tan \tfrac{1}{2}\gamma = -\frac{\sin \frac{1}{2}(a+b) \tan \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}(a-b)}.$$

To find c, solve (**44**), *p. 215, for* $\tan \frac{1}{2} c$, *namely,*

$$\text{(44)} \qquad \tan \tfrac{1}{2} c = -\frac{\sin \frac{1}{2}(\alpha + \beta) \tan \frac{1}{2}(a-b)}{\sin \frac{1}{2}(\alpha - \beta)}.$$

Check by the law of sines.

Ex. 1. Given $a = 58°$, $b = 137° 20'$, $B = 131° 20'$; find A, C, c.

Solution.

To find A use (**11**)

$$\begin{aligned} \log \sin a &= 9.9284 \\ \log \sin B &= \underline{9.8756} \\ & 19.8040 \\ \log \sin b &= \underline{9.8311} \\ \log \sin A &= 9.9729 \end{aligned}$$

$\therefore A_1 = 69° 58'$,

or, $A_2 = 180° - A_1 = 110° 2'$.

$$\begin{aligned} a &= 58° & a &= 58° \\ b &= \underline{137° 20'} & b &= \underline{137° 20'} \\ a+b &= 195° 20' & a-b &= -79° 20' \\ \tfrac{1}{2}(a+b) &= 97° 40'. & \tfrac{1}{2}(a-b) &= -39° 40'. \end{aligned}$$

$\beta = 180° - B = 48° 40'$.

Since $a < b$ and both A_1 and A_2 are $< B$, it follows that we have *two solutions.*

* Just as in the corresponding case in the solution of plane oblique triangles (Granville's *Plane Trigonometry*, pp. 105, 161), there may be *two solutions*, *one solution*, or *no solution*, depending on the given data.

† Since the angle A is here determined from its sine, it is necessary to consider both of the values found. If $a > b$ then $A > B$; and if $a < b$ then $A < B$. Hence [next page]

First solution. $\alpha_1 = 180° - A_1 = 110° \, 2'$.

$$\begin{aligned} \alpha_1 &= 110° \, 2' \\ \beta &= \underline{\;48° \, 40'} \\ \alpha_1 + \beta &= 158° \, 42' \\ \tfrac{1}{2}(\alpha_1 + \beta) &= 79° \, 21'. \end{aligned}$$

$$\begin{aligned} \alpha_1 &= 110° \, 2' \\ \beta &= \underline{\;48° \, 40'} \\ \alpha_1 - \beta &= \;61° \, 22' \\ \tfrac{1}{2}(\alpha_1 - \beta) &= 30° \, 41'. \end{aligned}$$

To find C_1 *use* **(46)**

$$\begin{aligned} \log \sin \tfrac{1}{2}(a + b) &= \;9.9961 \\ \log \tan \tfrac{1}{2}(\alpha_1 - \beta) &= \underline{\;9.7733} \\ &\;\;\, 19.7694 \\ \log \sin \tfrac{1}{2}(a - b) &= \underline{\;9.8050} \; (n) \\ \log \tan \tfrac{1}{2}\gamma_1 &= \;9.9644 \\ \tfrac{1}{2}\gamma_1 &= 42° \, 39'. \\ \gamma_1 &= 85° \, 18'. \\ \therefore \; C_1 = 180° - \gamma_1 &= 94° \, 42'. \end{aligned}$$

To find c_1 *use* **(44)**

$$\begin{aligned} \log \sin \tfrac{1}{2}(\alpha_1 + \beta) &= \;9.9924 \\ \log \tan \tfrac{1}{2}(a - b) &= \underline{\;9.9187} \; (n) \\ &\;\;\, 19.9111 \\ \log \sin \tfrac{1}{2}(\alpha_1 - \beta) &= \underline{\;9.7078} \\ \log \tan \tfrac{1}{2}c_1 &= 10.2033 \\ \tfrac{1}{2}c_1 &= 57° \, 57'. \\ \therefore \; c_1 &= 115° \, 54'. \end{aligned}$$

Check:

$$\begin{aligned} \log \sin a &= 9.9284 & \log \sin b &= 9.8311 & \log \sin c_1 &= 9.9541 \\ \log \sin A_1 &= \underline{9.9729} & \log \sin B &= \underline{9.8756} & \log \sin C_1 &= \underline{9.9985} \\ &\;\;\, 9.9555 & &\;\;\, 9.9555 & &\;\;\, 9.9556 \end{aligned}$$

Second solution. $\alpha_2 = 180° - A_2 = 69° \, 58'$.

$$\begin{aligned} \alpha_2 &= \;69° \, 58' \\ \beta &= \underline{\;48° \, 40'} \\ \alpha_2 + \beta &= 118° \, 38' \\ \tfrac{1}{2}(\alpha_2 + \beta) &= 59° \, 19'. \end{aligned}$$

$$\begin{aligned} \alpha_2 &= 69° \, 58' \\ \beta &= \underline{48° \, 40'} \\ \alpha_2 - \beta &= 21° \, 18' \\ \tfrac{1}{2}(\alpha_2 - \beta) &= 10° \, 39'. \end{aligned}$$

To find C_2 *use* **(46)**

$$\begin{aligned} \log \sin \tfrac{1}{2}(a + b) &= \;9.9961 \\ \log \tan \tfrac{1}{2}(\alpha_2 - \beta) &= \underline{\;9.2743} \\ &\;\;\, 19.2704 \\ \log \sin \tfrac{1}{2}(a - b) &= \underline{\;9.8050} \; (n) \\ \log \tan \tfrac{1}{2}\gamma_2 &= \;9.4654 \\ \tfrac{1}{2}\gamma_2 &= 16° \, 17'. \\ \gamma_2 &= 32° \, 34'. \\ \therefore \; C_2 = 180° - \gamma_2 &= 147° \, 26'. \end{aligned}$$

To find c_1 *use* **(44)**

$$\begin{aligned} \log \sin \tfrac{1}{2}(\alpha_2 + \beta) &= \;9.9345 \\ \log \tan \tfrac{1}{2}(a - b) &= \underline{\;9.9187} \; (n) \\ &\;\;\, 19.8532 \\ \log \sin \tfrac{1}{2}(\alpha_2 - \beta) &= \underline{\;9.2667} \\ \log \tan \tfrac{1}{2}c_2 &= 10.5865 \\ \tfrac{1}{2}c_2 &= 75° \, 28'. \\ \therefore \; c_2 &= 150° \, 56'. \end{aligned}$$

Check:

$$\begin{aligned} \log \sin a &= 9.9284 & \log \sin b &= 9.8311 & \log \sin c_2 &= 9.6865 \\ \log \sin A_2 &= \underline{9.9729} & \log \sin B &= \underline{9.8756} & \log \sin C_2 &= \underline{9.7310} \\ &\;\;\, 9.9555 & &\;\;\, 9.9555 & &\;\;\, 9.9555 \end{aligned}$$

If the side c or the angle C is wanted without first calculating the value of A, we may resolve the given triangle into two right triangles and then apply Napier's rules, as was illustrated under Cases II, (a), and II, (b), pp. 220, 223.

Theorem. *Only those values of A should be retained which are greater or less than B according as a is greater or less than b.*

If $\log \sin A$ = a positive number, there will be no solution.

EXAMPLES

Solve the following oblique spherical triangles :

No.	Given Parts			Required Parts		
1	$a=43°\,20'$	$b=48°\,30'$	$A=58°\,40'$	$B_1=68°\,47'$	$C_1=70°\,40'$	$c_1=49°\,18'$
				$B_2=111°\,13'$	$C_2=14°\,29'$	$c_2=11°\,36'$
2	$a=56°\,40'$	$b=30°\,50'$	$A=103°\,40'$	$B=36°\,36'$	$C=52°$	$c=42°\,39'$
3	$a=30°\,20'$	$b=46°\,30'$	$A=36°\,40'$	$B_1=59°\,4'$	$C_1=97°\,39'$	$c_1=56°\,57'$
				$B_2=120°\,57'$	$C_2=28°\,5'$	$c_2=23°\,28'$
4	$b=99°\,40'$	$c=64°\,20'$	$B=95°\,40'$	$C=65°\,30'$	$A=97°\,20'$	$a=100°\,45'$
5	$a=40°$	$b=118°\,20'$	$A=29°\,40'$	$B_1=42°\,40'$	$C_1=159°\,54'$	$c_1=153°\,30'$
				$B_2=137°\,20'$	$C_2=50°\,21'$	$c_2=90°\,10'$
6	$a=115°\,20'$	$c=146°\,20'$	$C=141°\,10'$	Impossible		
7	$a=109°\,20'$	$c=82°$	$A=107°\,40'$	$C=90°$	$B=113°\,37'$	$b=114°\,52'$
8	$b=108°\,30'$	$c=40°\,50'$	$C=39°\,50'$	$B_1=68°\,18'$	$A_1=132°\,34'$	$a_1=131°\,16'$
				$B_2=111°\,42'$	$A_2=77°\,5'$	$a_2=95°\,50'$
9	$a=162°\,20'$	$b=15°\,40'$	$B=125°$	Impossible		
10	$a=55°$	$c=138°\,10'$	$A=42°\,30'$	$C=146°\,38'$	$B=55°\,1'$	$b=96°\,34'$

21. Case III. **(*b*) Given two angles and the side opposite one of them, as *A*, *B*, *b* (ambiguous case *).**

From the law of sines, p. 207, we get

$$\text{(11)}\qquad \sin a = \frac{\sin A \sin b}{\sin B},$$

which gives a.† *To find c we use, from p. 215, the formula* ‡ **(44)**, *solved for* $\tan \frac{1}{2} c$, *namely,*

$$\text{(44)}\qquad \tan \tfrac{1}{2} c = -\frac{\sin \frac{1}{2}(\alpha + \beta) \tan \frac{1}{2}(a - b)}{\sin \frac{1}{2}(\alpha - \beta)}.$$

To find C, solve **(46)**, *p. 216, for* $\tan \frac{1}{2} \gamma$, *namely,*

$$\text{(46)}\qquad \tan \tfrac{1}{2} \gamma = -\frac{\sin \frac{1}{2}(a + b) \tan \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}(a - b)}.$$

Check by the law of sines.

* Just as in Case II, (*b*), we may have *two solutions*, *one solution*, or *no solution*, depending on the given data.

† Since the side is here determined from its sine, it is necessary to examine both of the values found. If $A > B$ then $a > b$; and if $A < B$ then $a < b$. Hence we have the

Theorem. *Only those values of a should be retained which are greater or less than b according as A is greater or less than B.*

If log sin a = a positive number, there will be no solution.

‡ Same as those used in Case III, (*a*), p. 224, when the Greek and Roman letters are interchanged.

Ex. 1. Given $A = 110°$, $B = 131° 20'$, $b = 137° 20'$; find a, b, C.

Solution. $\alpha = 180° - A = 70°$, and $\beta = 180° - B = 48° 40'$.

To find a use **(11)**

$$\begin{aligned}
\log\sin A &= 9.9730\\
\log\sin b &= \underline{9.8311}\\
& 19.8041\\
\log\sin B &= \underline{9.8756}\\
\log\sin a &= 9.9285
\end{aligned}$$

$\therefore a_1 = 58° 1'$,

or, $a_2 = 180° - a_1 = 121° 59'$.

$$\begin{aligned}
\alpha &= 70° & \alpha &= 70°\\
\beta &= \underline{48° 40'} & \beta &= \underline{48° 40'}\\
\alpha + \beta &= 118° 40' & \alpha - \beta &= 21° 20'
\end{aligned}$$

$\frac{1}{2}(\alpha + \beta) = 59° 20'$. $\frac{1}{2}(\alpha - \beta) = 10° 40'$.

Since $A < B$ and both a_1 and a_2 are $< b$, it follows that we have *two solutions.*

First solution.

$$\begin{aligned}
a_1 &= 58° 1' & a_1 &= 58° 1'\\
b &= \underline{137° 20'} & b &= \underline{137° 20'}\\
a_1 + b &= 195° 21' & a_1 - b &= -79° 19'\\
\tfrac{1}{2}(a_1 + b) &= 97° 41'. & \tfrac{1}{2}(a_1 - b) &= -39° 40'.
\end{aligned}$$

To find c_1 use **(44)**

$$\begin{aligned}
\log\sin\tfrac{1}{2}(\alpha + \beta) &= 9.9346\\
\log\tan\tfrac{1}{2}(a_1 - b) &= \underline{9.9187}\ (n)\\
& 19.8533\\
\log\sin\tfrac{1}{2}(\alpha - \beta) &= \underline{9.2674}\\
\log\tan\tfrac{1}{2}c_1 &= 10.5859
\end{aligned}$$

$\frac{1}{2}c_1 = 75° 27'$.

$\therefore c_1 = 150° 54'$.

To find C_1 use **(46)**

$$\begin{aligned}
\log\sin\tfrac{1}{2}(a_1 + b) &= 9.9961\\
\log\tan\tfrac{1}{2}(\alpha - \beta) &= \underline{9.2750}\\
& 19.2711\\
\log\sin\tfrac{1}{2}(a_1 - b) &= \underline{9.8050}\ (n)\\
\log\tan\tfrac{1}{2}\gamma_1 &= 9.4661
\end{aligned}$$

$\frac{1}{2}\gamma_1 = 16° 18'$.

$\gamma_1 = 32° 36'$.

$\therefore C_1 = 180° - \gamma_1 = 147° 24'$.

Check:

$\log\sin a_1 = 9.9285$	$\log\sin b = 9.8311$	$\log\sin c_1 = 9.6869$
$\log\sin A = \underline{9.9730}$	$\log\sin B = \underline{9.8756}$	$\log\sin C_1 = \underline{9.7314}$
9.9555	9.9555	9.9555

Second solution. This gives $c_2 = 64° 8'$, and $C_2 = 85° 18'$.

Remembering that $a_2 = 121° 59'$, we may now check the second solution.

Check:

$\log\sin a_2 = 9.9285$	$\log\sin b = 9.8311$	$\log\sin c_2 = 9.9542$
$\log\sin A = \underline{9.9730}$	$\log\sin B = \underline{9.8756}$	$\log\sin C_2 = \underline{9.9985}$
9.9555	9.9555	9.9557

Hence the two solutions are

$a_1 = 58° 1'$ $c_1 = 150° 54'$ $C_1 = 147° 23'$,

and $a_2 = 121° 59'$ $c_2 = 64° 8'$ $C_2 = 85° 18'$.

