

IDEMPOTENT-GENERATED SEMIGROUPS AND PSEUDOVARITIES

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Abstract The operator that constructs the pseudovariety generated by the idempotent-generated semigroups of a given pseudovariety is investigated. Several relevant examples of pseudovarieties generated by their idempotent-generated elements are given, as well as some properties of this operator. Particular attention is paid to the pseudovarieties in $\{J, R, L, DA\}$ concerning this operator and their generator ranks and idempotent-generator ranks.

Keywords: finite semigroup; pseudovariety; operator; idempotent-generated semigroup; generator rank; idempotent generator rank

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1. Introduction

Due to its applications in computer science, the theory of finite semigroups saw significant advancements in the 1960s driven by developments in the theory of finite automata. This connection with finite semigroups was first explored to obtain computational results. In parallel, combinatorial and algebraic properties of finite semigroups were studied. Eilenberg [6] established a correspondence between certain families of rational languages and certain classes of finite semigroups, called *pseudovarieties*, which provided a suitable framework for the bridges between the two theories.

There are many important pseudovarieties, often constructed from others by applying suitable operators. Some natural operators have been extensively studied. In this paper, we introduce a new one which constructs the subpseudovariety generated by the idempotent-generated semigroups of a given pseudovariety.

Several works have been dedicated to idempotent-generated semigroups. It is well known that any finite semigroup embeds into a finite regular idempotent-generated semigroup; this was proved by Howie [8] using full transformations semigroups. Howie [9]

also proved that the full transformations subsemigroup consisting of all order-preserving and contractive full transformations is idempotent generated. Laradji and Umar [10] improved this result and showed that, in fact, every order-preserving and contractive full transformation is expressible as a product of idempotents of the same type and with the same range. The analogous result for the subsemigroup of contractive full transformations also holds [10].

On the other hand, Pastijn [13] proved that every completely simple semigroup embeds into an idempotent-generated one. Furthermore, Petrich [14] presents a concrete model of the embedding due to Pastijn [13] of a semigroup into an idempotent-generated Rees matrix semigroup that preserves some properties.

In this paper, while we do not obtain a complete characterization of the pseudovarieties which are generated by their idempotent-generated semigroups, we prove that many familiar pseudovarieties have this property. The techniques used for this purpose include the representations of free profinite semigroups over R , J and DA due to Almeida and Weil [3], Almeida [1], and Moura [12], respectively. In the cases of R and J , we also observe an alternative approach based on some results concerning transformations of a finite chain due to Pin [15] and Straubing [19] and the results concerning idempotent-generated subsemigroups of full transformations from Howie and from Laradji and Umar. On the other hand, the work of Petrich [14] allows us to show that the pseudovarieties \bar{H} (where H is a pseudovariety of groups), CS and CR also have this property.

The new approach in the case of the pseudovarieties R and J is justified, as it gives a significant improvement in terms of the *generator rank* and the *idempotent generator rank*. More generally, we show that both ranks are infinite for every pseudovariety in the interval $[J, DS]$. We also prove that every semigroup in the subpseudovariety generated by all n -generated members of any of the pseudovarieties J , R , L , DA divides a semigroup in the same pseudovariety generated by at most $n + 1$ idempotents. We compare these results with the works of Umar [20], and Laradji and Umar [10] concerning the ranks and idempotent-ranks of the subsemigroups of all contractive full transformations, and contractive and order-preserving full transformations, respectively. We observe that, in fact, we decrease the number of idempotent generators of the idempotent-generated semigroups when we use the embeddings of the semigroups of R and J presented in this paper.

The paper is organized as follows. In §2, we recall some basics of the theory of pseudovarieties of semigroups and profinite semigroups and we introduce some notation concerning operators on pseudovarieties. We also present a list of the pseudovarieties, and bases of pseudoidentities defining them, that will be used in our study. In §3, we observe some properties of the operator \cdot_E , we determine some pseudovarieties of the form V_E and we give a short introduction to the main question addressed in the paper: what are the pseudovarieties that are generated by their idempotent-generated elements?

We present in the following sections some pseudovarieties having this property: in §4 using the embedding in a Rees matrix semigroup constructed by Petrich, and in §5 using representations of the free profinite semigroup.

Finally, in §6 we determine the generator rank and idempotent generator rank of every pseudovariety in the interval $[J, DS]$ and we also determine a lower bound for

the idempotent generator rank of the subpseudovarieties generated by all n -generated members of any pseudovariety in the interval $[J, DA]$. In combination with the results of § 5, we improve the last result, showing that the lower bound is the exact value in the case of the pseudovarieties J, R, L, DA . To introduce some relevant results in our study, we develop some existing techniques that need to be recalled. Rather than including them in § 2, we briefly introduce them when needed.

2. Preliminaries

We assume that readers are acquainted with notions concerning pseudovarieties of semigroups and profinite semigroups. We briefly recall some basics and we refer the reader to [1, 2, 15] for detailed information.

For a semigroup S , let S^1 be the monoid obtained by adjoining a neutral element 1 to S in case S does not already possess one, and $S^1 = S$ otherwise. We denote by $E(S)$ the set of idempotents of S and by $\langle E(S) \rangle$ the subsemigroup of S generated by $E(S)$. For $s \in S$, s^ω denotes the unique idempotent in the subsemigroup generated by s . We say that a semigroup S divides a semigroup T , and we write $S \prec T$, if there exists a surjective homomorphism of a subsemigroup of T onto S .

A *pseudovariety* of semigroups is a class of finite semigroups that is closed under taking subsemigroups, homomorphic images and finite direct products. Equivalently, a pseudovariety of semigroups is a class of finite semigroups closed under taking division and finite direct products. For example, \mathcal{S} is the pseudovariety of all finite semigroups.

There are many ways to construct new pseudovarieties from known ones, one of which is by applying operators to pseudovarieties. For example, given a pseudovariety V , the following classes of finite semigroups are pseudovarieties:

- EV consists of all $S \in \mathcal{S}$ such that $\langle E(S) \rangle \in V$;
- DV consists of all $S \in \mathcal{S}$ such that, for every regular \mathcal{D} -class D of S , $D \in V$;
- for a pseudovariety H of groups, \bar{H} consists of all $S \in \mathcal{S}$ such that every subgroup G of S belongs to H .

We also have other types of operators that construct new pseudovarieties by describing the generators. The new pseudovariety is then the smallest pseudovariety containing such semigroups. In this way, we introduce the operator \cdot_E , which is the topic of this paper. Given a pseudovariety V , we define V_E as the pseudovariety generated by the subsemigroups generated by the idempotents of each member of V , i.e.

$$V_E = \langle \langle E(S) \rangle \mid S \in V \rangle.$$

Note that $V_E \subseteq V$ as the indicated generators of V_E belong to V .

Because it will be useful in our study, we now present an obvious observation about the subsemigroup generated by a subset of idempotents of a given semigroup.

Lemma 2.1. *Let $S \in \mathcal{S}$ and $X \subseteq E(S)$. Then $\langle E(X) \rangle = \langle X \rangle$. In particular, we have $\langle E\langle E(S) \rangle \rangle = \langle E(S) \rangle$ and $\langle E\langle E(D) \rangle \rangle = \langle E(D) \rangle$ for every regular \mathcal{D} -class D of S .*

A semigroup equipped with a topology for which the multiplication is a continuous function is called a *topological semigroup*. Finite semigroups are endowed with the discrete topology. A topological semigroup S is a *profinite semigroup* (respectively, a *pro- V semigroup*) if it is a compact semigroup which is *residually finite* (respectively, *residually in V*), which means that, for any two distinct elements of S , there exists a continuous homomorphism into a finite semigroup (respectively, into a member of V) that separates them. Equivalently, profinite semigroups are compact *0-dimensional*, which means that the topology has an open basis consisting of clopen sets. The elements of a pseudovariety V are pro- V semigroups.

We denote by $\bar{\Omega}_A V$ the *free pro- V semigroup on A* , which is the unique (up to isomorphism of topological semigroups) pro- V semigroup such that every mapping $\mu: A \rightarrow S$ into a pro- V semigroup S can be extended to a unique continuous homomorphism $\hat{\mu}: \bar{\Omega}_A V \rightarrow S$ such that $\hat{\mu} \circ \iota = \mu$, where $\iota: A \rightarrow \bar{\Omega}_A V$ is the natural generating function (i.e. its image generates a dense subsemigroup of $\bar{\Omega}_A V$). The elements of $\bar{\Omega}_A V$ are called *implicit operations over V* . For $u \in \bar{\Omega}_A V$ the sequence $(u^{n!})_n$ converges and we denote the limit by u^ω , which is the unique idempotent in the closed subsemigroup generated by u .