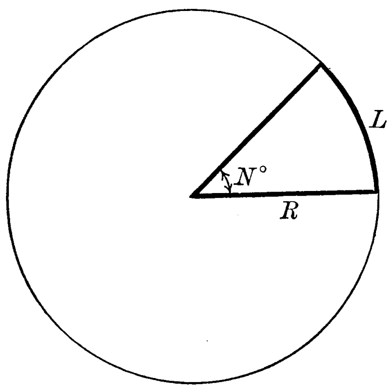
If the angle C or the side c is wanted without first computing a, we may resolve the given triangle into two right triangles and then apply Napier's rules, as was illustrated under Cases II, (a), and II, (b), pp. 220, 223.

EXAMPLES

Solve the following oblique spherical triangles :

No.	Given Parts			Required Parts		
1	$A=108° 40'$	$B=134° 20'$	$a=145° 36'$	$b=154° 45'$	$c=34° 9'$	$C=70° 18'$
2	$B=116°$	$C=80°$	$c=84°$	$b=114° 50'$	$A=79° 20'$	$a=82° 56'$
3	$A=132°$	$B=140°$	$b=127°$	$a_1=67° 24'$	$C_1=164° 6'$	$c_1=160° 6'$
				$a_2=112° 36'$	$C_2=128° 21'$	$c_2=103° 2'$
4	$A=62°$	$C=102°$	$a=64° 30'$	$c=90°$	$B=63° 43'$	$b=66° 26'$
5	$A=133° 50'$	$B=66° 30'$	$a=81° 10'$	Impossible		
6	$B=22° 20'$	$C=146° 40'$	$c=138° 20'$	$b=27° 22'$	$A=47° 21'$	$a=117° 9'$
7	$A=61° 40'$	$C=140° 20'$	$c=150° 20'$	$a_1=43° 3'$	$B_1=89° 24'$	$b_1=129° 8'$
				$a_2=136° 57'$	$B_2=26° 59'$	$b_2=20° 36'$
8	$B=73°$	$C=81° 20'$	$b=122° 40'$	Impossible		
9	$B=36° 20'$	$C=46° 30'$	$b=42° 12'$	$A_1=164° 44'$	$a_1=162° 38'$	$c_1=124° 41'$
				$A_2=119° 19'$	$a_2=81° 19'$	$c_2=55° 19'$
10	$A=110° 10'$	$B=133° 18'$	$a=147° 6'$	$b=155° 5'$	$c=33° 2'$	$C=70° 21'$

22. Length of an arc of a circle in linear units. From Geometry we know that *the length of an arc of a circle is to the circumference of the circle as the number of degrees in the arc is to 360.* That is



$$L : 2\,\pi R :: N : 360, \text{ or,}$$

$$L = \frac{\pi RN}{180}, \quad (52)$$

where L = length of arc,

N = number of degrees in arc,

R = length of radius.

In case the length of the arc is given to find the number of degrees in it, we instead solve for N, giving

$$N = \frac{180\,L}{\pi R}. \quad (53)$$

Considering the earth as a sphere, *an arc of one minute* on a great circle is called *a geographical mile* or *a nautical mile.** Hence there are 60 nautical miles in an arc of 1 degree, and $360 \times 60 = 21{,}600$ nautical miles in the circumference of a great circle of the earth. If we assume the radius of the earth to be 3960 statute miles, the length

* In connection with a ship's rate of sailing a *nautical mile* is also called a *knot.*

of a nautical mile ($= 1$ min. $= \frac{1}{60}$ of a degree) in statute miles will be, from (**52**),

$$L = \frac{3.1416 \times 3960 \times \frac{1}{60}}{180} = 1.15 \text{ mi.}$$

Ex. 1. *Find the length of an arc of* $22° \, 30'$ *in a circle of radius* 4 *in.*

Solution. Here $N = 22° \, 30' = 22.5°$, and $R = 4$ in.

Substituting in (**52**), $L = \frac{3.1416 \times 4 \times 22.5}{180} = 1.57$ in. *Ans.*

Ex. 2. A ship has sailed on a great circle for $5\frac{1}{2}$ hr. at the rate of 12 statute miles an hour. How many degrees are there in the arc passed over?

Solution. Here $L = 5\frac{1}{2} \times 12 = 66$ mi., and $R = 3960$ mi.

Substituting in (**53**), $N = \frac{180 \times 66}{3.1416 \times 3960} = .955° = 57.3'$. *Ans.*

23. Area of a spherical triangle. From Spherical Geometry we know that *the area of a spherical triangle is to the area of the surface of the sphere as the number of degrees in its spherical excess* is to 720.* That is,

$$\text{Area of triangle} : 4\pi R^2 :: E : 720, \text{ or,}$$

$$\textbf{(54)} \qquad \textbf{Area of spherical triangle} = \frac{\pi R^2 E}{180}.$$

In case the three angles of the triangle are not given, we should first find them by solving the triangle. Or, if the three sides of the triangle are given, we may find E directly by Lhuilier's formula,† namely,

$$\textbf{(55)} \qquad \tan \tfrac{1}{4} E = \sqrt{\tan \tfrac{1}{2} s \tan \tfrac{1}{2}(s-a) \tan \tfrac{1}{2}(s-b) \tan \tfrac{1}{2}(s-c)},$$

where a, b, c denote the sides and $s = \frac{1}{2}(a + b + c)$.

The area of a spherical polygon will evidently be the sum of the areas of the spherical triangles formed by drawing arcs of great circles as diagonals of the polygon.

Ex. 1. The angles of a spherical triangle on a sphere of 25-in. radius are $A = 74° \, 40'$, $B = 67° \, 30'$, $C = 49° \, 50'$. Find the area of the triangle.

Solution. Here $E = (A + B + C) - 180° = 12°$.

Substituting in (**54**), Area $= \frac{3.1416 \times (25)^2 \times 12}{180} = 130.9$ sq. in. *Ans.*

* The spherical excess (usually denoted by E) of a spherical triangle is the excess of the sum of the angles of the triangle over 180°. Thus, if A, B, and C are the angles of a spherical triangle,

$$E = A + B + C - 180°.$$

† Derived in more advanced treatises.

EXAMPLES

1. Find the length of an arc of 5° 12′ in a circle whose radius is 2 ft. 6 in. *Ans.* 2.72 in.

2. Find the length of an arc of 75° 30′ in a circle whose radius is 10 yd. *Ans.* 13.17 yd.

3. How many degrees are there in a circular arc 15 in. long, if the radius is 6 in.? *Ans.* 143° 18′.

4. A ship sailed over an arc of 4 degrees on a great circle of the earth each day. At what rate was the ship sailing? *Ans.* 11.515 mi. per hour.

5. Find the perimeter in inches of a spherical triangle of sides 48°, 126°, 80°, on a sphere of radius 25 in. *Ans.* 110.78 in.

6. The course of the boats in a yacht race was in the form of a triangle having sides of length 24 mi., 20 mi., 18 mi. If we assume that they sailed on arcs of great circles, how many minutes of arc did they describe? *Ans.* 53.85 min.

7. The angles of a spherical triangle are $A = 63°$, $B = 84° 21'$, $C = 79°$; the radius of the sphere is 10 in. What is the area of the triangle? *Ans.* 80.88 sq. in.

8. The sides of a spherical triangle are $a = 6.47$ in., $b = 8.39$ in., $c = 9.43$ in.; the radius of the sphere is 25 in. What is the area of the triangle? *Ans.* 26.9 sq. in.

Hint. Find E by using formula (**55**).

9. In a spherical triangle $A = 75° 16'$, $B = 39° 20'$, $c = 26$ ft.; the radius of the sphere is 14 ft. Find the area of the triangle. *Ans.* 158.45 sq. ft.

10. Two ships leave Boston at the same time. One sails east 441 mi. and the other 287 mi. E. 38° 21′ N. the first day. If we assume that each ship sailed on an arc of a great circle, what is the area of the spherical triangle whose vertices are at Boston and at the positions of the ships at the end of the day? *Ans.* 41,040 sq. mi.

11. A steamboat traveling at the rate of 15 knots per hour skirts the entire shore line of an island having the approximate shape of an equilateral triangle in 18 hr. What is the approximate area of the island? *Ans.* 34,960 sq. mi.

12. Find the areas of the following spherical triangles, having given

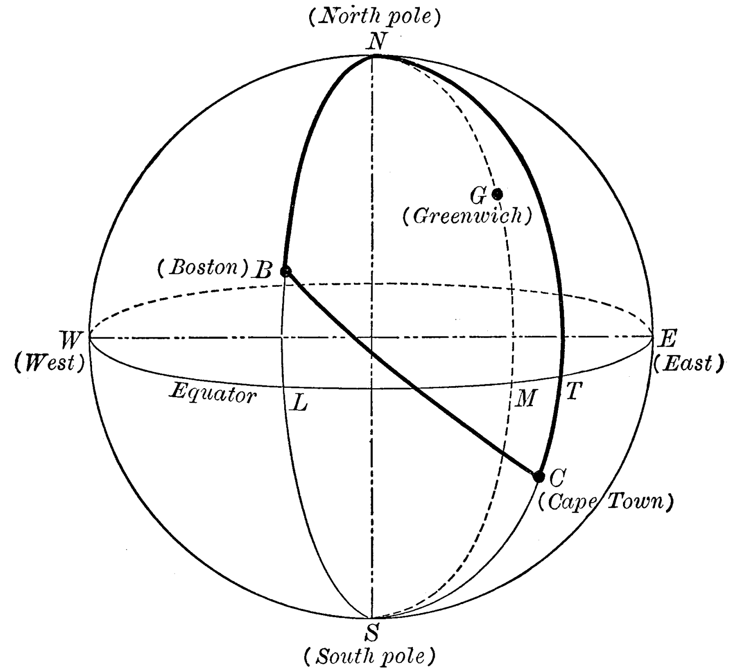
(a) $a = 47° 30'$, $b = 55° 40'$, $c = 60° 10'$; $R = 10$ ft. *Ans.* 42.96 sq. ft.
(b) $a = 43° 30'$, $b = 72° 24'$, $c = 87° 50'$; $R = 10$ in. 59.19 sq. in.
(c) $A = 74° 40'$, $B = 67° 30'$, $C = 49° 50'$; $R = 100$ yd. 2094 sq. yd.
(d) $A = 112° 30'$, $B = 83° 40'$, $C = 70° 10'$; $R = 25$ cm. 941.2 sq. cm.
(e) $a = 64° 20'$, $b = 42° 30'$, $C = 50° 40'$; $R = 12$ ft. 46.73 sq. ft.
(f) $C = 110°$, $A = 94°$, $b = 44°$; $R = 40$ rd. 709.2 sq. rd.
(g) $a = 43° 20'$, $b = 48° 30'$, $A = 58° 40'$; $R = 100$ rd. 24.88 acres.
(h) $A = 108° 40'$, $B = 134° 20'$, $a = 145° 36'$; $R = 3960$ mi. 36,460,000 sq. mi.

CHAPTER III

APPLICATIONS OF SPHERICAL TRIGONOMETRY TO THE CELESTIAL AND TERRESTRIAL SPHERES

24. Geographical terms. In what follows we shall assume the earth to be a sphere of radius 3960 statute miles.

The **meridian** of a place on the earth is that great circle of the earth which passes through the place and the north and south poles.



Thus, in the figure representing the earth, NGS is the meridian of Greenwich, NBS is the meridian of Boston, and NCS is the meridian of Cape Town.

The **latitude** of a place is the arc of the meridian of the place extending from the equator to the place. Latitude is measured north or south of the equator from 0° to 90°. Thus, in the figure, the arc LB measures the north latitude of Boston, and the arc TC measures the south latitude of Cape Town.

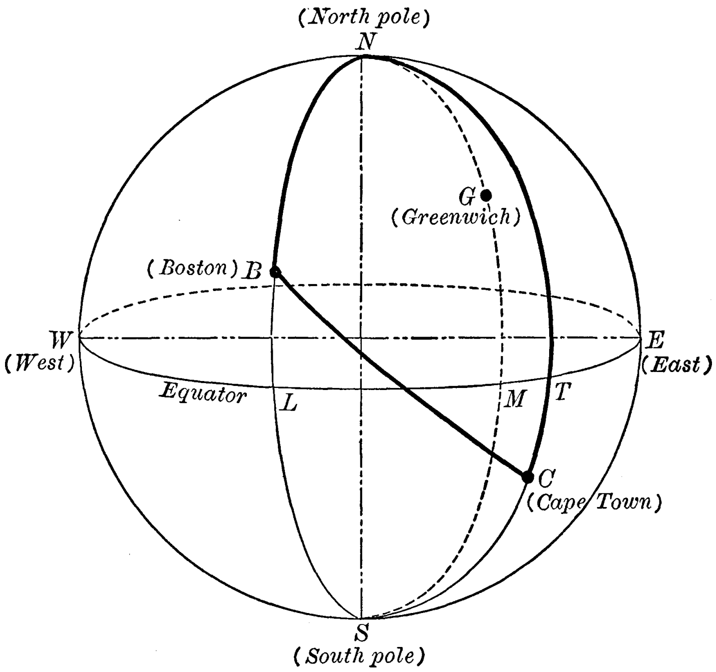
The **longitude** of a place is the arc of the equator extending from the zero meridian* to the meridian of the place. Longitude is

* As in this case, the zero meridian, or reference meridian, is usually the meridian passing through Greenwich, near London. The meridians of Washington and Paris are also used as reference meridians.

measured east or west from the Greenwich meridian from 0° to 180°. Thus, in the figure, the arc *MT* measures the east longitude of Cape Town, while the arc *ML* measures the west longitude of Boston. Since the arcs *MT* and *ML* are the measures of the angles *MNT* and *MNL* respectively, it is evident that we can also define the longitude of a place as the angle between the reference meridian and the meridian of the place. Thus, in the figure, the angle *MNT* is the east longitude of Cape Town, while the angle *MNL* is the west longitude of Boston.

The **bearing** of one place from a second place is the angle between the arc of a great circle drawn from the second place to the first place, and the meridian of the second place. Thus, in the figure, the bearing of Cape Town from Boston is measured by the angle *CBN* or the angle *CBL*, while the bearing of Boston from Cape Town is measured by the angle *NCB* or the angle *SCB*.*

25. Distances between points on the surface of the earth. Since we know from Geometry that *the shortest distance on the surface of a*



sphere between any two points on that surface is the arc, not greater than a semicircumference, of the great circle that joins them, it is evident that the shortest distance between two places on the earth is measured in the same way. Thus, in the figure, the shortest

* The *bearing* or *course* of a ship at any point is the angle the path of the ship makes with the meridian at that point.

distance between Boston and Cape Town is measured on the arc BC of a great circle. We observe that this arc BC is one side of a spherical triangle of which the two other sides are the arcs BN and CN. Since

$$\text{arc } BN = 90° - \text{arc } LB = 90° - \text{north latitude of Boston},$$

$$\text{arc } CN = 90° + \text{arc } TC = 90° + \text{south latitude of Cape Town},$$

and
$$\begin{aligned} \text{angle } BNC &= \text{angle } MNL + \text{angle } MNT \\ &= \text{west longitude of Boston} \\ &\quad + \text{east longitude of Cape Town} \\ &= \text{difference in longitude of Boston and Cape Town,} \end{aligned}$$

it is evident that if we know the latitudes and longitudes of Boston and Cape Town, we have all the data necessary for determining two sides and the included angle of the triangle BNC. The third side BC, which is the shortest distance between Boston and Cape Town, may then be found as in Case II, (*a*), p. 219.

In what follows, north latitude will be given the sign + and south latitude the sign −.

Rule for finding the shortest distance between two points on the earth and the bearing of each from the other, the latitude and longitude of each point being given.

First step. *Subtract the latitude of each place from 90°.* The results will be the two sides of a spherical triangle.*

Second step. *Find the difference of longitude of the two places by subtracting the lesser longitude from the greater if both are E. or both are W., but add the two if one is E. and the other is W. This gives the included angle of the triangle.†*

Third step. *Solving the triangle by Case II, (a), p. 219, the third side gives the shortest distance between the two points in degrees of arc,‡ and the angles give the bearings.*

* Note that this is *algebraic* subtraction. Thus, if the two latitudes were 25° N. and 42° S., we would get as the two sides of the triangle,

$$90° - 25° = 65° \quad \text{and} \quad 90° - (-42°) = 90° + 42° = 132°.$$

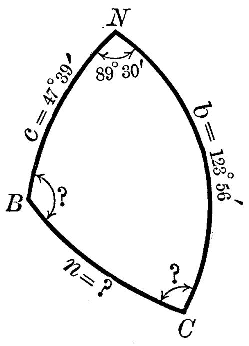
† If the difference of longitude found is greater than 180°, we should subtract it from 360° and use the remainder as the included angle.

‡ The number of minutes in this arc will be the distance between the two places in geographical (nautical) miles. The distance between the two places in statute miles is given by the formula

$$L = \frac{3.1416 \times 3960 \times N}{180},$$

where N = the number of degrees in the arc.

Ex. 1. Find the shortest distance along the earth's surface between Boston (lat. 42° 21′ N., long. 71° 4′ W.) and Cape Town (lat. 33° 56′ S., long. 18° 26′ E.), and the bearing of each city from the other.



Solution. Draw a spherical triangle in agreement with the figure on p. 232.

First step.

$$c = 90° - 42° 21' = 47° 39',$$

$$b = 90° - (-33° 56') = 123° 56'.$$

Second step.

$$N = 71° 4' + 18° 26' = 89° 30' = \text{difference in long.}$$

Third step. Solving the triangle by Case II, (*a*), p. 219, we get

$$n = 68° 14' = 68.23° = 4094 \text{ nautical miles,}$$

$$C = 52° 43' = \text{bearing of Boston from Cape Town,}$$

and

$$B = 116° 43' = \text{bearing of Cape Town from Boston.}$$

Hence a ship sailing from Boston to Cape Town on the arc of a great circle *sets out from Boston* on a course S. 63° 17′ E. and *approaches Cape Town* on a course S. 52° 43′ E.*

EXAMPLES

1. Find the shortest distance between Baltimore (lat. 39° 17′ N., long. 76° 37′ W.) and Cape Town (lat. 33° 56′ S., long. 18° 26′ E.), and the bearing of each from the other. *Ans.* Distance = 65° 48′ = 3947 nautical miles,
S. 64° 58′ E. = bearing of Cape Town from Boston,
N. 57° 42′ W. = bearing of Boston from Cape Town.

2. What is the distance from New York (lat. 40° 43′ N., long. 74° W.) to Liverpool (lat. 53° 24′ N., 3° 4′ W.)? Find the bearing of each place from the other. In what latitude will a steamer sailing on a great circle from New York to Liverpool cross the meridian of 50° W., and what will be her course at that point? *Ans.* Distance = 47° 50′ = 2870 nautical miles,
N. 75° 7′ W. = bearing of New York from Liverpool,
N. 49° 29′ E. = bearing of Liverpool from New York.
Lat. 51° 13′ N., with course N. 66° 54′ E.

3. Find the shortest distance between the following places in geographical miles:

(a) New York (lat. 40° 43′ N., long. 74° W.) and San Francisco (lat. 37° 48′ N., long. 122° 28′ W.). *Ans.* 2230.

(b) Sandy Hook (lat. 40° 28′ N., long. 74° 1′ W.) and Madeira (lat. 32° 28′ N., long. 16° 55′ W.). *Ans.* 2749.

(c) San Francisco (lat. 37° 48′ N., long. 122° 28′ W.) and Batavia (lat. 6° 9′ S., long. 106° 53′ E.). *Ans.* 7516.

(d) San Francisco (lat. 37° 48′ N., long. 122° 28′ W.) and Valparaiso (lat. 33° 2′ S., long. 71° 41′ W.) *Ans.* 5109.

* A ship that sails on a great circle (except on the equator or a meridian) must be continually changing her course. If the ship in the above example keeps constantly on the course S. 63° 17′ E., she will never reach Cape Town.

4. Find the shortest distance in statute miles (taking diameter of earth as 7912 mi.) between Boston (lat. 42° 21′ N., long. 71° 4′ W.) and Greenwich (lat. 51° 29′ N.), and the bearing of each place from the other.

Ans. Distance = 3276 mi.,
N. 53° 7′ E. = bearing of Greenwich from Boston,
N. 71° 39′ W. = bearing of Boston from Greenwich.

5. As in last example, find the shortest distance between and bearings for Calcutta (lat. 22° 33′ N., long. 88° 19′ E.) and Valparaiso (lat. 33° 2′ S., long. 71° 42′ W.).

Ans. Distance = 10,860 mi.,
S. 64° 20.5′ E. = bearing of Calcutta from Valparaiso,
S. 54° 54.5′ W. = bearing of Valparaiso from Calcutta.

6. Find the shortest distance in statute miles from Oberlin (long. 82° 14′ W.) to New Haven (long. 72° 55′ W.), the latitude of each place being 41° 17′ N.

Ans. 483.2 mi.

7. From a point whose latitude is 17° N. and longitude 130° W. a ship sailed an arc of a great circle over a distance of 4150 statute miles, starting S. 54° 20′ W. Find its latitude and longitude, if the length of 1° is $69\frac{1}{6}$ statute miles.

Ans. Lat. 19° 42′ S., long. 178° 21′ W.

26. Astronomical problems. One of the most important applications of Spherical Trigonometry is to Astronomy. In fact, Trigonometry was first developed by astronomers, and for centuries was studied only in connection with Astronomy. We shall take up the study of a few simple problems in Astronomy.

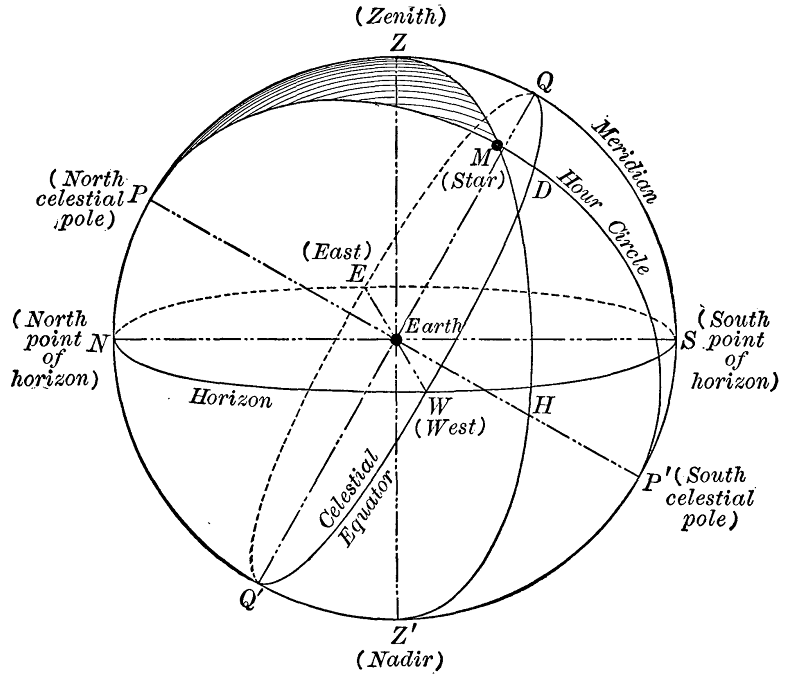
27. The celestial sphere. When there are no clouds to obstruct the view, the sky appears like a great hemispherical vault, with the observer at the center. The stars seem to glide upon the inner surface of this sphere from east to west,* their paths being parallel circles whose planes are perpendicular to the polar axis of the earth, and having their centers in that axis produced. Each star † makes a complete revolution, called its diurnal (daily) motion, in 23 hr. 56 min., ordinary clock time. We cannot estimate the distance of the surface of this sphere from us, further than to perceive that it must be very far away indeed, because it lies beyond even the remotest terrestrial objects. To an observer the stars all seem to be at the same enormous distance from him, since his eyes can judge their *directions only* and not their *distances*. It is therefore natural, and it is extremely convenient from a mathematical point of view, to regard this imaginary sphere on which all the heavenly bodies seem to be projected, as having a radius of unlimited length. This

* This apparent turning of the sky from east to west is in reality due to the rotation of the earth in the opposite direction, just as to a person on a swiftly moving train the objects outside seem to be speeding by, while the train appears to be at rest. The sky is really motionless, while the earth is rotating from west to east.

† By stars we shall mean fixed stars and nebulæ whose relative positions vary so slightly that it takes centuries to make the change perceptible.

sphere, called the **celestial sphere,** is conceived of as having such enormous proportions that the whole solar system (sun, earth, and planets) *lies at its center*, like a few particles of dust at the center of a great spherical balloon. The stars seem to retain the same relative positions with respect to each other, being in this respect like places on the earth's surface. As viewed from the earth, the sun, moon, planets, and comets are also projected on the celestial sphere, but they are changing their apparent positions with respect to the stars and with respect to each other. Thus, the sun appears to move eastward with respect to the stars about one degree each day, while the moon moves about thirteen times as far.

The following figure represents the celestial sphere, with the earth at the center showing as a mere dot.



The **zenith** of an observer is the point on the celestial sphere directly overhead. A plumb line held by the observer and extended upwards will pierce the celestial sphere at his zenith (Z in figure).

The **nadir** is the point on the celestial sphere which is diametrically opposite to the zenith (Z' in the figure).

The **horizon** of an observer is the great circle on the celestial sphere having the observer's zenith for a pole; hence every point on the horizon ($SWNE$ in the figure) will be 90° from the zenith and from the nadir. A plane tangent* to a surface of still water

* On account of the great distance, a plane passed tangent to the earth at the place of the observer will cut the celestial sphere in a great circle which (as far as we are concerned) coincides with the observer's horizon.

at the place of the observer will cut the celestial sphere in his horizon.

All points on the earth's surface have different zeniths and horizons.

Every great circle passing through the zenith will be perpendicular to the horizon; such circles are called **vertical circles** (as $ZMHZ'$ and $ZQSP'Z'$ in figure).

The **celestial equator** or **equinoctial** is the great circle in which the plane of the earth's equator cuts the celestial sphere ($EQWQ'$ in the figure).

The **poles** of the celestial equator are the points (P and P' in the figure) where the earth's axis, if produced, would pierce the celestial sphere. The poles may also be defined as those two points on the sky where a star would have no diurnal (daily) motion. The Pole Star is near the north celestial pole, being about $1\frac{1}{4}°$ from it. Every point on the celestial equator is 90° from each of the celestial poles.

All points on the earth's surface have the same celestial equator and poles.

The geographical meridian of a place on the earth was defined as that great circle of the earth which passes through the place and the north and south poles. The **celestial meridian** of a point on the earth's surface is the great circle in which the plane of the point's geographical meridian cuts the celestial sphere ($ZQSP'Z'Q'NP$ in the figure). It is evidently that vertical circle of an observer which passes through the north and south points of his horizon. All points on the surface of the earth which do not lie on the same north-and-south line have different celestial meridians.

The **hour circle** of a heavenly body is that great circle of the celestial sphere which passes through the body * and through the north and south celestial poles. In the figure $PMDP'$ is the hour circle of the star M. The hour circles of all the heavenly bodies are continually changing with respect to any observer.

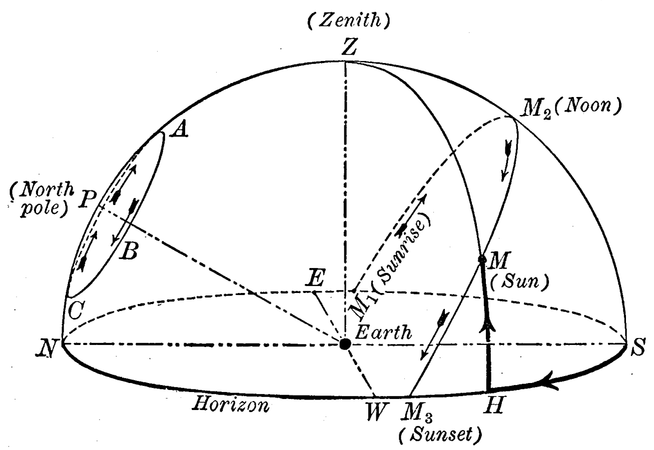
The spherical triangle PZM, having the north pole, the zenith, and a heavenly body at its three vertices, is a very important triangle in Astronomy. It is called the ***astronomical triangle.***

28. Spherical coördinates. When learning how to draw (or plot) the graph of a function, the student has been taught how to locate a point in a plane by measuring its distances from two fixed and mutually perpendicular lines called the axes of coördinates, the two distances being called the rectangular coördinates of the point.