An equality of the form $u = v$, with $u, v \in \bar{\Omega}_A S$, is called a *pseudoidentity* and $|A|$ is its *arity*. The pseudoidentity is *valid* in a profinite semigroup S , and we write $S \models u = v$, if, for every continuous homomorphism $\varphi: \bar{\Omega}_A S \rightarrow S$, we have $\varphi(u) = \varphi(v)$. It is easy to see that the validity of a pseudoidentity in a finite semigroup is preserved under taking homomorphic images, subsemigroups and finite direct products. Conversely, Reiterman's Theorem [17] says that every pseudovariety is defined by some set of finitary pseudoidentities. We end this section with a list of pseudovarieties that are used in this paper and some corresponding bases of pseudoidentities defining them:

$J = \llbracket (xy)^\omega x = (xy)^\omega = y(xy)^\omega \rrbracket$	\mathcal{J} -trivial semigroups;
$R = \llbracket (xy)^\omega x = (xy)^\omega \rrbracket$	\mathcal{R} -trivial semigroups;
$L = \llbracket y(xy)^\omega = (xy)^\omega \rrbracket$	\mathcal{L} -trivial semigroups;
$A = \llbracket x^{\omega+1} = x^\omega \rrbracket$	aperiodic (or \mathcal{H} -trivial) semigroups;
$G = \llbracket x^\omega = 1 \rrbracket$	groups;
$LG = \llbracket (x^\omega y)^\omega x^\omega = x^\omega \rrbracket$	local groups;
$CR = \llbracket x^{\omega+1} = x \rrbracket$	completely regular semigroups;
$CS = \llbracket x^{\omega+1} = x, (xyx)^\omega = x^\omega \rrbracket$	completely simple semigroups;
$RB = \llbracket x^2 = x, xyx = x \rrbracket$	rectangular bands;
$DA = \llbracket ((xy)^\omega x)^2 = (xy)^\omega x \rrbracket$	regular \mathcal{D} -classes are aperiodic semigroups;
$DG = \llbracket (xy)^\omega = (yx)^\omega \rrbracket$	regular \mathcal{D} -classes are groups;
$DO = \llbracket (xy)^\omega (yx)^\omega (xy)^\omega = (xy)^\omega \rrbracket$	regular \mathcal{D} -classes are orthodox semigroups;
$DS = \llbracket ((xy)^\omega x)^{\omega+1} = (xy)^\omega x \rrbracket$	regular \mathcal{D} -classes are semigroups.

We reserve the letter H to denote an arbitrary pseudovariety of groups.

3. Properties of the operator \cdot_E

We now establish some basic properties of the operator \cdot_E . We start by observing that the definition given for V_E , where V is a pseudovariety, is equivalent to V_E being generated by the idempotent-generated semigroups of V .

Lemma 3.1 and its corollaries below can be extracted from [4], where the operators \cdot_E and E are studied from the viewpoint of congruences on the lattice of pseudovarieties of finite semigroups. The proofs are presented here for the sake of completeness.

Lemma 3.1. *The operator \cdot_E has the following properties, where V and W are arbitrary pseudovarieties:*

- (i) $V_E = \langle T \in V \mid T = \langle E(T) \rangle \rangle$;
- (ii) $V \subseteq W$ implies $V_E \subseteq W_E$;
- (iii) $(V \cap W)_E \subseteq V_E \cap W_E$;
- (iv) $(V_E)_E = V_E$;
- (v) $(EV)_E = V_E$;
- (vi) $E(V_E) = EV$.

Proof. (i) Let $T = \langle E(S) \rangle$ with $S \in V$. Since $\langle E(S) \rangle$ is a subsemigroup of S , it follows that $T \in V$. By Lemma 2.1, we have that $\langle E(T) \rangle = \langle E\langle E(S) \rangle \rangle = \langle E(S) \rangle = T$. Hence, the generators of the two pseudovarieties are the same.

(ii) This is immediate from the definition of V_E , and (iii) follows from (ii).

(iv) The direct inclusion follows from $V_E \subseteq V$ and (ii). Conversely, since the generators of V_E are the semigroups $\langle E(S) \rangle$, with $S \in V$, it suffices to show that $\langle E(S) \rangle \in (V_E)_E$, for all $S \in V$. Indeed, since $\langle E(S) \rangle \in V_E$, by Lemma 2.1 and the definition of \cdot_E we have $\langle E(S) \rangle = \langle E\langle E(S) \rangle \rangle \in (V_E)_E$.

(v) We have $(EV)_E = \langle \langle E(S) \rangle \mid S \in EV \rangle = \langle \langle E(S) \rangle \mid \langle E(S) \rangle \in V \rangle$. Let us see that the generators of $(EV)_E$ are in V_E . In fact, as $\langle E(S) \rangle \in V$, Lemma 2.1 yields $\langle E(S) \rangle = \langle E\langle E(S) \rangle \rangle \in V_E$. The reverse inclusion follows from $V \subseteq EV$ and (ii).

(vi) Since $V_E \subseteq V$, applying the increasing operator E , we obtain $E(V_E) \subseteq EV$. If $S \in EV$, i.e. $\langle E(S) \rangle \in V$, then Lemma 2.1 gives that $\langle E(S) \rangle = \langle E\langle E(S) \rangle \rangle$ is one of the generators of V_E so that, in particular, $S \in E(V_E)$. \square

A natural question, for which we have no answer, is whether we always have equality in part (iii) of Lemma 3.1. More generally, one may ask whether the operator \cdot_E preserves arbitrary intersections, while it is easy to see that it preserves arbitrary joins in the lattice of all pseudovarieties of finite semigroups. The more general question can be viewed as one of many similar questions arising from [4]. See [16] for related questions.

Corollary 3.2. *Let V and W be pseudovarieties such that $V_E = V$ and $EV = EW$. Then $V \subseteq W$.*

Proof. Applying the operator \cdot_E to $EV = EW$ and using property (v) of Lemma 3.1, it follows that $V = V_E = (EV)_E = (EW)_E = W_E \subseteq W$. \square

Corollary 3.3. *Given two pseudovarieties V and W , the following conditions are equivalent:*

- (a) $V_E = W_E$;
- (b) $EV = EW$;
- (c) $V_E \subseteq W \subseteq EV$.

Proof. (a) \Rightarrow (b). From (a) and property (vi) of Lemma 3.1, it follows that $EV = E(V_E) = E(W_E) = EW$.

(b) \Rightarrow (c). The second inclusion in (c) follows from $W \subseteq EW = EV$. To show the first inclusion, we recall that, by properties (iv)–(vi) of Lemma 3.1, $((EV)_E)_E = (EV)_E$ and $E((EV)_E) = EV = EW$. Hence, by Corollary 3.2, $(EV)_E \subseteq W$. Moreover, also by property (v) of Lemma 3.1, $V_E = (EV)_E \subseteq W$.

(c) \Rightarrow (a). Applying the operator \cdot_E to (c), by properties (ii), (iv) and (v) of Lemma 3.1, we obtain $V_E = (V_E)_E \subseteq W_E \subseteq (EV)_E = V_E$ and, therefore, $W_E = V_E$. \square

In other words, given a pseudovariety V , the equations $X_E = V_E$ and $EX = EV$ in the variable X are equivalent and the class of its solutions is the interval $[V_E, EV]$.

It is natural to ask: for which pseudovarieties V is V_E equal to V ? As an obvious example, for every pseudovariety V of bands, since its semigroups consist only of idempotents, we have $V_E = V$. But, there are pseudovarieties that do not satisfy the equality $V = V_E$. Let us give some examples.

Example 3.4. For every pseudovariety H of groups, we have $H_E = I$, where $I = \llbracket x = y \rrbracket$ is the trivial pseudovariety.

Example 3.5. It is well known that LG is the class of all finite semigroups such that all idempotents are \mathcal{J} -equivalent and, therefore, they are in the minimal ideal of the semigroup. So we have $(LG)_E = \langle \langle E(S) \rangle \mid S \in LG \rangle \subseteq CS \subsetneq LG$.

Example 3.6. It is well known that $RB \vee G = CS \cap O$, where $O = \llbracket (x^\omega y^\omega)^\omega = x^\omega y^\omega \rrbracket$ is the class of all finite orthodox semigroups. So we have $(RB \vee G)_E = (CS \cap O)_E = RB$.