* By this is meant that the hour circle passes through that point on the celestial sphere where we see the heavenly body projected.

If we now consider the surface to be spherical instead of plane, a similar system of locating points on it may be employed, two fixed and mutually perpendicular great circles being chosen as reference circles, and the *angular* distances of a point from these reference circles being used as the *spherical coördinates* of the point. Since the reference circles are perpendicular to each other, each one of them passes through the poles of the other.

In his study of Geography the student has already employed such a system for locating points on the earth's surface, for the latitude and longitude of a point on the earth are really the *spherical coördinates* of the point, the two reference circles being the equator and the zero meridian (usually the meridian of Greenwich). Thus, in the figure on p. 231, we may consider the spherical coördinates of Boston to be the arcs ML (west longitude) and LB (north latitude); and of Cape Town the spherical coördinates would be the arcs MT (east longitude) and TC (south latitude). Similarly, we have systems of spherical coördinates for determining the position of a point on the celestial sphere, and we shall now take up the study of the more important of these.



29. The horizon and meridian system. In this case the two fixed and mutually perpendicular great circles of reference are the *horizon* of the observer ($SHWNE$) and his *meridian* (SM_2ZPN), and the spherical coördinates of a heavenly body are its *altitude* and *azimuth.*

The **altitude** of a heavenly body is its angular distance above the horizon measured on a vertical circle from 0° to 90°.* Thus the altitude of the sun M is the arc HM. The distance of a heavenly body from the zenith is called its *zenith distance* (ZM in the figure), and it is evidently the complement of its altitude. The altitude of the zenith is 90°. The altitude of the sun at sunrise or sunset is zero.

The **azimuth** of a heavenly body is the angle between its vertical circle and the meridian of the observer. This angle is usually

* At sea the altitude is usually measured by the sextant, while on land a surveyor's transit is used.

measured along the horizon from the south point westward to the foot of the body's vertical circle.* Thus the azimuth of the sun M is the angle SZH, which is measured by the arc SH. The azimuth of the sun at noon is zero and at midnight 180°. The azimuth of a star directly west of an observer is 90°, of one north 180°, and of one east 270°.

Knowing the azimuth and altitude (spherical coördinates) of a heavenly body, we can locate it on the celestial sphere as follows. From the south point of the horizon, as S (which may be considered the origin of coördinates, since it is an intersection of the reference circles), lay off the azimuth, as SH. Then on the vertical circle passing through H lay off the altitude, as HM. The body is then located at M.

Ex. 1. In each of the following examples draw a figure of the celestial sphere and locate the body from the given spherical coördinates.

	Azimuth	*Altitude*		*Azimuth*	*Altitude*
(a)	45°	45°	(j)	0°	0°
(b)	60°	30°	(k)	180°	0°
(c)	90°	60°	(l)	0°	90°
(d)	120°	75°	(m)	90°	0°
(e)	180°	55°	(n)	270°	0°
(f)	225°	0°	(o)	360°	0°
(g)	300°	60°	(p)	330°	45°
(h)	315°	15°	(q)	75°	75°
(i)	178°	82°	(r)	90°	90°

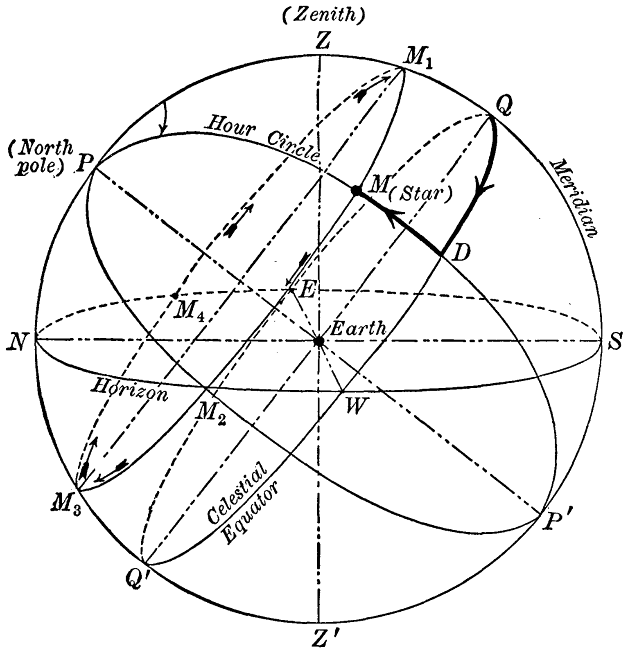
Since any two places on the earth have, in general, different meridians and different horizons, it is evident that this system of spherical coördinates is purely local. The sun rises at M_1 on the eastern horizon (altitude zero), mounts higher and higher in the sky, on a circle ($M_1M_2M_3$) parallel to the celestial equator, until it reaches the observer's meridian M_2 (at noon, when its altitude is a maximum), then sinks downward to M_3 and sets on the western horizon.

Similarly, for any other heavenly body, so that all are continually changing their altitudes and azimuths. To an observer having the zenith shown in the figure, a star in the northern sky near the north pole will not set at all, and to the same observer a star near the south pole will not rise at all. If its path for one day were traced on the celestial sphere, it would be a circle (as ABC) with its center in the polar axis and lying in a plane parallel to the plane of the equator.

* That is, azimuth is measured from 0° to 360° clockwise.

30. The equator and meridian system. In this case the two fixed and mutually perpendicular great circles of reference are the *celestial equator* ($EQDWQ'$) and the *meridian* of the observer ($NPZQSP'Z'Q'$); and the spherical coördinates of a heavenly body are its *declination* and *hour angle.*

The **declination** of a heavenly body is its angular distance north or south of the celestial equator measured on the hour circle of the body from 0° to 90°.* Thus, in the figure, the arc DM is a measure of the north declination of the star M. *North declination* is always considered *positive* and *south declination negative.* Hence the declination of the north pole is + 90°, while that of the south pole is − 90°.



The declinations of the sun, moon, and planets are continually changing, but the declination of a fixed star changes by an exceedingly small amount in the course of a year. The angular distance of a heavenly body from the north celestial pole, measured on the hour circle of the body, is called its *north polar distance* (PM in figure). The north polar distance of a star is evidently the complement of its declination.

The **hour angle** of a heavenly body is the angle between the meridian of the observer and the hour circle of the star measured *westward* from the meridian from 0° to 360°. Thus, in the figure, the hour angle of the star M is the angle QPD (measured by the arc QD). This angle is commonly used as a measure of time, hence the name *hour angle.* Thus the star M makes a complete circuit in 24 hours; that is, the hour angle QPD continually increases at the uniform rate of 360° in 24 hours, or 15° an hour. For this reason the hour angle of a heavenly body is usually reckoned in hours from

* The declinations of the sun, moon, planets, and some of the fixed stars, for any time of the year, are given in the *Nautical Almanac* or *American Ephemeris*, published by the United States government.

0 to 24, one hour being equal to 15°.* When the star is at M_1 (on the observer's meridian) its hour angle is zero. Then the hour angle increases until it becomes the angle M_1PM (when the star is at M). When the star sets on the western horizon its hour angle becomes M_1PM_2. Twelve hours after the star is at M_1 it will be at M_3, when its hour angle will be 180° (= 12 hours). Continuing on its circuit, the star rises at M_4 and finally reaches M_1, when its hour angle has become 360° (= 24 hours), or 0° again.

Knowing the hour angle and declination (spherical coördinates) of a heavenly body, we can locate it on the celestial sphere as follows. From the point, as Q, where the reference circles intersect, lay off the hour angle (or arc), as QD. Then on the hour circle passing through D lay off the declination, as DM. The body is then located at M.

Ex. 1. In each of the following examples draw a figure of the celestial sphere and locate the body from the given spherical coördinates.

	Hour angle	*Declination*		*Hour angle*	*Declination*
(a)	45°	N. 30°	(j)	60°	S. 45°
(b)	60°	N. 60°	(k)	0°	0°
(c)	90°	S. 45°	(l)	180°	0°
(d)	120°	S. 30°	(m)	90°	N. 90°
(e)	180°	N. 50°	(n)	270°	0°
(f)	5 hr.	N. 75°	(o)	12 hr.	S. 10°
(g)	15 hr.	− 25°	(p)	3 hr.	+ 80°
(h)	6 hr.	+ 79°	(q)	9 hr.	− 45°
(i)	0 hr.	− 90°	(r)	20 hr.	+ 60°

31. Practical applications. Among the practical applications of Astronomy the most important are:

(*a*) *To determine the position of an observer on the surface of the earth* (*i.e. his latitude and longitude*).

(*b*) *To determine the meridian of a place on the surface of the earth.*

(*c*) *To ascertain the exact time of day at the place of the observer.*

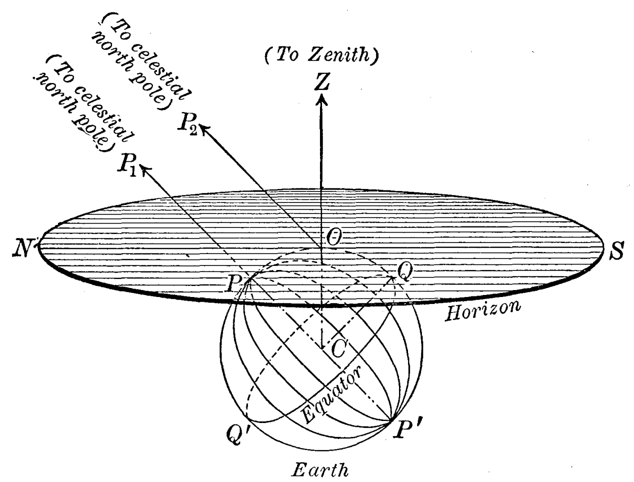
(*d*) *To determine the position of a heavenly body.*

The first of these, when applied to the determination of the place of a ship at sea, is the problem to which Astronomy mainly owes its economic importance. National astronomical observatories have been

* On account of the yearly revolution of the earth about the sun, it takes the sun about 4 minutes longer to make the circuit than is required by any particular fixed star. Hence the solar day is about 4 minutes longer than the sidereal (star) day, but each is divided into 24 hours; the first giving hours of ordinary clock time, while the second gives sidereal hours, which are used extensively in astronomical work. When speaking of the sun's hour angle it shall be understood that it is measured in hours of ordinary clock time, while the hour angle of a fixed star is measured in sidereal hours. In either case 1 hour = 15°.

established, and yearly nautical almanacs are being published by the principal nations controlling the commerce of the world, in order to supply the mariner with the data necessary to determine his position accurately and promptly.

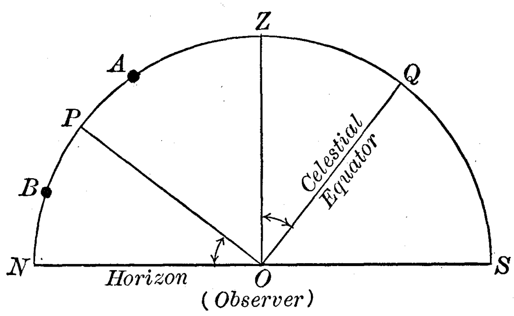
32. Relation between the observer's latitude and the altitude of the celestial pole. To an observer on the earth's equator (latitude zero) the pole star is on the horizon; that is, the altitude of the star is zero. If the observer is traveling northward, the pole star will gradually rise; that is, the latitude of the observer and the altitude of the star are both increasing. Finally, when the observer reaches the north pole of the earth his latitude and the altitude of the star have both increased to 90°. The place of the pole in the sky then



depends in some way on the observer's latitude, and we shall now prove that **the altitude of a celestial pole is equal to the latitude of the observer.**

Let O be the place of observation, say some place in the northern hemisphere; then the angle QCO (or arc QO) measures its north latitude. Produce the earth's axis CP until it pierces the celestial sphere at the celestial north pole. A line drawn from O in the direction (as OP_2) of the celestial north pole will be parallel to CP_1, since the celestial north pole is at an unlimited distance from the earth (see § 27, p. 235). The angle NOP_2 measures the altitude of the north pole. But CO is perpendicular to ON and CQ is perpendicular to OP_2 (since it is perpendicular to the parallel line CP_1); hence the angles NOP_2 and QCO are equal, and we find that *the altitude of the pole as observed at O is equal to the latitude of O.*

33. To determine the latitude of a place on the surface of the earth. If we project that part of the celestial sphere which lies above the horizon on the plane of the observer's celestial meridian, the horizon will be projected into a line (as NS), and the upper half of the celestial equator will also be projected into a line (as OQ). From the last section we know that the latitude of the observer equals the altitude of the elevated celestial pole (arc NP in figure), or, what amounts to the same thing, equals the angular distance between the zenith and the celestial equator (arc ZQ in figure). If then the elevated pole could be seen as a definitely marked point in the sky, the observer's latitude would be found by simply measuring the angular distance of that pole above the horizon. But there are no fixed stars visible at the exact points where the polar axis pierces the celestial sphere, the so-called polar star being about $1\frac{1}{4}°$ from the celestial north pole. Following are some methods for determining the latitude of a place on the surface of the earth.



First method. *To determine latitude by observations on circumpolar stars.* The most obvious method is to observe with a suitable instrument the altitude of some star near the pole (so near the pole that it never sets; as, for instance, the star whose path in the sky is shown as the circle ABC in figure, p. 238) at the moment when it crosses the meridian above the pole, and again 12 hours later, when it is once more on the meridian but below the pole. In the first case its elevation will be the greatest possible; in the second, the least possible. *The mean of the two observed altitudes is evidently the latitude of the observer.* Thus, in the figure on this page, if NA is the maximum altitude and NB the minimum altitude of the star, then

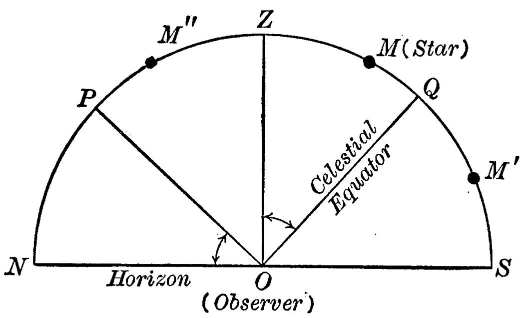
$$\frac{NA + NB}{2} = NP = \text{altitude of pole}$$
$$= \text{latitude of place of observation.}$$

Ex. 1. The maximum altitude of a star near the pole star was observed to be $54° 16'$, and 12 hours later its minimum altitude was observed to be $40° 24'$. What is the latitude of the place of observation?

Solution. $54° 16' + 40° 24' = 94° 40'$.

Therefore $\dfrac{94° 40'}{2} = 47° 20' =$ altitude of north pole
$=$ north latitude of place of observation.

Second method. *To determine latitude from the meridian altitude of a celestial body whose declination is known.* The altitude of a star M is measured when it is on the observer's meridian. If we subtract this meridian altitude (arc SM in figure) from 90°, we get the star's zenith distance (ZM). In the Nautical Almanac we now look up the star's declination at the same instant; this gives us the arc QM. Adding the declination of the star to its zenith distance, we get



$$QM + MZ = QZ = NP = \text{altitude of pole} = \text{latitude of place.}$$

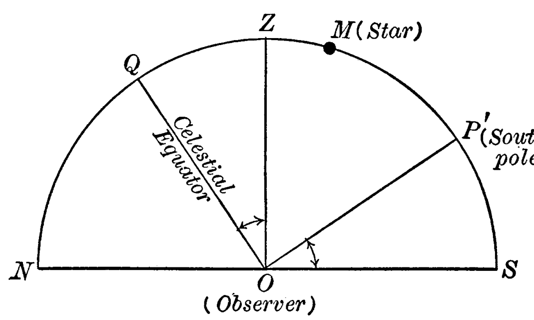
Therefore, when the observer is on the northern hemisphere and the star is on the meridian south of zenith,

*North latitude = zenith distance + declination.**

If the star is on the meridian between the zenith and the pole (as at M'' †), we will have

$$\begin{aligned} \textit{North latitude} &= NP = ZQ = QM'' - ZM'' \\ &= \textit{declination} - \textit{zenith distance.} \end{aligned}$$

If the observer is on the southern hemisphere and the star M is on his meridian between the zenith and south pole, we would have



$$\begin{aligned} &\textit{South latitude} \\ &= SP' = SM - MP' \\ &= SM - (90° - QM) \\ &= \textit{altitude} - \textit{co-declination}, \end{aligned}$$

if we consider only the numerical value of the declination.

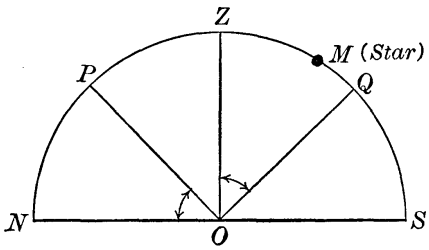
In working out examples the student should depend on the figure rather than try to memorize formulas to cover all possible cases.

Ex. 2. An observer in the northern hemisphere measured the altitude of a star at the instant it crossed his celestial meridian south of zenith, and found it to be 63° 40′. The declination of the star for the same instant was given by the Nautical Almanac as 21° 15′ N. What was the latitude of the observer?

* If the star is south of the celestial equator (as at M'), the same rule will hold, for then the declination is negative (south), and the algebraic sum of the zenith distance and declination will still give the arc QZ.

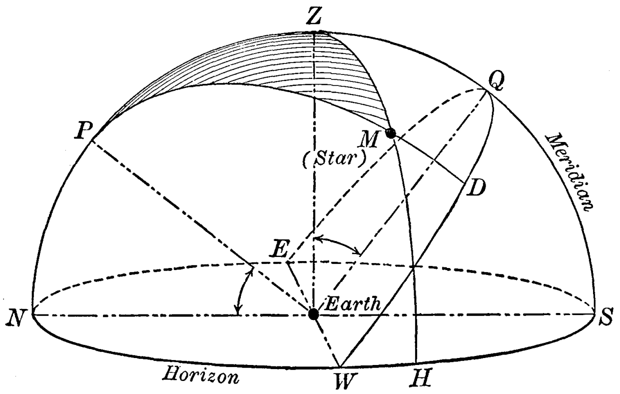
† Maximum altitude, if a circumpolar star.

Solution. Draw the semicircle $NZSO$. Lay off the arc SM = altitude = 63° 40′, which locates the star at M. Since the declination of the star is north, the celestial equator may be located by laying off the arc MQ = declination = 21° 15′ towards the south. The line QO will then be the projection of the celestial equator, and OP, drawn perpendicular to QO, will locate the north pole P.



$$\text{Zenith distance} = ZM = 90° - SM \text{ (alt.)}$$
$$= 90° - 63° 40' = 26° 20'.$$
$$\therefore \text{ North latitude of observer} = NP = ZQ = ZM \text{ (zen. dist.)} + MQ \text{ (dec.)}$$
$$= 26° 20' + 21° 15' = 47° 35'.$$

Third method. *To determine latitude when the altitude, declination, and hour angle of a celestial body are known.* Referring to the astronomical (spherical) triangle PZM, we see that



side MZ
= 90° − HM (alt.)
= co-altitude,

the altitude of the star being found by measurement. Also

side PM
= 90° − DM (dec.)
= co-declination,

the declination of the star being found from the Nautical Almanac.

Angle ZPM = hour angle, which is given. This hour angle will be the local time when the observation is made on the sun. We then have two sides and the angle opposite one of them given in the spherical triangle PZM. Solving this for the side PZ, by Case III, (*a*), p. 224, we get

$$\textit{Latitude of observer} = NP = 90° - PZ.$$

Ex. 3. The declination of a star is 69° 42′ N. and its hour angle 60° 44′. What is the north latitude of the place if the altitude of the star is observed to be 49° 40′?

Solution. Referring to the above figure, we have, in this example,

$$\text{side } MZ = \text{co-alt.} = 90° - 49° 40' = 40° 20',$$
$$\text{side } PM = \text{co-dec.} = 90° - 69° 42' = 20° 18',$$
$$\text{angle } ZPM = \text{hour angle} = 60° 44'.$$

Solving for the side PZ by Case III, (*a*), p. 224, we get side PZ = 47° 9′ = co-lat.

$$\therefore 90° - 47° 9' = 42° 51' = \text{north latitude of place.}$$

The angle MZP is found to be 27° 53′; hence the azimuth of the star (angle SZH) is 180° − 27° 53′ = 152° 7′.

EXAMPLES

1. The following observations for altitude have been made on some north circumpolar star. What is the latitude of each place?

	Maximum altitude	*Minimum altitude*	*North latitude*
(a) New York	50° 46′	30° 40′	*Ans.* 40° 43′
(b) Boston	44° 22′	40° 20′	42° 21′
(c) New Haven	58° 24′	24° 10′	41° 17′
(d) Greenwich	64° 36′	38° 22′	51° 29′
(e) San Francisco	55° 6′	20° 30′	37° 48′
(f) Calcutta	24° 18′	20° 48′	22° 33′

2. In the following examples the altitude of some heavenly body has been measured at the instant when it crossed the observer's celestial meridian. What is the latitude of the observer in each case, the declination being found from the Nautical Almanac?

	Hemisphere	*Meridian altitude*	*Declination*	*Body is*	*Latitude*
(a)	Northern	60°	N. 20°	S. of zenith	*Ans.* 50° N.
(b)	Northern	75° 40′	N. 32° 13′	S. of zenith	46° 33′ N.
(c)	Northern	43° 27′	S. 10° 52′	S. of zenith	35° 41′ N.
(d)	Northern	38° 6′	S. 44° 26′	S. of zenith	7° 28′ N.
(e)	Northern	50°	N. 62°	N. of zenith	22° N.
(f)	Northern	28° 46′	N. 73° 16′	N. of zenith	12° 2′ N.
(g)	Southern	67°	S. 59°	S. of zenith	36° S.
(h)	Southern	45° 26′	S. 81° 48′	S. of zenith	37° 14′ S.
(i)	Southern	72°	S. 8°	N. of zenith	26° S.
(j)	Southern	22° 18′	N. 46° 25′	N. of zenith	21° 17′ S.

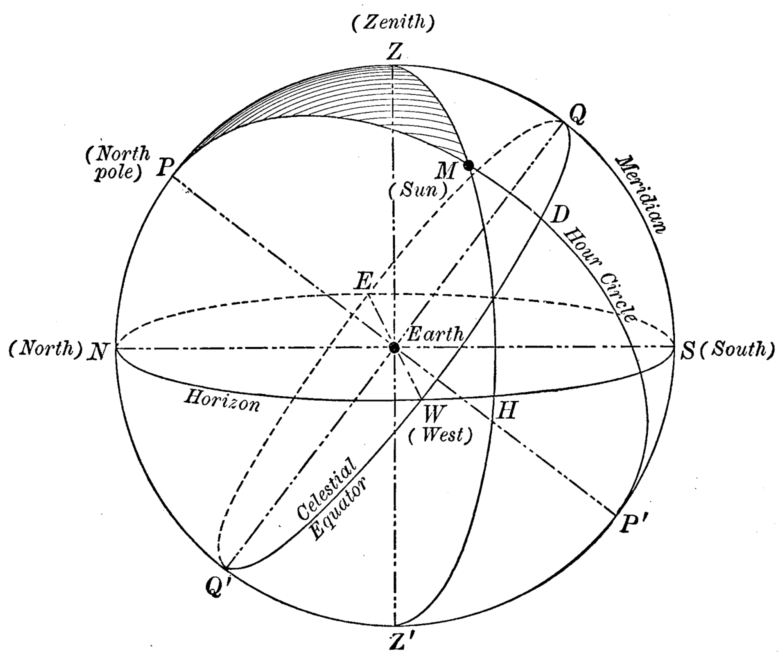
3. In the following examples the altitude of some heavenly body not on the observer's celestial meridian has been measured. The hour angle and declination is known for the same instant. Find the latitude of the observer in each case.

	Hemisphere	*Altitude*	*Declination*	*Hour angle*	*Latitude*
(a)	Northern	40°	N. 10°	50°	*Ans.* 27° 2′ N.
(b)	Northern	15°	S. 8°	65°	35° 38′ N.
(c)	Northern	52°	N. 19°	2 hr.	48° 16′ N.
(d)	Northern	64° 42′	N. 24° 20′	345°	3° 34′ N. or 46° 36′ N.
(e)	Northern	0°	S. 5°	5 hr.	77° 37′ N.
(f)	Northern	25°	0°	21 hr.	53° 18′ N.
(g)	Northern	0°	N. 11° 14′	68° 54′	No solution
(h)	Northern	9° 26′	0°	72° 22′	57° 14′ N.
(i)	Southern	38°	S. 12°	52°	33° 56′ S. or 4° 8′ S.
(j)	Southern	19°	N. 7°	3 hr.	52° 56′ S.
(k)	Southern	46° 18′	S. 15° 23′	326°	49° 14′ S.
(l)	Southern	0°	N. 14°	38°	72° 26′ S.
(m)	Southern	57° 36′	0°	2 hr.	12° 50′ S.

34. To determine the time of day. A very simple relation exists between the hour angle of the sun and the time of day at any place. The sun appears to move from east to west at the uniform rate of 15° per hour, and when the sun is on the meridian of a place it is apparent noon at that place. Comparing,

Hour angle of sun	*Time of day*
0°	Noon
15°	1 P.M.
30°	2 P.M.
45°	3 P.M.
90°	6 P.M.
180°	Midnight
195°	1 A.M.
210°	2 A.M.
270°	6 A.M.
300°	8 A.M.
360°	Noon

The hour angle of the sun M is the angle at P in the astronomical (spherical) triangle PZM. We may find this hour angle (time of



day) by solving the astronomical triangle for the angle at P, provided we know three other elements of the triangle.

DM = declination of sun, and is found from the Nautical Almanac.

$\therefore$ *Side* $PM = 90° - DM$ = *co-declination of sun.*

HM = altitude of sun, and is found by measuring the angular distance of the sun above the horizon with a sextant or transit.

$\therefore$ *side* $MZ = 90° - HM$ = *co-altitude of sun.*

NP = altitude of the celestial pole
= latitude of the observer (p. 243).

$\therefore$ *Side* $PZ = 90° - NP$ = *co-latitude of observer.*

Hence we have

Rule for determining the time of day at a place whose latitude is known, when the declination and altitude of the sun at that time and place are known.

First step. *Take for the three sides of a spherical triangle*

the co-altitude of the sun,
the co-declination of the sun,
the co-latitude of the place.

Second step. *Solve this spherical triangle for the angle opposite the first-mentioned side. This will give the* ***hour angle in degrees*** *of the sun, if the observation is made in the afternoon. If the observation is made in the forenoon, the hour angle will be* ***360° — the angle found.***

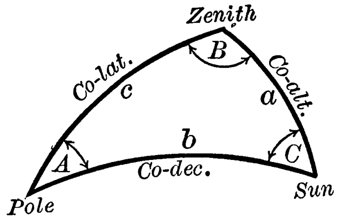
Third step. *When the observation is made in the afternoon the time of day will be*

$$\frac{\text{hour angle}}{15} \text{ P.M.}$$

When the observation is made in the forenoon the time of day will be

$$\left(\frac{\text{hour angle}}{15} - 12\right) \text{ A.M.}$$

Ex. 1. In New York (lat. 40° 43′ N.) the sun's altitude is observed to be 30° 40′. Having given that the sun's declination is 10° N. and that the observation is made in the afternoon, what is the time of day?



Solution. First step. Draw the triangle.

Side a = co-alt. = $90° - 30° 40' = 59° 20'$.
Side b = co-dec. = $90° - 10° = 80°$.
Side c = co-lat. = $90° - 40° 43' = 49° 17'$.