The notion of E -local pseudovariety, introduced in [11], enables us to determine $(DO)_E$ and $(DH)_E$, as we see in the following examples. Recall that a pseudovariety V is E -local if it satisfies the following property: given $S \in \mathcal{S}$, $\langle E(S) \rangle \in V$ if and only if $\langle E(D) \rangle \in V$, for every regular \mathcal{D} -class D of S .

Example 3.7. Let $S \in DH$. Since every regular \mathcal{D} -class D of S is a group, it follows that $\langle E(D) \rangle$ is trivial and, therefore, $\langle E(D) \rangle \in J$. Since J is E -local [11, Example 3.6], we have $\langle E(S) \rangle \in J$. Hence, $J \subseteq DH \subseteq EJ$, where the first inclusion is trivial. Thus, it follows from Corollary 3.3 that $(DH)_E = J_E = J$, where the last equality follows from Corollary 3.14, which is proved below.

Example 3.8. We observe that $DO \subseteq EDA$. Indeed, for $S \in DO$ and a regular \mathcal{D} -class D of S , $\langle E(D) \rangle \in DA$. Since DA is E -local [11, Proposition 3.5], we have $\langle E(S) \rangle \in DA$. Hence, $S \in EDA$. Since $DA \subseteq DO \subseteq EDA$, it follows from Corollary 3.3 that $(DO)_E = (DA)_E = DA$, where the last equality follows from Corollary 5.6.

In an attempt to identify the pseudovarieties that are generated by their idempotent-generated elements, we present the results below and in §§ 4 and 5. We start by suggesting, as an easy exercise, the result from Howie [8] that states that any finite semigroup embeds into a finite regular idempotent-generated semigroup, so that, in particular, we have the following result.

Proposition 3.9 (Howie [8]). $S_E = S$.

On the other hand, Pin [15] and Straubing [19] obtained the following representation theorems for \mathcal{R} -trivial monoids and \mathcal{J} -trivial monoids, respectively.

Theorem 3.10 (Pin [15, Theorem IV.3.6]). *A finite monoid is \mathcal{R} -trivial if and only if it is a submonoid of \mathcal{E}_X , the submonoid consisting of all contractive total transformations of some finite chain X .*

Theorem 3.11 (Straubing [19]). *A finite monoid is \mathcal{J} -trivial if and only if it divides \mathcal{C}_X , the submonoid of all order-preserving and contractive transformations of some finite chain X .*

Combining these theorems with the following results about idempotent-generated subsemigroups of total transformations due, respectively, to Laradji and Umar [10] and to Howie [9], we obtain Corollary 3.14.

Theorem 3.12 (Laradji and Umar [10, Theorem 1.3]). *The monoid \mathcal{E}_X is idempotent generated.*

Theorem 3.13 (Howie [9, Theorem 3.2]). *The monoid \mathcal{C}_X is idempotent generated.*

Corollary 3.14. *The equality $V_E = V$ holds if V is any of the pseudovarieties R, L, J .*

Similarly, using embeddings into idempotent-generated semigroups of the same type from Pastijn [13] concerning completely simple semigroups and completely regular semigroups, we obtain the following results.

Proposition 3.15 (Pastijn [13, Corollary 3.7]). $(CS)_E = CS$.

Proposition 3.16 (Pastijn [13, Theorem 3.5]). $(CR)_E = CR$.

Using Proposition 3.15 we may establish an equality in Example 3.5, as we see below.

Example 3.17. By Proposition 3.15 and property (ii) of Lemma 3.1 and since $CS \subseteq LG$, it follows that $CS = (CS)_E \subseteq (LG)_E$. Thus, and by Example 3.5, we have $(LG)_E = CS$.

In §4 we return to the last two results and we show how to prove them using the general construction of Petrich [14]. In §5, we prove that the pseudovarieties R , L and J are fixed points of the $\cdot_{\mathcal{E}}$ operator by a different approach, namely by using implicit operations. While, using transformation semigroups, the number of idempotent generators of the idempotent-generated semigroup depends on the cardinality of the embedded semigroup, in this method the number of idempotent generators of the idempotent-generated semigroup is controlled by the number of generators of the embedded semigroup. However, for the case $V = R$, the first method enables us to show that, in fact, there exists an embedding of an \mathcal{R} -trivial semigroup into an idempotent-generated \mathcal{R} -trivial semigroup. In the other cases, we just prove a division property. The second method is also used to prove the equality for the pseudovariety DA , while we do not know how to apply the first method.

4. An embedding into a Rees matrix semigroup

In [5] one can find a proof of Howie's result that every (finite) semigroup can be embedded into an idempotent-generated (finite) semigroup. The proof is attributed to J.-F. Perrot and depends on a construction that was later used and extended by Pastijn [13] to obtain such embedding results within special classes of semigroups. Petrich [14] reformulated Perrot's construction in a new notation that made it considerably more transparent. We reformulate Pastijn's embedding results for completely simple [13, Corollary 3.7] and completely regular [13, Theorem 3.5] semigroups in Petrich's language.

We briefly recall the construction of this embedding. Let S be a semigroup (not necessarily finite). We consider the Rees matrix semigroup $\Phi S = \mathcal{M}(S^1, S^1, \Sigma; Q)$, with $\Sigma = \{\sigma, \tau\}$, where σ and τ are two distinct symbols that are not in S , and $Q = (q_{\alpha s})$ is the sandwich matrix with entries

$$q_{\sigma s} = 1, \quad q_{\tau s} = s, \quad s \in S^1.$$

The mapping

$$\varphi_S: s \rightarrow (1, s, \sigma), \quad s \in S,$$

embeds S into ΦS , although it is not the unique embedding from S into ΦS . Petrich determined the set of idempotents of ΦS , which is

$$E(\Phi S) = \{(s, t, \sigma) \in \Phi S \mid t \in E(S^1)\} \cup \{(s, t, \tau) \in \Phi S \mid t = tst\}, \quad (4.1)$$

and described Green's relations on ΦS as follows.

Lemma 4.1 (Petrich [14, Lemma 4.3]). *Let $(s, t, \alpha), (u, v, \beta) \in \Phi S$. Then*

- (i) $(s, t, \alpha) \mathcal{L} (u, v, \beta)$ if and only if $t \mathcal{L} v$ and $\alpha = \beta$,
- (ii) $(s, t, \alpha) \mathcal{R} (u, v, \beta)$ if and only if $s = u$ and $t \mathcal{R} v$,
- (iii) $(s, t, \alpha) \mathcal{H} (u, v, \beta)$ if and only if $s = u$, $t \mathcal{H} v$, and $\alpha = \beta$,
- (iv) $(s, t, \alpha) \mathcal{D} (u, v, \beta)$ if and only if $t \mathcal{D} v$.

Thus, ΦS has the same number of \mathcal{D} -classes as S^1 and each \mathcal{D} -class D' of ΦS , which corresponds to a \mathcal{D} -class D of S , has the following number of \mathcal{L} -classes and \mathcal{R} -classes, respectively: $2 \cdot |\mathcal{L}\text{-classes of } D|$ and $|S^1| \cdot |\mathcal{R}\text{-classes of } D|$. Each \mathcal{H} -class of ΦS has the same number of elements of the corresponding \mathcal{H} -class in S^1 .

It is obvious that if S is a finite semigroup, so is ΦS . Note that ΦS is generated by the set of idempotents $\{(s, 1, \sigma) \mid s \in S^1\} \cup \{(1, 1, \tau)\}$, which gives an immediate proof of the result from Howie (see Proposition 3.9). Petrich also showed that this embedding preserves other properties of S [14, Theorem 5.4]. In particular, he proves the following corollary.

Corollary 4.2 (Petrich [14, Theorem 5.4]). *Every semigroup of \bar{H} , where H is a pseudovariety of groups, embeds into an idempotent-generated semigroup of \bar{H} .*

Choosing some specific subsemigroups of ΦS , we can prove the following results.

Proposition 4.3. *Every semigroup of CS embeds into an idempotent-generated semigroup of CS .*

Proof. Note that, if $S \in CS$, then ΦS has at most two \mathcal{D} -classes: the one corresponding to D, D' , and the other corresponding to the neutral element added to S . If we show that D' is generated by its idempotents, then it suffices to consider the embedding $\varphi'_S: s \mapsto (1, s, \sigma)$ from S into the semigroup D' .

Let $(1, s, \sigma)$, with $s \in S$, be any element of $\varphi_S(S)$ and let $e \in E(S)$ be such that $e \mathcal{H} s$. Then one can compute $(1, e, \tau) \cdot (s, e, \sigma) = (1, s, \sigma)$ with $(1, e, \tau), (s, e, \sigma) \in E(D')$. Hence, the group \mathcal{H} -class $\{(1, s, \sigma) \mid s \in S\}$ is contained in $\langle E(D') \rangle$. Since all \mathcal{H} -classes of $\langle E(D') \rangle$ have the same number of elements, we conclude that $\langle E(D') \rangle = D'$. Hence, S embeds into $\langle E(D') \rangle$, which is an idempotent-generated completely simple semigroup. \square

Proposition 4.4. *Every semigroup of CR embeds into an idempotent-generated semigroup of CR .*

Proof. Let $S \in CR$. We want to determine an idempotent-generated completely regular subsemigroup of ΦS where S embeds. Let H be an \mathcal{H} -class of S . Since H is a group, the \mathcal{H} -classes of ΦS of the form $\{s\} \times H \times \{\sigma\}$, with $s \in S^1$, and $\{1\} \times H \times \{\tau\}$ are groups. Let t be any element of S and let H_t be the \mathcal{H} -class of S containing this element. We observe that the \mathcal{H} -classes of the form $\{s\} \times H_t \times \{\tau\}$, with $s \geq_{\mathcal{J}} t$, are groups. Let $e \in E(S)$ be such that $e \mathcal{H} t$. Then there exist $x, y \in S$ such that $e = xsy = exsye$. Hence, $e \mathcal{L} sye$ and, since $L_e \cap R_{sye}$ is a group, we have that $esye \mathcal{H} e$ and $es \mathcal{R} e$. By Green's Lemma, it follows that

$$\begin{aligned} \mu_s: H_t &\rightarrow H_{ts} \\ u &\mapsto us \end{aligned}$$

is a bijection. Let $v \in H_t$ be such that vs is the idempotent of H_{ts} . Since $vsv \mathcal{H} v$ and $\mu_s(vsv) = vsvs = vs = \mu_s(v)$, then $vsv = v$ and (s, v, τ) is the idempotent of $\{s\} \times H_t \times \{\tau\}$.

Now, we observe that if an \mathcal{H} -class of ΦS of the form $\{s\} \times H_t \times \{\tau\}$ is a group, then $s \geq_{\mathcal{J}} t$. Thus, if there exists $u \in H_t$ such that (s, u, τ) is an idempotent, then $usu = u$, and so $s \geq_{\mathcal{J}} u \mathcal{H} t$. So, we have identified all maximal subgroups of ΦS .

We consider the subsemigroup T of ΦS generated by the following idempotents:

$$(s, t, \sigma) \quad \text{with } t \in E(S^1) \text{ and } s \geq_{\mathcal{J}} t; \quad (s, t, \tau) \quad \text{with } t = tst.$$

Basically, we chose all idempotents of the \mathcal{R} -classes whose \mathcal{H} -classes are groups. Thus, the product of any two idempotents in the same \mathcal{D} -class is also in this \mathcal{D} -class and, specifically, in an \mathcal{H} -class containing an idempotent of the set of generators. Let us see what the product is of two idempotents of the set of generators that are not in the same \mathcal{D} -class. Let (s, t, α) and (u, v, β) be two such idempotents. We have $(s, t, \alpha) \cdot (u, v, \beta) = (s, tq_{\alpha u}v, \beta)$. As $s \geq_{\mathcal{J}} t$, we have $s \geq_{\mathcal{J}} tq_{\alpha u}v$. Hence, this product is in an \mathcal{H} -class that contains an idempotent of the set of generators of T .

Note that the \mathcal{H} -classes of the form $\{1\} \times H \times \{\sigma\}$, where H is an \mathcal{H} -class of S^1 are in T . In fact, given $a \in S^1$, we have $(1, a, \sigma) = (1, e, \tau) \cdot (a, e, \sigma)$, where $e \in E(S^1)$ is such that $e \mathcal{H} a$, and $(1, e, \tau)$ and (a, e, σ) are idempotents of T . It follows that T is the subsemigroup of ΦS consisting of the \mathcal{R} -classes of ΦS whose \mathcal{H} -classes are groups.

Hence, T is a completely regular semigroup and $\varphi''_S: s \mapsto (1, s, \sigma)$ is an embedding of S into T . □

In the above proof, we may reduce the choice of the idempotents and we may consider the subsemigroup of T generated by the following idempotents:

$$(1, e, \tau), (a, e, \sigma) \quad \text{with } a \in S \text{ and } e \in E(S) \text{ such that } e \mathcal{H} a.$$

This subsemigroup is also a union of \mathcal{R} -classes of ΦS whose \mathcal{H} -classes are groups; the \mathcal{H} -classes of the form $\{1\} \times H \times \{\sigma\}$, where H is an \mathcal{H} -class of S^1 , are also in this subsemigroup. However, to simplify the proof, we considered the subsemigroup T .

Example 4.5. Consider the completely regular semigroup

$$S = \langle a, b, c, d \mid a^3 = a, b^2 = b, c^3 = c, d^2 = d, ab = ba, cb = bc, ada = a, ac = ca = bd = db = cd = dc = 0 \rangle.$$

We present in Figure 1 a \mathcal{D} -class of ΦS to illustrate the distribution of the idempotents. We also observe that T consists of the \mathcal{R} -classes of ΦS whose \mathcal{H} -classes are groups, as we have mentioned previously.

Corollary 4.6. *The pseudovarieties \bar{H} , CS and CR satisfy the equality $V_E = V$.*

When we work with the pseudovarieties DS and DA , and since the regular \mathcal{D} -classes of the semigroups of these pseudovarieties are completely simple semigroups, one may ask whether the construction used in Proposition 4.4 may lead to a proof of existence of an embedding from every semigroup of any of these pseudovarieties into an idempotent-generated semigroup of the same pseudovariety. However, in the following example, we observe that this is not the case.

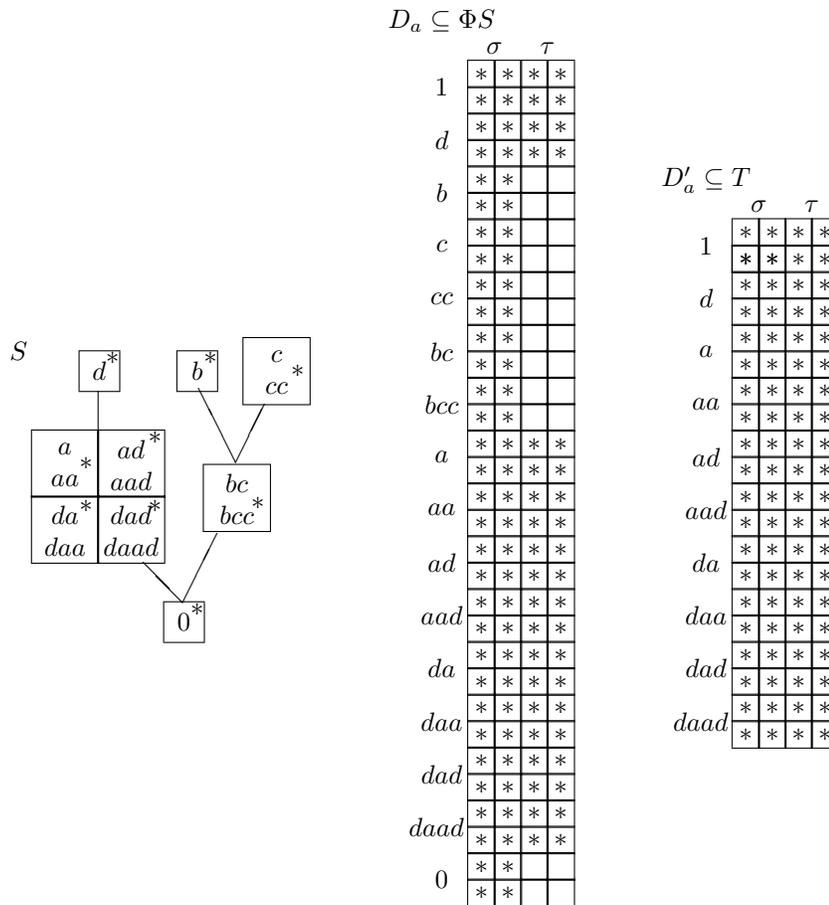


Figure 1. An embedding for a completely regular semigroup

Example 4.7. Consider the semigroup $S = \langle a \mid a^3 = a^2 \rangle$. We look at the subsemigroup T of ΦS generated by the idempotents of the same type as those of Proposition 4.4 (see Figure 2). Note that neither the element $(1, a, \sigma)$ nor any element of the \mathcal{D} -class D_a is in T . We have to choose the idempotent $(a, 1, \sigma)$ to be a generator of T , but, in that case, T is no longer an element of DS (and, consequently, of DA).