Second step. As we have three sides given, the solution of this triangle comes under Case I, (a), p. 217. But as we only want the angle A (hour angle), some labor may be saved by using one of the formulas **(18)**, **(19)**, **(20)**, pp. 211, 212. Let us use **(18)**,

$$\begin{array}{rl|l} a = & 59^\circ 20' & \\ b = & 80^\circ & \sin \frac{1}{2}\alpha = \sqrt{\dfrac{\sin s \sin (s-a)}{\sin b \sin c}}, \\ c = & \underline{49^\circ 17'} & \\ 2s = & 188^\circ 37' & \\ s = & 94^\circ 19'. & \log \sin \frac{1}{2}\alpha = \frac{1}{2}[\log \sin s + \log \sin (s-a) - \{\log \sin b + \log \sin c\}]. \\ s - a = & 34^\circ 59'. & \end{array}$$

$$\begin{array}{rl} \log \sin s = & 9.9988 \\ \log \sin (s-a) = & \underline{9.7584} \\ \log \text{numerator} = & 19.7572 \\ \log \text{denominator} = & \underline{19.8731} \\ & 9.8841 \\ & 2\,|\underline{19.8841} \\ \log \sin \frac{1}{2}\alpha = & 9.9421 \\ \frac{1}{2}\alpha = & 61^\circ 4'. \\ \alpha = & 122^\circ 8'. \end{array} \qquad \begin{array}{rl} \log \sin b = & 9.9934 \\ \log \sin c = & \underline{9.8797} \\ \log \text{denominator} = & 19.8731 \end{array}$$

$$\therefore\ A = 180^\circ - \alpha = 57^\circ 52' = \text{hour angle of sun.}$$

Third step. Time of day $= \dfrac{\text{hour angle}}{15}$ P.M. = 3 hr. 51 min. P.M. *Ans.*

EXAMPLES

1. In Milan (lat. 45° 30′ N.) the sun's altitude at an afternoon observation is 26° 30′. The sun's declination being 8° S., what is the time of day?
Ans. 2 hr. 33 min. P.M.

2. In New York (lat. 40° 43′ N.) a forenoon observation on the sun gives 30° 40′ as the altitude. What is the time of day, the sun's declination being 10° S.? *Ans.* 9 hr. 46 min. A.M.

3. A mariner observes the altitude of the sun to be 60°, its declination at the time of observation being 6° N. If the latitude of the vessel is 12° S., and the observation is made in the morning, find the time of day. *Ans.* 10 hr. 24 min. A.M.

4. A navigator observes the altitude of the sun to be 35° 23′, its declination being 10° 48′ S. If the latitude of the ship is 26° 13′ N., and the observation is made in the afternoon, find the time of day. *Ans.* 2 hr. 45 min. P.M.

5. At a certain place in latitude 40° N. the altitude of the sun was found to be 41°. If its declination at the time of observation was 20° N., and the observation was made in the morning, how long did it take the sun to reach the meridian? *Ans.* 3 hr. 31 min.

6. In London (lat. 51° 31′ N.) at an afternoon observation the sun's altitude is 15° 40′. Find the time of day, given that the sun's declination is 12° S.
Ans. 2 hr. 59 min. P.M.

7. A government surveyor observes the sun's altitude to be 21°. If the latitude of his station is 27° N. and the declination of the sun 16° N., what is the time of day if the observation was made in the afternoon? *Ans.* 4 hr. 57 min. P.M.

8. The captain of a steamship observes that the altitude of the sun is 26° 30′. If he is in latitude 45° 30′ N. and the declination of the sun is 18° N., what is the time of day if the observation was made in the afternoon? *Ans.* 4 hr. 41 min. P.M.

35. To find the time of sunrise or sunset. If the latitude of the place and the declination of the sun is known, we have a special case of the preceding problem; for at sunrise or sunset the sun is on the horizon and its altitude is zero. Hence the co-altitude, which is one side of the astronomical triangle, will be 90°, and the triangle will be a quadrantal triangle (p. 204). The triangle may then be solved by the method of the last section or as a quadrantal triangle.

EXAMPLES

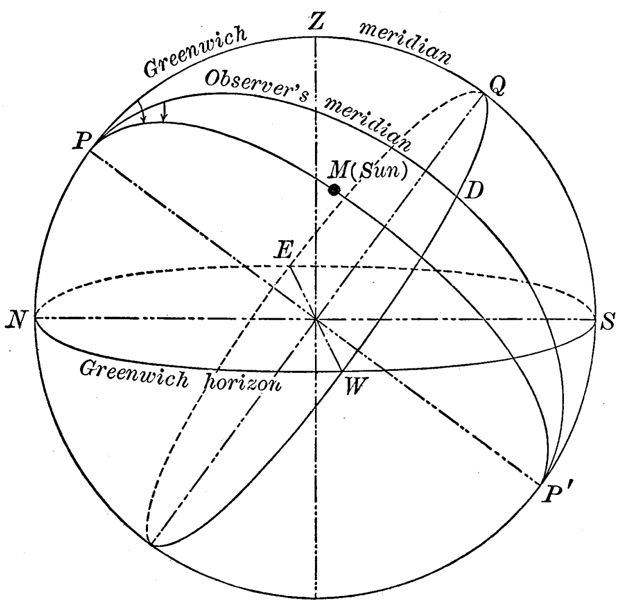
1. At what hour will the sun set in Montreal (lat. 45° 30′ N.), if its declination at sunset is 18° N.? *Ans.* 7 hr. 17 min. P.M.

2. At what hour will the sun rise in Panama (lat. 8° 57′ N.), if its declination at sunrise is 23° 2′ S.? *Ans.* 6 hr. 15 min. A.M.

3. About the first of April of each year the declination of the sun is 4° 30′ N. Find the time of sunrise on that date at the following places:

(a) New York (lat. 40° 43′ N.). *Ans.* 5 hr. 45 min. A.M.
(b) London (lat. 51° 31′ N.). 5 hr. 37 min. A.M.
(c) St. Petersburg (lat. 60° N.). 5 hr. 29 min. A.M.
(d) New Orleans (lat. 29° 58′ N.). 5 hr. 50 min. A.M.
(e) Sydney (lat. 33° 52′ S.). 5 hr. 48 min. A.M.

36. To determine the longitude of a place on the earth. From the definition of terrestrial longitude given on p. 231 it is evident that the meridians on the earth are projected into hour circles on the celestial sphere. Hence the same angle (or arc) which measures the angle between the celestial meridians (hour circles) of the place of observation and of Greenwich may be taken as a measure of the longitude of the place. Thus, in the figure, if PQP' is the meridian (hour circle) of Greenwich and PDP' the meridian (hour circle) of the place of observation, then the angle QPD (or arc QD) measures the west longitude of the place. If PMP' is the hour circle of the sun, it is evident that



angle QPM = hour angle of sun for Greenwich
= local time at Greenwich;

angle DPM = hour angle of sun for observer
= local time at place of observation.

Also, angle QPM − angle DPM = angle QPD = longitude of place.

Hence *the longitude of the place of observation equals the difference* of local times between the standard meridian and the place in question.* Or, in general, we have the following

Rule for finding longitude: *The observer's longitude is the amount by which noon at Greenwich is earlier or later than noon at the place of observation. If Greenwich has the earlier time, the longitude of the observer is east; if it has the later time, then the longitude is west.*

We have already shown (p. 248) how the observer may find his own local time. It then remains *to determine the Greenwich time without going there.* The two methods which follow are those in general use.

First method. *Find Greenwich time by telegraph (wire or wireless).* By far the best method, whenever it is available, is to make a direct telegraphic comparison between the clock of the observer and that of some station the longitude of which is known. The difference between the two clocks will be the difference in longitude of the two places.

Ex. 1. The navigator on a battleship has determined his local time to be 2 hr. 25 min. P.M. By wireless he finds the mean solar time at Greenwich to be 4 hr. 30 min. P.M. What is the longitude of the ship?

Solution. Greenwich having the later time,

	4 hr. 30 min.
	2 hr. 25 min.
Subtracting,	2 hr. 5 min. = west longitude of the ship.

Reducing this to degrees and minutes of arc,

	2 hr. 5 min.
	15
Multiplying,	31° 15′ = west longitude of ship.

Second method. *Find Greenwich time from a Greenwich chronometer.* The chronometer is merely a very accurate watch. It has been set to Greenwich time at some place whose longitude is known, and thereafter keeps that time wherever carried.

* This difference in time is not taken greater than 12 hours. If a difference in time between the two places is calculated to be more than 12 hours, we subtract it from 24 hours and use the remainder instead as the difference.

Ex. 2 An exploring party have calculated their local time to be 10 hr. A.M. The Greenwich chronometer which they carry gives the time as 8 hr. 30 min. A.M. What is their longitude?

Solution. Greenwich has here the earlier time.

10 hr.
8 hr. 30 min.

Subtracting, 1 hr. 30 min. = 22° 30′ = east longitude.

EXAMPLES

1. In the following examples we have given the local time of the observer and the Greenwich time at the same instant. Find the longitude of the observer in each case.

	Observer's local time	*Corresponding Greenwich time*	*Longitude of observer*
(a)	Noon.	3 hr. 30 min. P.M.	*Ans.* 52° 30′ W.
(b)	Noon.	7 hr. 20 min. A.M.	70° E.
(c)	Midnight.	10 hr. 15 min. P.M.	26° 15′ E.
(d)	4 hr. 10 min. P.M.	Noon.	62° 30′ E.
(e)	8 hr. 25 min. A.M.	Noon.	53° 45′ W.
(f)	9 hr. 40 min. P.M.	Midnight.	35° W.
(g)	2 hr. 15 min. P.M.	11 hr. 20 min. A.M.	43° 45′ E.
(h)	10 hr. 26 min. A.M.	5 hr. 16 min. A.M.	77° 30′ E.
(i)	1 hr. 30 min. P.M.	7 hr. 45 min. P.M.	93° 45′ W.
(j)	Noon.	Midnight.	180° W. or E.
(k)	6 hr. P.M.	6 hr. A.M.	180° E. or W.
(l)	5 hr. 45 min. A.M.	7 hr. 30 min. P.M.	153° 45′ E.
(m)	10 hr. 55 min. P.M.	8 hr. 35 min. A.M.	145° W.

2. If the Greenwich time is 9 hr. 20 min. P.M., January 24, at the same instant that the time is 3 hr. 40 min. A.M., January 25, at the place of observation, what is the observer's longitude? *Ans.* 95° E.

3. The local time is 4 hr. 40 min. A.M., March 4, and the corresponding Greenwich time is 8 hr. P.M., March 3. What is the longitude of the place?

Ans. 130° E.

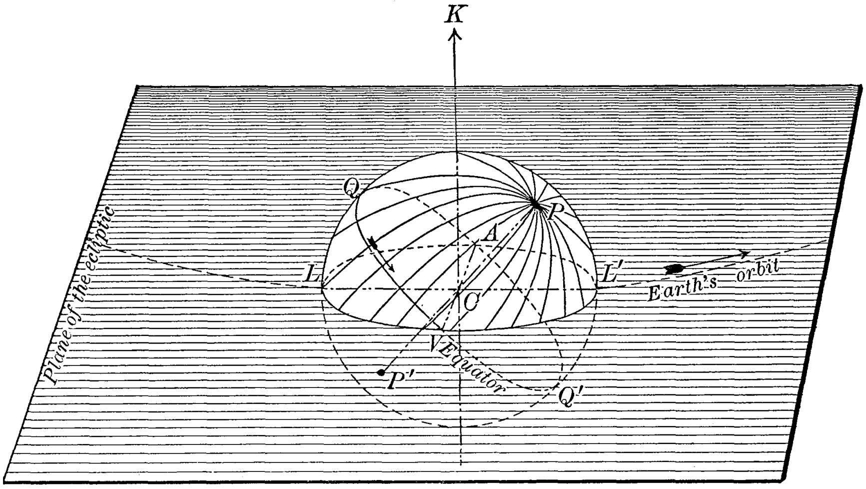
4. In the following examples we have given the local time of the observer and the local time at the same instant of some other place whose longitude is known. Find the longitude of the observer in each case.

	Observer's local time	*Corresponding time and longitude of the other place*	*Longitude of observer*
(a)	2 hr. P.M.	5 hr. P.M. at Havana (long. 82° 23′ W.)	*Ans.* 127° 23′ W.
(b)	10 hr. A.M.	3 hr. P.M. at Yokohama (long. 139° 41′ E.)	64° 41′ E.
(c)	5 hr. 20 min. P.M.	11 hr. 30 min. P.M. at Glasgow (long. 4° 16′ W.)	96° 46′ W.
(d)	8 hr. 25 min. A.M.	6 hr. 35 min. A.M. at Vera Cruz (long. 96° 9′ W.)	68° 39′ W.
(e)	9 hr. 45 min. P.M.	Midnight at Batavia (long. 106° 52′ E.)	73° 7′ E.
(f)	7 hr. 40 min. P.M.	Noon at Gibraltar (long. 5° 21′ W.)	109° 39′ E.
(g)	4 hr. 50 min. P.M.	Noon at Auckland (long. 174° 50′ E.)	112° 40′ W.

5. What is the longitude of each place mentioned in the examples on p. 249, the Greenwich time for the same instant being given below?

Example, p. 249	*Greenwich time*	*Longitude of place*
(a) Ex. 3	2 hr. 12 min. P.M.	*Ans.* 57° W. long. (vessel)
(b) Ex. 4	4 hr. 52 min. P.M.	31° 45′ W. long. (vessel)
(c) Ex. 5	5 hr. 9 min. A.M.	50° E. long. (observer)
(d) Ex. 7	10 hr. 33 min. P.M.	84° W. long. (surveyor)
(e) Ex. 8	6 hr. 25 min. P.M.	26° W. long. (ship)

37. The ecliptic and the equinoxes. The earth makes a complete circuit around the sun in one year. To us, however, it appears as if the sun moved and the earth stood still, the (apparent) yearly path of the sun among the stars being a great circle of the celestial sphere which we call the **ecliptic.** Evidently the plane of the earth's orbit



cuts the celestial sphere in the ecliptic. The plane of the equator and the plane of the ecliptic are inclined to each other at an angle of about $23\frac{1}{2}°$ ($= e$), called the *obliquity of the ecliptic* (angle LVQ in figure).