We end this subsection with no answer for the question: does $V_E = V$ for any of the pseudovarieties DS or DA ? In the following section we see that, in fact, the pseudovariety DA satisfies such equality.

5. Representations by implicit operations

We refer the reader to [2] for detailed information about profinite semigroups and to standard references for the basics of topology. By an *embedding* of topological semigroups

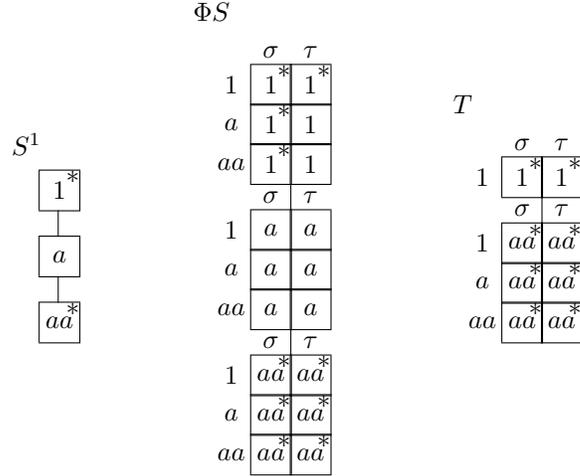
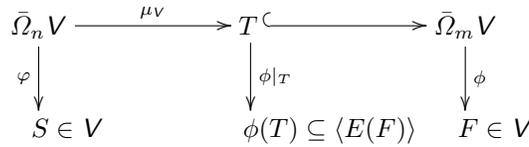


Figure 2. An embedding for a monogenic monoid

we mean a semigroup homomorphism that is simultaneously a homeomorphism with the image subspace. A *clopen* subset of a topological space is one that is simultaneously closed and open.

Theorem 5.1. *Let V be a pseudovariety such that, for every n , there exists m such that $\bar{\Omega}_n V$ embeds in $\langle X \rangle$ for some $X \subseteq E(\bar{\Omega}_m V)$. Then $V_E = V$.*

Proof. Let V be a pseudovariety satisfying the above conditions. Let $S \in V$ and let $\varphi: \bar{\Omega}_n V \rightarrow S$ be a continuous surjective homomorphism. Let $\mu_V: \bar{\Omega}_n V \rightarrow \langle X \rangle \subseteq \bar{\Omega}_m V$ be an embedding, with $X \subseteq E(\bar{\Omega}_m V)$. We consider the following diagram:



where T is the image of μ_V .

We claim that there is a family of clopen subsets $(U_s)_{s \in S}$ of $\bar{\Omega}_m V$, pairwise disjoint, such that $U_s \cap T = \mu_V(\varphi^{-1}(s))$. We proceed to prove the claim. For each $s \in S$, let $A_s = \mu_V(\varphi^{-1}(s))$ and $A'_s = T \setminus A_s$. Since $\{s\}$ is a clopen subset of S , $\varphi^{-1}(s)$ is a clopen subset of $\bar{\Omega}_n V$ and $\mu_V(\varphi^{-1}(s))$ is a clopen subset of T . Since A_s and A'_s are closed sets of the closed subspace T of $\bar{\Omega}_m V$, A_s and A'_s are closed subsets of $\bar{\Omega}_m V$. Moreover, since $\bar{\Omega}_m V$ is compact and zero dimensional, then A_s and A'_s are, respectively, separated by some disjoint clopen sets V_s and V'_s . We choose an arbitrary ordering for the elements of S : $s_1, \dots, s_{|S|}$. Let $U_{s_1} = V_{s_1}$ and recursively, for $i = 2, \dots, |S|$, let $U_{s_i} = V_{s_i} \setminus (\bigcup_{j < i} U_{s_j})$. Note that, for every i , U_{s_i} is also a clopen subset of $\bar{\Omega}_m V$ and $(U_s)_{s \in S}$ is a family satisfying the claim.

Since $\bar{\Omega}_m V$ is a pro- V semigroup and, for all $s \in S$, U_s is a clopen subset, there exists a continuous homomorphism $\phi_s: \bar{\Omega}_m V \rightarrow F_s$ with $F_s \in V$ such that $U_s = \phi_s^{-1}(\phi_s(U_s))$ (see [1, 2]). Let $\phi: \bar{\Omega}_m V \rightarrow F = \prod_s F_s$ be the continuous homomorphism such that $\phi = (\phi_s)_{s \in S}$. Then $U_s = \phi^{-1}(\phi(U_s))$ for all $s \in S$. We consider the diagram

$$\begin{array}{ccc} \bar{\Omega}_n V & & \\ \varphi \downarrow & \searrow \phi \circ \mu_V & \\ S & \xleftarrow{\rho} \phi(T) \hookrightarrow & \langle E(F) \rangle \end{array}$$

We show that S is a homomorphic image of $\phi(T)$: more precisely, we show that there exists $\rho: \phi(T) \rightarrow S$ such that the diagram commutes. It suffices to show that, for $w, z \in \bar{\Omega}_n V$, if $(\phi \circ \mu_V)(w) = (\phi \circ \mu_V)(z)$, then $\varphi(w) = \varphi(z)$. Let $s_1 = \varphi(w)$, $s_2 = \varphi(z)$ and suppose that $s_1 \neq s_2$. Since $U_{s_1} \cap U_{s_2} = \emptyset$, we have that $\phi(U_{s_1}) \cap \phi(U_{s_2}) = \emptyset$. Now, $\mu_V(w) \in \mu_V(\varphi^{-1}(s_1)) = U_{s_1} \cap T$ and, therefore, $\phi(\mu_V(w)) \in \phi(U_{s_1} \cap T)$. Similarly, we obtain $\phi(\mu_V(z)) \in \phi(U_{s_2} \cap T)$. It follows that $\phi(\mu_V(w)) \neq \phi(\mu_V(z))$.

We conclude that S divides $\langle E(F) \rangle$. Since $F \in V$, it follows that $\langle E(F) \rangle \in V_E$ and, therefore, $S \in V_E$. This shows that $V \subseteq V_E$, while the reverse inclusion is always verified. \square

From Theorem 5.1, to conclude that $V_E = V$, it suffices to exhibit an embedding $\mu_V: \bar{\Omega}_n V \rightarrow \overline{\langle X \rangle}$ with $X \subseteq E(\bar{\Omega}_m V)$, for every integer n . We do not know if, conversely, such an embedding always exists in the case when $V_E = V$.

For $V \in \{R, L, J, DA\}$, we consider the unique continuous homomorphism μ_V such that

$$\begin{aligned} \mu_V: \bar{\Omega}_n V &\rightarrow \bar{\Omega}_{n+1} V \\ x_i &\mapsto x_i^\omega y^\omega, \end{aligned}$$

where y is a new variable and we prove that μ_V is an embedding. In each case, we depend heavily on a suitable representation of the profinite semigroup $\bar{\Omega}_n V$.

Let us start with the pseudovariety R . We use the representation of implicit operations over R by means of labelled ordinals due to Almeida and Weil [3]. We briefly recall it. Let A be an alphabet with $|A| = n$ and let $\mathbf{rLO}(A)$ be the set of reduced A -labelled ordinals. Recall that an A -labelled ordinal is a pair (α, l) , where α is an ordinal and $l: \alpha \rightarrow A$ is a labelling function. The *content* of (α, l) , $c(\alpha, l)$, is the range of l . The *cumulative content* of a limit ordinal $\beta \leq \alpha$, $\overleftarrow{c}(\beta)$, is the set of all letters $a \in A$ such there exists a sequence $(\gamma_k)_k$ of ordinals with $\bigcup_k \gamma_k = \beta$, $\gamma_k < \beta$ and $l(\gamma_k) = a$ for all k . An A -labelled ordinal (α, l) is said to be *reduced* if $l(\beta) \notin \overleftarrow{c}(\beta)$ for each limit ordinal $\beta < \alpha$.

Let $(\alpha, l) \in \mathbf{rLO}(A)$. For each $a \in A$, let γ_a be the smallest ordinal such that $\gamma_a < \alpha$ and $l(\gamma_a) = a$ (i.e. γ_a is the position of the first occurrence of a). We set $\gamma_a = 0$ if $l(\gamma) \neq a$, for all $\gamma < \alpha$. Let $\alpha_1 = \max\{\gamma_a \mid a \in A\}$ (i.e. the first occurrence of the last appearing letter) and let β_1 be such that $\alpha = \alpha_1 + 1 + \beta_1$, with $(\alpha_1, l_1), (\beta_1, m_1) \in \mathbf{rLO}(A)$, $l_1 = l|_{\alpha_1}$ and $m_1(\gamma) = l(\alpha_1 + 1 + \gamma)$, where $\gamma < \beta_1$. We call the equality $\alpha = \alpha_1 + 1 + \beta_1$ the *left basic partition* of (α, l) . We iterate this process while $\beta_i \neq 0$. Let $\beta_0 = \alpha$, $m_0 = l$

and $\beta_i = \alpha_{i+1} + 1 + \beta_{i+1}$ with α_{i+1} and β_{i+1} constructed in the same way. While $\beta_i \neq 0$, we obtain ordinals (α_i, l_i) and (β_i, m_i) , with $i \geq 1$, where $l_{i+1} = m_{i|\alpha_{i+1}}$ and $m_{i+1}(\gamma) = m_i(\alpha_{i+1} + 1 + \gamma)$, $\gamma < \beta_{i+1}$. Almeida and Weil showed that

$$\alpha = \sum_{i \geq 1} (\alpha_i + 1)$$

and they define the product in $\mathbf{rLO}(A)$ that follows. Let $(\alpha, l), (\beta, m) \in \mathbf{rLO}(A)$. If α is not a limit ordinal, then

$$(\alpha, l)(\beta, m) = (\alpha + \beta, l'), \tag{5.1}$$

where $l'(\gamma) = l(\gamma)$ if $\gamma < \alpha$ and $l'(\alpha + \gamma) = m(\gamma)$ if $\gamma < \beta$. If α is a limit ordinal, then we write $\beta = \beta_1 + \beta_2$, where β_1 is the smallest ordinal such that $m(\beta_1) \notin \overleftarrow{c}(\alpha)$. The product is given by

$$(\alpha, l)(\beta, m) = (\alpha + \beta_2, l'), \tag{5.2}$$

where $l'(\gamma) = l(\gamma)$ if $\gamma < \alpha$ and $l'(\alpha + \gamma) = m(\beta_1 + \gamma)$ if $\gamma < \beta_2$. Almeida and Weil proved that $\mathbf{rLO}(A)$ equipped with this operation is isomorphic to $\bar{\Omega}_n R$.

Proposition 5.2. *The function $\mu_R: \bar{\Omega}_n R \rightarrow \bar{\Omega}_{n+1} R$ is an embedding.*

Proof. We denote by $\psi_A: \bar{\Omega}_n R \rightarrow \mathbf{rLO}(A)$ the isomorphism defined by Almeida and Weil [3], where $|A| = n$. Let $B = \{a, b: a \in A\}$ with $b \notin A$. We consider the following diagram:

$$\begin{array}{ccc} \bar{\Omega}_n R & \xrightarrow{\mu_R} & \bar{\Omega}_{n+1} R \\ \psi_A \downarrow & & \downarrow \psi_B \\ \mathbf{rLO}(A) & \xrightarrow{\nu} & \mathbf{LO}(B) \end{array}$$

with ν defined by

$$\begin{aligned} \nu: \mathbf{rLO}(A) &\rightarrow \mathbf{LO}(B) \\ (\alpha, l) &\mapsto ((\omega + \omega)\alpha, l'), \end{aligned}$$

where $\mathbf{LO}(B)$ is the set of B -labelled ordinals and

$$\begin{aligned} l' &: (\omega + \omega)\alpha \rightarrow B, \\ \beta &\mapsto \begin{cases} l(\gamma) & \text{if } \beta = (\omega + \omega)\gamma + k \text{ with } \gamma < \alpha, k \in \omega, \\ b & \text{if } \beta = (\omega + \omega)\gamma + \omega + k \text{ with } \gamma < \alpha, k \in \omega. \end{cases} \end{aligned}$$

We prove that the diagram commutes, i.e. that $\nu = \psi_B \circ \mu_R \circ \psi_A^{-1}$. Let

$$\left(\alpha = \sum_{i \geq 1} (\alpha_i + 1), l \right) \in \mathbf{rLO}(A).$$

Since ψ_A^{-1} , ψ_B and μ_R are homomorphisms, we proceed by induction on $|c(\alpha_i, l_i)|$, which is finite and less than $|c(\alpha, l)|$, and we obtain

$$\begin{array}{ccc} \prod_{i \geq 1} (\psi_A^{-1}(\alpha_i, l_i) a_i) & \xrightarrow{\mu_R} & \prod_{i \geq 1} (\mu_R(\psi_A^{-1}(\alpha_i, l_i)) a_i^\omega b^\omega) \\ \uparrow \psi_A^{-1} & & \downarrow \psi_B \\ \left(\alpha = \sum_{i \geq 1} (\alpha_i + 1), l \right) & & (\delta, m) = \sum_{i \geq 1} ((\omega + \omega)\alpha_i, l'_i) + (\omega + \omega, g_i) \end{array}$$

where $a_i = m_{i-1}(\alpha_i)$ and

$$g_i: \omega + \omega \rightarrow B$$

$$\beta \mapsto \begin{cases} a_i & \text{if } \beta < \omega, \\ b & \text{if } \beta = \omega + \gamma \text{ with } \gamma < \omega. \end{cases}$$

We want to prove that $(\delta, m) = \nu(\alpha, l) = ((\omega + \omega)\alpha, l')$. Indeed, we have

$$\begin{aligned} \delta &= \sum_{i \geq 1} ((\omega + \omega)\alpha_i + (\omega + \omega)) \\ &= (\omega + \omega)\alpha_1 + (\omega + \omega) + (\omega + \omega)\alpha_2 + (\omega + \omega) + \dots \\ &= (\omega + \omega)(\alpha_1 + 1 + \alpha_2 + 1 + \dots) \\ &= (\omega + \omega) \sum_{i \geq 1} (\alpha_i + 1) \\ &= (\omega + \omega)\alpha, \end{aligned}$$

where the third equality follows from [18, Exercise 1.41], and

$$m: (\omega + \omega)\alpha \rightarrow B$$

$$\beta \mapsto \begin{cases} l'_i(\gamma) & \text{if } \beta = (\omega + \omega) \left(\sum_{j=1}^{i-1} (\alpha_j + 1) \right) + \gamma, \\ & \text{with } \gamma < (\omega + \omega)\alpha_i, \\ g_i(\gamma) & \text{if } \beta = (\omega + \omega) \left(\sum_{j=1}^{i-1} (\alpha_j + 1) \right) + (\omega + \omega)\alpha_i + \gamma, \\ & \text{with } \gamma < (\omega + \omega), \end{cases}$$

where we set $\sum_{j=1}^0 (\alpha_j + 1) = 0$. In the first case, it follows that

$$\begin{aligned} m(\beta) &= l'_i(\gamma) \\ &= \begin{cases} l'_i(\delta) & \text{if } \gamma = (\omega + \omega)\delta + k, \text{ with } \delta < \alpha_i, k \in w, \\ b & \text{if } \gamma = (\omega + \omega)\delta + \omega + k, \text{ with } \delta < \alpha_i, k \in w, \end{cases} \end{aligned}$$

$$\begin{aligned}
&= \begin{cases} l(\alpha_1 + 1 + \cdots + \alpha_{i-1} + 1 + \delta) & \text{if } \gamma = (\omega + \omega)\delta + k, \\ b & \text{if } \gamma = (\omega + \omega)\delta + \omega + k, \end{cases} \\
&= \begin{cases} l(\eta) & \text{if } \beta = (\omega + \omega)\eta + k, \\ b & \text{if } \beta = (\omega + \omega)\eta + \omega + k, \end{cases} \\
&= l'(\beta),
\end{aligned}$$

where $\eta = \sum_{j=1}^{i-1}(\alpha_j + 1) + \delta$. In the second case, for $\eta = \sum_{j=1}^{i-1}(\alpha_j + 1) + \alpha_i$, we have

$$\begin{aligned}
m(\beta) &= g_i(\gamma) \\
&= \begin{cases} a_i & \text{if } \gamma < w, \\ b & \text{if } \gamma = w + \delta \text{ with } \delta < w, \end{cases} \\
&= \begin{cases} l(\alpha_1 + 1 + \cdots + \alpha_{i-1} + 1 + \alpha_i) & \text{if } \beta = (\omega + \omega)\eta + \delta, \\ b & \text{if } \beta = (\omega + \omega)\eta + \omega + \delta, \end{cases} \\
&= \begin{cases} l(\eta) & \text{if } \beta = (\omega + \omega)\eta + \delta, \\ b & \text{if } \beta = (\omega + \omega)\eta + \omega + \delta, \end{cases} \\
&= l'(\beta)
\end{aligned}$$

and, therefore, $m = l'$. It follows that the diagram commutes and ν is a homomorphism from $\mathbf{rLO}(A)$ into $\mathbf{rLO}(B)$, where the product involved is the one defined by formulae (5.1) and (5.2). Thus, μ_R is injective if and only if ν is injective. Let (α, l) and (β, m) be reduced labelled ordinals such that $\nu(\alpha, l) = \nu(\beta, m)$. By [18, Exercise 3.41], we have

$$(\omega + \omega)\alpha = (\omega + \omega)\beta \implies \alpha = \beta$$

and

$$l' = m' \implies l(\gamma) = m(\gamma) \quad \text{for all } \gamma < \alpha \implies l = m.$$

Hence, $(\alpha, l) = (\beta, m)$ and ν is injective. \square

The dual result for the pseudovariety L follows by duality.