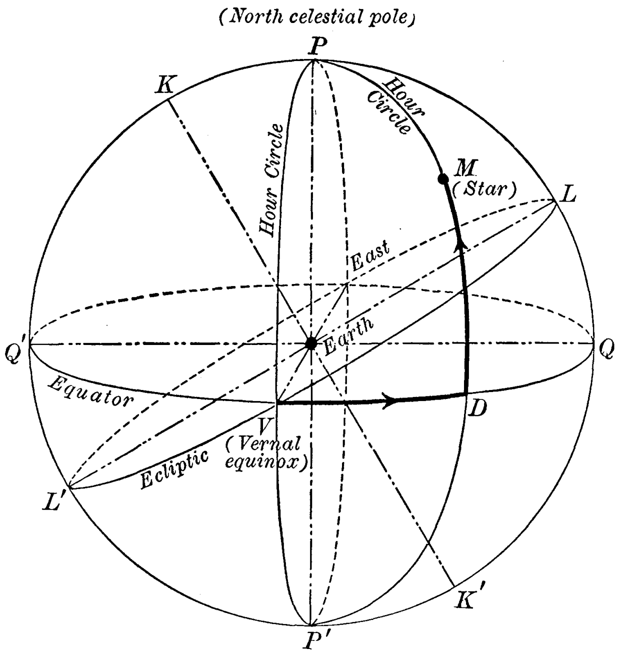
The points where the ecliptic intersects the celestial equator are called the **equinoxes.** The point where the sun crosses the celestial equator when moving northward (in the spring, about March 21) is called the **vernal equinox,** and the point where it crosses the celestial equator when moving southward (in the fall, about September 21) is called the **autumnal equinox.**

If we project the points V and A in our figure on the celestial sphere, the point V will be projected in the vernal equinox and the point A in the autumnal equinox.

38. The equator and hour circle of vernal equinox system.* The two fixed and mutually perpendicular great circles of reference are in

* Sometimes called the *equator system.*

this case the celestial equator (QVQ') and the hour circle of the vernal equinox (PVP'), also called the *equinoctial colure ;* and the spherical coördinates of a heavenly body are its *declination* and *right ascension.*



The **declination** of a heavenly body has already been defined on p. 240 as its angular distance north or south of the celestial equator measured on the hour circle of the body from 0° to 90°, positive if north and negative if south. In the figure DM is the north declination of the star M.

The **right ascension** of a heavenly body is the angle between the hour circle of the body and the hour circle of the vernal equinox measured *eastward* from the latter circle from 0° to 360°, or in hours from 0 to 24. In the figure, the angle VPD (or the arc VD) is the right ascension of the star M. The right ascensions of the sun, moon, and planets are continually changing.* The angle LVQ ($= e$) is the obliquity of the ecliptic ($= 23\frac{1}{2}°$).

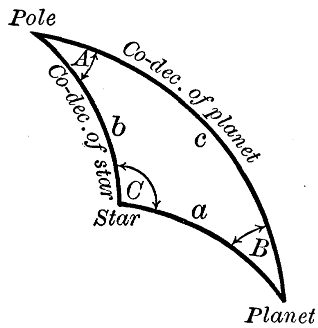
Ex. 1. In each of the following examples draw a figure of the celestial sphere and locate the body from the given spherical coördinates.

Right ascension	*Declination*	*Right ascension*	*Declination*
(a) 0°	0°	(j) 90°	0°
(b) 180°	0°	(k) 270°	0°
(c) 90°	N. 90°	(l) 90°	S. 90°
(d) 45°	N. 45°	(m) 45°	S. 45°
(e) 60°	N. 60°	(n) 90°	S. 30°
(f) 120°	+ 30°	(o) 240°	+ 60°
(g) 300°	– 60°	(p) 330°	– 45°
(h) 12 hr.	+ 45°	(q) 6 hr.	+ 15°
(i) 20 hr.	0°	(r) 9 hr.	– 75°

* The right ascensions of the sun, moon, and planets may be found in the Nautical Almanac for any time of the year.

Ex. 2. The right ascension of a planet is 10 hr. 40 min. and its declination S. 6°. Find the angular distance from this planet to a fixed star whose right ascension is 3 hr. 20 min. and declination N. 48°.

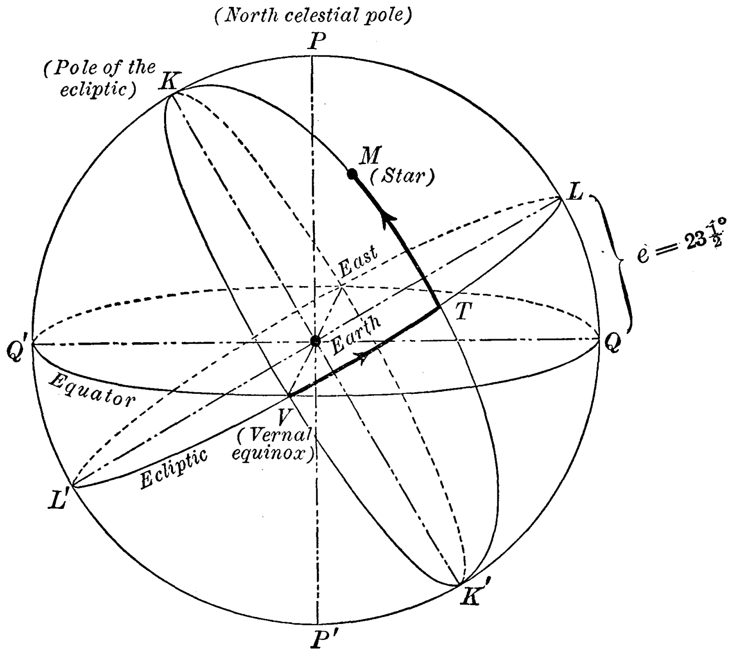
Solution. Locate the planet and the star on the celestial sphere. Draw the spherical triangle whose vertices are at the north pole, the planet, and the fixed star. Then



Angle A = difference of right ascensions
= 10 hr. 40 min. − 3 hr. 20 min.
= 7 hr. 20 min. = 110°.
Side b = co-declination of star
$= 90° - 48° = 42°$.
Side c = co-declination of planet
$= 90° - (-6°) = 96°$. To find side a.

As we have two sides and the included angle given, the solution of this triangle comes under Case II, (a), p. 219. Since a only is required, the shortest method is that illustrated on p. 220, the solution depending on the solution of right spherical triangles. On solving, we get $a = 107° 48'$. *Ans.*

39. The system having for reference circles the ecliptic and the great circle *KVK'* passing through the pole of the ecliptic and the vernal



equinox.* The spherical coördinates of a heavenly body in this case are its *latitude and longitude.*†

The **latitude** of a heavenly body is its angular distance north or south of the ecliptic, measured on the great circle passing through

* Sometimes called the *ecliptic system.*

† Sometimes called *celestial latitude and longitude* in contradistinction to the latitude and longitude of places on the earth's surface (*terrestrial latitude and longitude*), which were defined on p. 231, and which have different meanings.

the body and the pole of the ecliptic. Thus, in the figure, the arc TM measures the north latitude of the star M.

The **longitude** of a heavenly body is the angle between the great circle passing through the body and the pole of the ecliptic, and the great circle passing through the vernal equinox and the pole of the ecliptic, measured *eastward* from the latter circle from 0° to 360°. In the figure, the angle VKT (or the arc VT) is the longitude of the star M. The latitudes and longitudes of the sun, moon, and planets are continually changing. The angle LVQ ($= e$) is the obliquity of the ecliptic ($= 23\frac{1}{2}° =$ arc KP).

Since the ecliptic is the apparent yearly path of the sun, the celestial latitude of the sun is always zero. The declination of the sun, however, varies from N. $23\frac{1}{2}°$ ($=$ arc QL) on the longest day of the year in the northern hemisphere (June 21), the sun being then the highest in the sky (at L), to S. $23\frac{1}{2}°$ (arc $Q'L'$) on the shortest day of the year (December 22), the sun being then the lowest in the sky (at L'). The declination of the sun is zero at the equinoxes (March 21 and September 21).

Ex. 1. In each of the following examples draw a figure of the celestial sphere and locate the body from the given spherical coördinates.

Celestial longitude	*Celestial latitude*	*Celestial longitude*	*Celestial latitude*
(a) 0°	0°	(j) 90°	0°
(b) 90°	N. 90°	(k) 180°	0°
(c) 180°	N. 45°	(l) 0°	S. 60°
(d) 270°	0°	(m) 60°	N. 30°
(e) 45°	S. 30°	(n) 120°	N. 45°
(f) 135°	+ 15°	(o) 270°	− 75°
(g) 315°	+ 60°	(p) 30°	− 60°
(h) 6 hr.	− 45°	(q) 9 hr.	0°
(i) 15 hr.	+ 45°	(r) 18 hr.	+ 30°

Ex. 2. Given the right ascension of a star 2 hr. 40 min. and its declination 24° 20′ N., find its celestial latitude and longitude.

Solution. Locate the star on the celestial sphere. Consider the spherical triangle KPM on the next page.

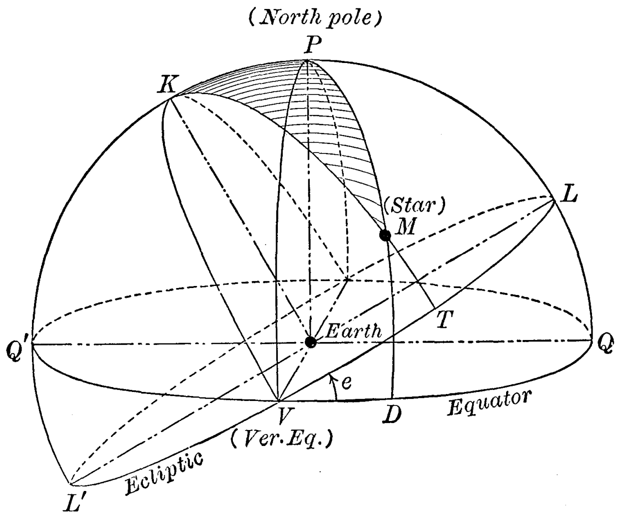
$$\begin{aligned}\text{Angle } KPM &= \angle Q'PV + \angle VPD \\ &= 90° + \text{right ascension} \\ &= 90° + 2 \text{ hr. } 40 \text{ min.} \\ &= 90° + 40° = 130°.\end{aligned}$$

$$\begin{aligned}\text{Side } PM &= \text{co-declension} \\ &= 90° - 24° \, 20' \\ &= 65° \, 40'.\end{aligned}$$

Side $KP = LQ = e = 23° 30'$.

To find side KM = co-latitude of the star,

and angle PKM = co-longitude of the star.



As we have two sides and the included angle given, the solution of this triangle comes under Case II, (*a*), p. 219. Solving, we get

$$\text{Side } KM = 81° 52' \text{ and } \angle PKM = 44° 52'.$$

$$\therefore\ 90° - KM = 90° - 81° 52' = 8° 8' = TM = \text{latitude of star,}$$

and $$90° - \angle PKM = 90° - 44° 52' = 45° 8' = VT = \text{longitude of star.}$$

EXAMPLES

1. Find the distance in degrees between the sun and the moon when their right ascensions are respectively 12 hr. 39 min., 6 hr. 56 min., and their declinations are 9° 23′ S., 22° 50′ N. *Ans.* 90°.

2. Find the distance between Regulus and Antares, the right ascensions being 10 hr. and 16 hr. 20 min., and the polar distances 77° 19′ and 116° 6′. *Ans.* 99° 56′.

3. Find the distance in degrees between the sun and the moon when their right ascensions are respectively 15 hr. 12 min., 4 hr. 45 min., and their declinations are 21° 30′ S., 5° 30′ N. *Ans.* 154° 19′.

4. The right ascension of Sirius is 6 hr. 39 min., and his declination is 16° 31′ S.; the right ascension of Aldebaran is 4 hr. 27 min., and his declination is 16° 12′ N. Find the angular distance between the stars. *Ans.* 46° 2′.

5. Given the right ascension of a star 10 hr. 50 min., and its declination 12° 30′ N., find its latitude and longitude. Take $e = 23° 30'$.

Ans. Latitude = 18° 24′ N., longitude = 281° 7′.

6. If the moon's right ascension is 4 hr. 15 min. and its declination 6° 20′ N., what is its latitude and longitude?

Ans. Latitude = 14° 43′ N., longitude = 62° 58′.

7. The sun's longitude was 59° 40′. What was its right ascension and declination? Take $e = 23° 27'$.

Ans. Right ascension = 3 hr. 50 min., declination = 20° 38′ N.

Hint. The latitude of the sun is always zero, since it moves in the ecliptic. Hence in the triangle KPM (figure, p. 257), $KM = 90°$, and it is a quadrantal triangle. This triangle may then be solved by the method explained on p. 204.

8. Given the sun's declination 16° 1′ N., find the sun's right ascension and longitude. Take $e = 23° 27'$.

Ans. Right ascension = 9 hr. 14 min., longitude = 136° 6′.

9. The sun's right ascension is 14 hr. 8 min.; find its longitude and declination. Take $e = 23° 27'$. *Ans.* Longitude = 214° 16′, declination = 12° 56′ S.

10. Find the length of the longest day of the year in latitude 42° 17′ N.

Ans. 15 hr. 6 min.

Hint. This will be the time from sunrise to sunset when the sun is the highest in the sky, that is, when its declination is 23° 27′ N.

11. Find the length of the shortest day in lat. 42° 17′ N. *Ans.* 8 hr. 54 min.

Hint. The sun will then be the lowest in the sky, that is, its declination will be 23° 27′ S.