Proposition 5.3. *The function $\mu_L: \bar{\Omega}_n L \rightarrow \bar{\Omega}_{n+1} L$ is an embedding.* \square

Now, we consider the pseudovariety J of \mathcal{J} -trivial semigroups. We use the representation by canonical form of implicit operations over J obtained by the first author [1, § 8.2]. Consider the variety \mathcal{V} of type $(2, 1)$ defined by the set of identities

$$\Sigma = \{(xy)z = x(yz), (xy)^\omega = (yx)^\omega = (x^\omega y^\omega)^\omega, x^\omega x = x^\omega = xx^\omega, (x^\omega)^\omega = x^\omega\}.$$

We may reduce any term in the variables x_1, x_2, \dots using the following Noetherian and confluent system of *reduction rules*:

- (rr1) to eliminate parentheses concerning the application of the operation of multiplication;
- (rr2) to substitute any subterm of the form t^ω by u^ω , where u is the product, in increasing order of the indices, of the variables occurring in t ;
- (rr3) to absorb in factors of the form u^ω any adjacent factors in which only occur variables of u .

A term of \mathcal{V} is called a *word* if it does not involve the unary operation \cdot^ω , and it is called *idempotent* if it is of the form t^ω , for some term t . The *content* $c(t)$ of a term t is the set of variables occurring in t . The factorization in *canonical form* of a term t is $t = t_1 \cdots t_n$, where

- (cf1) each t_i is a word or an idempotent,
- (cf2) each idempotent t_i is of the form u^ω , where u is a product of variables with the indices in strictly increasing order,
- (cf3) given two consecutive idempotents t_i and t_{i+1} , the sets $c(t_i)$ and $c(t_{i+1})$ are incomparable,
- (cf4) two consecutive terms t_i and t_{i+1} are not both words,
- (cf5) if t_i is a word and t_{i+1} is an idempotent, then the last letter of t_i is not in $c(t_{i+1})$,
- (cf6) if t_{i+1} is a word and t_i is an idempotent, then the first letter of t_{i+1} is not in $c(t_i)$.

Let $F_n\mathcal{V}$ be the \mathcal{V} -free algebra on $\{x_1, \dots, x_n\}$. The semigroup $\bar{\Omega}_n J$ may be seen as an algebra of type $(2, 1)$, where all elements are constructed using the operations of multiplication and omega power and the variables $\{x_1, \dots, x_n\}$. Then we have a natural surjective homomorphism

$$\begin{aligned} \psi : F_n\mathcal{V} &\rightarrow \bar{\Omega}_n J \\ x_i &\mapsto x_i, \quad i = 1, \dots, n, \end{aligned}$$

and [1, Theorem 8.2.7] establishes that ψ is, in fact, an isomorphism. We are now able to prove the desired proposition.

Proposition 5.4. *The function $\mu_J : \bar{\Omega}_n J \rightarrow \bar{\Omega}_{n+1} J$ is an embedding.*

Proof. By the above, showing that μ_J is injective is equivalent to establishing that

$$\begin{aligned} \nu : F_n\mathcal{V} &\rightarrow F_{n+1}\mathcal{V} \\ x_i &\mapsto x_i^\omega y^\omega \end{aligned}$$

is injective. Let $w, z \in F_n\mathcal{V}$ be such that $\nu(w) = \nu(z)$ and let $w = w_1 \cdots w_m$ and $z = z_1 \cdots z_n$ be the respective factorizations in canonical form. We want to determine the factorizations in canonical form of $\nu(w)$ and $\nu(z)$. Let $i \in \{1, \dots, m\}$. Suppose that w_i

is a word, i.e. $w_i = x_{i_1} \cdots x_{i_k}$. Then $\nu(w_i) = x_{i_1}^\omega y^\omega \cdots x_{i_k}^\omega y^\omega$. Note that this factorization is in canonical form, because it is a product such that two consecutive idempotents have incomparable contents. Suppose now that w_i is an idempotent, i.e. $w_i = (x_{i_1} \cdots x_{i_l})^\omega$ with $i_1 < \cdots < i_l$. Then $\nu(w_i) = (x_{i_1}^\omega y^\omega \cdots x_{i_l}^\omega y^\omega)^\omega = (x_{i_1} \cdots x_{i_l} y)^\omega$ applying the reduction rule (rr2). Note that $i_1 < \cdots < i_l < y$ (assuming that the new letter y is larger than any of the others) and, therefore, the last presented factorization of $\nu(w_i)$ is in canonical form. Therefore, an idempotent of $F_n \mathcal{V}$ has as image an idempotent of $F_{n+1} \mathcal{V}$ and a word of length k has as image a product of $2k$ idempotents of $F_{n+1} \mathcal{V}$, in canonical form.

Consider now the product $w_i w_j$ with $j = i+1$. Note that, by the definition of canonical form, w_i and w_j are not both words. Suppose that w_i is a word and w_j is an idempotent. Then

$$\nu(w_i w_j) = x_{i_1}^\omega y^\omega \cdots x_{i_k}^\omega y^\omega \cdot (x_{j_1} \cdots x_{j_l} y)^\omega = x_{i_1}^\omega y^\omega \cdots x_{i_k}^\omega (x_{j_1} \cdots x_{j_l} y)^\omega,$$

applying the reduction rule (rr3). By hypothesis, $x_{i_k} \notin c(w_j)$ and we conclude that the last factorization of $\nu(w_i w_j)$ is in canonical form. If w_i is an idempotent and w_j is a word, or if both w_i and w_j are idempotents, then we have, respectively, the following canonical forms for $w_i w_j$:

$$\nu(w_i w_j) = (x_{i_1} \cdots x_{i_k} y)^\omega x_{j_1}^\omega y^\omega \cdots x_{j_l}^\omega y^\omega$$

and

$$\nu(w_i w_j) = (x_{i_1} \cdots x_{i_k} y)^\omega (x_{j_1} \cdots x_{j_l} y)^\omega.$$

Let $\nu(w) = \bar{w}_1 \cdots \bar{w}_{m'}$ and let $\nu(z) = \bar{z}_1 \cdots \bar{z}_{n'}$ be the factorizations in canonical form of $\nu(w)$ and $\nu(z)$, respectively. Since $\nu(w) = \nu(z)$, by [1, Theorem 8.2.8] we have $m' = n'$ and $\bar{w}_i = \bar{z}_i$, for all i . Three cases can occur for each factor \bar{w}_i : $\bar{w}_i = x_j^\omega$, $\bar{w}_i = y^\omega$ or $\bar{w}_i = (x_{j_1} \cdots x_{j_l} y)^\omega$, for some j, l . Note that the content of the idempotent in the last case has cardinal bigger than 1, while in the other cases it is 1. We recover w as follows. In the first two cases, we substitute \bar{w}_i by x_j and by 1, respectively. In the last case, we substitute \bar{w}_i by $(x_{j_1} \cdots x_{j_l})^\omega$. It is easy to see that the canonical forms of w and z are recovered and they are equal. It follows that $w = z$ and ν is injective. \square

Finally, we treat the case of DA using the representation of implicit operations over DA by means of labelled orderings obtained by Moura [12], which is similar to the case of the pseudovariety R . So, we omit most details and we refer the reader to [12] as needed. In that paper, we proved that there is a bijection between the free profinite semigroup over DA , $\bar{\Omega}_A DA$, and the set of all reduced A -labelled $*$ -linear orderings, $\mathbf{rLO}^*(A)$.