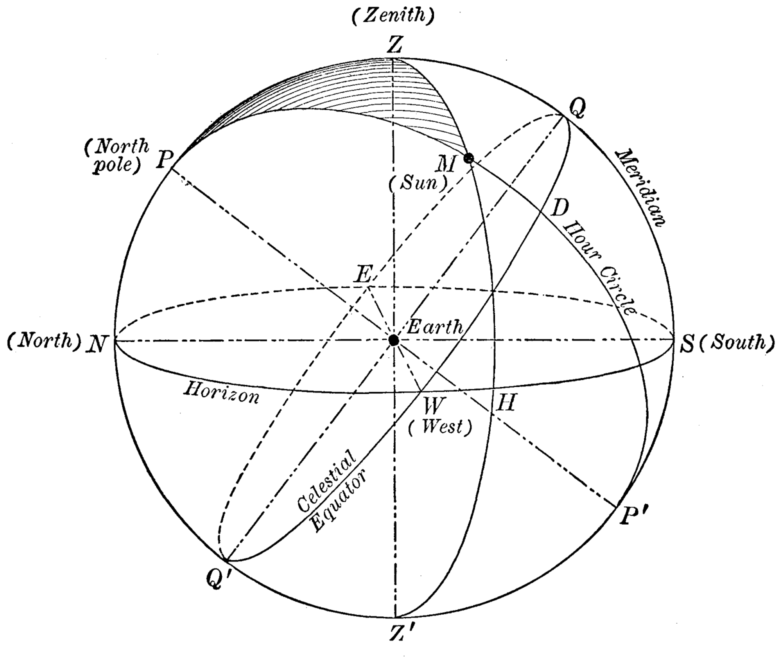
12. Find the length of the longest day in New Haven (lat. 41° 19′ N.). Take $e = 23° 27'$. *Ans.* 15 hr.

13. Find the length of the shortest day in New Haven. *Ans.* 9 hr.

14. Find the length of the longest day in Stockholm (lat. 59° 21′ N.). Take $e = 23° 27'$. *Ans.* 18 hr. 16 min.

15. Find the length of the shortest day in Stockholm. *Ans.* 5 hr. 48 min.

40. The astronomical triangle. We have seen that many of our most important astronomical problems depend on the solution of



the astronomical triangle PZM. In any such problem the first thing to do is to ascertain which parts of the astronomical triangle are given or can be obtained directly from the given data, and which are required. The different magnitudes which may enter into such problems are

$$HM = \text{altitude of the heavenly body,}$$
$$DM = \text{declination of the heavenly body,}$$
$$\text{angle } ZPM = \text{hour angle of the heavenly body,}$$
$$\text{angle } SZM = \text{azimuth of the heavenly body,}$$
$$NP = \text{altitude of the celestial pole}$$
$$= \text{latitude of the observer.}$$

As parts of the astronomical triangle PZM we then have

$$\text{side } MZ = 90° - HM = \text{co-altitude,}$$
$$\text{side } PM = 90° - DM = \text{co-declination,}$$
$$\text{side } PZ = 90° - NP = \text{co-latitude,}$$
$$\text{angle } ZPM = \text{hour angle,}$$
$$\text{angle } PZM = 180° - \text{azimuth (angle } SZM).^*$$

The student should be given practice in picking out the known and unknown parts in examples involving the astronomical triangle, and in indicating the case under which the solution of the triangle comes.

For instance, let us take Ex. 15, p. 261.

Given parts
- Latitude $= 51° \ 32'$ N.
- $\therefore$ side $PZ = 90° - 51° \ 32' = 38° \ 28'$.
- Altitude $= 35° \ 15'$.
- $\therefore$ side $MZ = 90° - 35° \ 15' = 54° \ 45'$.
- Declination $= 21° \ 27'$ N.
- $\therefore$ side $MP = 90° - 21° \ 27' = 68° \ 33'$.

Required: Local time = hour angle = angle ZPM.

Since we have three sides given to find an angle, the solution of the triangle comes under Case I, (*a*), p. 217. This gives angle $ZPM = 59° \ 45' = 3$ hr. 59 min. P.M.

41. Errors arising in the measurement of physical quantities.† Errors of some sort will enter into all data obtained by measurement. For instance, if the length of a line is measured by a steel tape, account must be taken of the expansion due to heat as well as the sagging of the tape under various tensions. Or, suppose the navigator of a ship

* When the heavenly body is situated as in the figure. If the body is east of the observer's meridian, we would have angle $PZM =$ azimuth $- 180°$.

† In this connection the student is advised to read § 93 in Granville's *Plane Trigonometry*.

at sea is measuring the altitude of the sun by means of a sextant. The observed altitude should be corrected for errors due to the following causes:

1. Dip. Owing to the observer's elevation above the sea level (on the deck or bridge of the ship), the observed altitude will be too great on account of the dip (or lowering) of the horizon.

2. Index error of sextant. As no instrument is perfect in construction, each one is subject to a certain constant error which is determined by experiment.

3. Refraction of light. Celestial bodies appear higher than they really are because of the refraction of light by the earth's atmosphere. This refraction will depend on the height of the celestial body above the horizon, and also on the state of the barometer and thermometer, since changes in the pressure and temperature of the air affect its density.

4. Semidiameter of the sun. As the observer cannot be sure where the center of the sun is, the altitude of (say) the lower edge of the sun is observed and to that is added the known semidiameter of the sun for that day found from the Nautical Almanac.

5. Parallax. The parallax of a celestial body is the angle subtended by the radius of the earth passing through the observer, as seen from the body. As viewed from the earth's surface, a celestial body appears lower than it would be if viewed from the center, and this may be shown to depend on the parallax of the body.

We shall not enter into the detail connected with these corrections, as that had better be left to works on Field Astronomy; our purpose here is merely to call the attention of the student to the necessity of eliminating as far as possible the errors that arise when measuring physical quantities.

For the sake of simplicity we have assumed that the necessary corrections have been applied to the data given in the examples found in this book.

MISCELLANEOUS EXAMPLES

1. The continent of Asia has nearly the shape of an equilateral triangle. Assuming each side to be 4800 geographical miles and the radius of the earth to be 3440 geographical miles, find the area of Asia.

Ans. About 13,333,000 sq. mi.

2. The distance between Paris (lat. 48° 50′ N.) and Berlin (lat. 52° 30′ N.) is 472 geographical miles, measured on the arc of a great circle. What time is it at Berlin when it is noon at Paris? *Ans.* 44 min. past noon.

3. The altitude of the north pole is 45°, and the azimuth of a star on the horizon is 135°. Find the polar distance of the star. *Ans.* 60°.

4. What will be the altitude of the sun at 9 A.M. in Mexico City (lat. 19° 25′ N.), if its declination at that time is 8° 23′ N.? *Ans.* 37° 41′.

5. Find the altitude of the sun at 6 hr. A.M. at Munich (lat. 48° 9′ N.) on the longest day of the year. *Ans.* Altitude = 17° 15′.

6. Find the time of day when the sun bears due east and due west on the longest day of the year at St. Petersburg (lat. 59° 56′ N.).

Ans. 6 hr. 58 min. A.M., 5 hr. 2 min. P.M.

7. What is the direction of a wall in lat. 52° 30′ N. which casts no shadow at 6 A.M. on the longest day of the year?

Ans. 75° 11′, reckoned from the north point of the horizon.

8. Find the latitude of the place at which the sun rises exactly in the north-east on the longest day of the year. *Ans.* 55° 45′ N.

9. Find the latitude of the place at which the sun sets at 10 hr. P.M. on the longest day. *Ans.* 63° 23′ N. or S.

10. Given the latitude of the place of observation 52° 30′ N., the declination of a star 38°, its hour angle 28° 17′. Find the altitude of the star.

Ans. Altitude = 65° 33′.

11. Given the latitude of the place of observation 51° 19′ N., the polar distance of a star 67° 59′, its hour angle 15° 8′. Find the altitude and azimuth of the star. *Ans.* Altitude = 58° 22′, azimuth = 27° 48′.

12. Given the declination of a star 7° 54′ N., its altitude 22° 45′, its azimuth 50° 14′. Find the hour angle of the star and the latitude of the observer.

Ans. Hour angle = 45° 41′, latitude = 67° 59′ N.

13. The latitude of a star is 51° N., and its longitude 315°. Find its declination. Take $e = 23° 27'$. *Ans.* Declination = 32° 23′ N.

14. Given the latitude of the observer 44° 50′ N., the azimuth of a star 41° 2′, its hour angle 20°. Find its declination. *Ans.* Declination = 20° 49′ N.

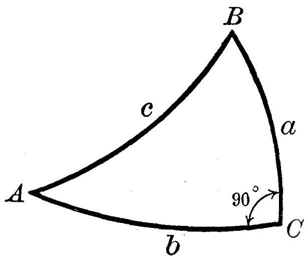
15. Given the latitude of the place of observation 51° 32′ N., the altitude of the sun west of the meridian 35° 15′, its declination 21° 27′ N. Find the local time. *Ans.* 3 hr. 59 min. P.M.

CHAPTER IV

RECAPITULATION OF FORMULAS

SPHERICAL TRIGONOMETRY

42. Right spherical triangles, pp. 196–197.



(1) $\cos c = \cos a \cos b,$

(2) $\sin a = \sin c \sin A,$

(3) $\sin b = \sin c \sin B,$

(4) $\cos A = \cos a \sin B,$

(5) $\cos B = \cos b \sin A,$

(6) $\cos A = \tan b \cot c,$

(7) $\cos B = \tan a \cot c,$

(8) $\sin b = \tan a \cot A,$

(9) $\sin a = \tan b \cot B,$

(10) $\cos c = \cot A \cot B.$

General directions for solving right spherical triangles by Napier's rules of circular parts are given on p. 200.

Spherical isosceles and quadrantal triangles are discussed on p. 204.

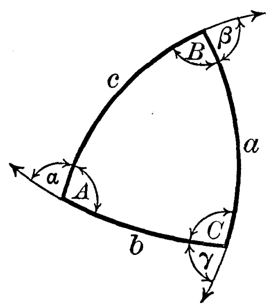
43. Relations between the sides and angles of oblique spherical triangles, pp. 206–216.

$$\alpha = 180° - A, \qquad \beta = 180° - B, \qquad \gamma = 180° - C.$$

$$s = \tfrac{1}{2}(a + b + c), \qquad \sigma = \tfrac{1}{2}(\alpha + \beta + \gamma).$$

d = diameter of inscribed circle.

$\delta = 180°$ − diameter of circumscribed circle.



Law of sines, p. 207.

(11) $$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C},$$

or, $$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.$$

Law of cosines for the sides, p. 209.

(12) $$\cos a = \cos b \cos c - \sin b \sin c \cos \alpha.$$

Law of cosines for the angles, p. 209.

(15) $$\cos\alpha = \cos\beta\cos\gamma - \sin\beta\sin\gamma\cos a.$$

Functions of $\frac{1}{2}\alpha$, $\frac{1}{2}\beta$, $\frac{1}{2}\gamma$ in terms of the sides, pp. 211–213.

(18) $$\sin\tfrac{1}{2}\alpha = \sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}.$$

(19) $$\cos\tfrac{1}{2}\alpha = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}.$$

(20) $$\tan\tfrac{1}{2}\alpha = \sqrt{\frac{\sin s\sin(s-a)}{\sin(s-b)\sin(s-c)}}.$$

(27) $$\tan\tfrac{1}{2}d = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}}.$$

(28) $$\tan\tfrac{1}{2}\alpha = \frac{\sin(s-a)}{\tan\frac{1}{2}d}.$$

(29) $$\tan\tfrac{1}{2}\beta = \frac{\sin(s-b)}{\tan\frac{1}{2}d}.$$

(30) $$\tan\tfrac{1}{2}\gamma = \frac{\sin(s-c)}{\tan\frac{1}{2}d}.$$

Functions of the half sides in terms of α, β, γ, p. 214.

(31) $$\sin\tfrac{1}{2}a = \sqrt{\frac{\sin\sigma\sin(\sigma-\alpha)}{\sin\beta\sin\gamma}}.$$

(32) $$\cos\tfrac{1}{2}a = \sqrt{\frac{\sin(\sigma-\beta)\sin(\sigma-\gamma)}{\sin\beta\sin\gamma}}.$$

(33) $$\tan\tfrac{1}{2}a = \sqrt{\frac{\sin\sigma\sin(\sigma-\alpha)}{\sin(\sigma-\beta)\sin(\sigma-\gamma)}}.$$

(40) $$\tan\tfrac{1}{2}\delta = \sqrt{\frac{\sin(\sigma-\alpha)\sin(\sigma-\beta)\sin(\sigma-\gamma)}{\sin\sigma}}.$$

(41) $$\tan\tfrac{1}{2}a = \frac{\sin(\sigma-\alpha)}{\tan\frac{1}{2}\delta}.$$

(42) $$\tan\tfrac{1}{2}b = \frac{\sin(\sigma-\beta)}{\tan\frac{1}{2}\delta}.$$

(43) $$\tan\tfrac{1}{2}c = \frac{\sin(\sigma-\gamma)}{\tan\frac{1}{2}\delta}.$$

Napier's Analogies, p. 215.

(44) $$\tan\tfrac{1}{2}(a-b) = -\frac{\sin\frac{1}{2}(\alpha-\beta)}{\sin\frac{1}{2}(\alpha+\beta)}\tan\tfrac{1}{2}c.$$

(45) $$\tan\tfrac{1}{2}(a+b) = -\frac{\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)}\tan\tfrac{1}{2}c.$$

(46) $$\tan\tfrac{1}{2}(\alpha-\beta) = -\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\tan\tfrac{1}{2}\gamma.$$

(47) $$\tan\tfrac{1}{2}(\alpha+\beta) = -\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\tan\tfrac{1}{2}\gamma.$$

44. General directions for the solution of oblique spherical triangles, pp. 216–227.

CASE I. (*a*) *Given the three sides, p. 217.*
(*b*) *Given the three angles, p. 218.*

CASE II. (*a*) *Given two sides and their included angle, p. 219.*
(*b*) *Given two angles and their included side, p. 222.*

CASE III. (*a*) *Given two sides and the angle opposite one of them, p. 224.*
(*b*) *Given two angles and the side opposite one of them, p. 226.*

45. Length of an arc of a circle in linear units, p. 228.

(52) $$L = \frac{\pi R N}{180}.$$

$N =$ number of degrees in angle.

46. Area of a spherical triangle, p. 229.

(54) $$\text{Area} = \frac{\pi R^2 E}{180}.$$

$$E = A + B + C - 180^\circ.$$

(55) $$\tan\tfrac{1}{4}E = \sqrt{\tan\tfrac{1}{2}s\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)}.$$

FOUR-PLACE TABLES OF LOGARITHMS

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