Proposition 5.5. *The function $\mu_{DA}: \bar{\Omega}_n DA \rightarrow \bar{\Omega}_{n+1} DA$ is an embedding.*

Proof. Since $\bar{\Omega}_A DA$ and $\mathbf{rLO}^*(A)$ are isomorphic, it suffices to prove that the following mapping is injective:

$$\begin{aligned} \nu: \mathbf{rLO}^*(A) &\rightarrow \mathbf{rLO}^*(B) \\ (o, l) &\mapsto ((\omega + \omega^*)2o, l'). \end{aligned}$$

By [12], $\nu(o, l)$ is constructed from (o, l) in the following way: each position of o is replaced by the ordering $(\omega + \omega^*)2$ and, if this position is labelled $a \in A$, the label of each position on the resulting ordering is a or b , depending on whether the position is in the first or second term of the form $\omega + \omega^*$. Thus, given two consecutive positions $p < q$ from $\nu(o, l)$, one and only one of the following cases can occur (for some $a \in A$):

- $l'(p) = l'(q) = a$;
- $l'(p) = l'(q) = b$;
- $l'(p) = a$ and $l'(q) = b$;
- $l'(p) = b$ and $l'(q) = a$.

In the first three cases, both positions are in the same interval $(\omega + \omega^*)2$ of o , resulting from the replacement of a position of o labelled a , for some $a \in A$. In the fourth case, the positions are in consecutive intervals corresponding to the replacement of consecutive positions of o . We split $(\omega + \omega^*)2o$ into intervals $I_p, p \in o$, of the form $(\omega + \omega^*)2$, that are maximal for the following condition:

$$I_p = I_{p,1} \cup I_{p,2},$$

where $I_{p,1}$ is an interval whose elements are labelled with the same letter of A and $I_{p,2}$ is an interval whose elements are labelled with b . It follows that

$$(\omega + \omega^*)2o = \bigcup_{p \in o} I_p$$

and $l'(\bar{p}) = l(p)$, for all $\bar{p} \in I_p$, with $I_p = I_{p,1} \cup I_{p,2}$ satisfying the above condition. Thus, we may recover (o, l) from $((\omega + \omega^*)2o, l')$ considering the ordering of such intervals with the labelling function that labels each interval by a if the set of labels of its positions is $\{a, b\}$. □

Combining Propositions 5.2–5.5 with Theorem 5.1, we obtain the following result, of which only the case of DA has not already been proved by the alternative approach referred to in § 3.

Corollary 5.6. *The pseudovarieties R, L, J and DA satisfy the equation $V_E = V$.*

It remains an open problem whether the pseudovariety DS satisfies the equality $V_E = V$. This motivates the study of the free profinite semigroup over DS , for which no representation result is currently known.

6. Ranks

For a pseudovariety V we consider the following associated parameters:

- (i) rank V is the least positive integer n such that V is defined by a set of pseudoidentities on at most n variables, unless there is no such n , in which case we let rank $V = \infty$;

- (ii) for a positive integer n , $V(n)$ denotes the pseudovariety generated by all n -generated members of V , that is the class of all finite continuous homomorphic images of $\bar{\Omega}_n V$;
- (iii) the *generator rank* of V , denoted $\text{grank } V$, is the least positive integer n such that $V = V(n)$, unless there is no such n , in which case we set $\text{grank } V = \infty$;
- (iv) the *idempotent generator rank* of V , denoted $\text{idgrank } V$, is the least positive integer n such that V is generated by its members which are generated by at most n idempotents, unless there is no such n , in which case we set $\text{idgrank } V = \infty$.

The following are simple observations concerning these parameters.

Lemma 6.1. *Let V be a pseudovariety and n be a positive integer. Denote by Σ_n the set of all pseudoidentities in at most n variables that are valid in V . Then the following hold:*

- (i) $\text{rank } V \leq n$ if and only if $V = \llbracket \Sigma_n \rrbracket$;
- (ii) $\text{grank } V \leq n$ if and only if $V = V(n)$;
- (iii) $\text{idgrank } V \leq n$ implies $V = V_E(n)$;
- (iv) $\text{grank } V \leq \text{idgrank } V$.

Lemma 6.2. *Let x_1, \dots, x_n be n distinct variables and consider the word $u_i = x_1 \cdots x_{i-1} x_{i+1} \cdots x_n$. Then the pseudoidentity*

$$(u_1^\omega \cdots u_n^\omega)^{\omega+1} = u_1^\omega \cdots u_n^\omega \tag{6.1}$$

holds in $DS(n-1)$ but not in $J(n)$.

Proof. Ordering the variables by increasing order of the indices, the canonical form of the implicit operation over J determined by the left-hand side of the pseudoidentity (6.1) is $(x_1 \cdots x_n)^\omega$, while the right-hand side is already in canonical form. By [1, Theorem 8.2.7] it follows that $\bar{\Omega}_n J$ fails (6.1), whence we see that $J(n)$ fails.

Let $\varphi: \bar{\Omega}_n \mathcal{S} \rightarrow \bar{\Omega}_{n-1} DS$ be any continuous homomorphism. We need to show that φ identifies the two sides of (6.1), that is that $\varphi(u_1^\omega \cdots u_n^\omega)$ is regular. Now, by [1, Theorems 8.1.10 and 8.2.7], $\varphi(u_1^\omega \cdots u_n^\omega)$ is regular if and only if it has the same content as some $\varphi(u_i^\omega)$. Since $c(\varphi(u_1^\omega \cdots u_n^\omega))$ contains at most $n-1$ variables, there is an index $i \in \{1, \dots, n\}$ such that

$$c(\varphi(u_1^\omega \cdots u_n^\omega)) = \bigcup_{j \in \{1, \dots, n\} \setminus \{i\}} c(\varphi(x_j)),$$

whence $c(\varphi(u_1^\omega \cdots u_n^\omega)) = c(\varphi(u_i^\omega))$. □

The following result is an immediate application of Lemma 6.2.

Proposition 6.3. *Every pseudovariety in the interval $[J, DS]$ has infinite grank.*

Proof. Let V be a pseudovariety in the interval $[J, DS]$ and suppose that it has finite grank m . Then for $n > m$ we have $V = V(n-1) = V(n)$. This entails that V is contained in $DS(n-1)$ and contains $J(n)$, which contradicts Lemma 6.2. \square

In contrast, the pseudoidentity definitions of the pseudovarieties J, R, L, DA, DG, DO, DS given at the end of §2 show that the rank of each of them is 2. Indeed, the smallest pseudovarieties defined by one-variable pseudoidentities containing the first four, respectively the last three, of these are respectively A and S .

There are pseudovarieties whose generator rank is smaller than its rank. As an example, we consider the pseudovariety $V = J(2)$. It is obvious that $V = V(2)$ and so the generator rank of V is 2 (note that, for example, the semigroup $\langle e, f \mid e^2 = e, f^2 = f, fe = 0 \rangle$ is in $J(2) \setminus J(1)$). Since, by Proposition 6.3, J has infinite generator rank, it follows that $J(2) \subsetneq J$. Now, by Lemma 6.4, we conclude that $J(2)$ has rank bigger than 2.

Lemma 6.4. *Let Σ_2 be the set of all pseudoidentities in at most two variables that are valid in $J(2)$. Then $\llbracket \Sigma_2 \rrbracket = J$.*

Proof. Let $u, v \in \bar{\Omega}_2 S$ be such that the pseudoidentity $u = v$ holds in $J(2)$. For every semigroup $S \in J$ and every continuous homomorphism $\varphi: \bar{\Omega}_2 S \rightarrow S$, the elements $\varphi(u)$ and $\varphi(v)$ belong to a 2-generated subsemigroup of S , which in turn is in $J(2)$. Thus, we have the equality of $\varphi(u)$ and $\varphi(v)$ and so the pseudoidentity $u = v$ holds in J .

For the direct inclusion, it suffices to note that, if $u = v$ is a pseudoidentity in at most two variables that is valid in J , then it is obviously valid in $J(2)$, and so it belongs to Σ_2 . Since $\text{rank } J = 2$, it follows that $\llbracket \Sigma_2 \rrbracket \subseteq J$. \square

At this point, we do not know what the rank of the pseudovariety $J(2)$ is, or even whether it is finite. Of course, if $J(2)$ is finitely based, then it has finite rank; but we also do not know if that is the case.

For the idgrank, we can prove the following results.

Lemma 6.5. *We define, recursively, two sequences of implicit operations as follows: for $n \geq 3$, we put*

$$\begin{aligned} v_3 &= (x_1 x_2)^\omega x_3 \cdot x_1^2 \cdot x_3 (x_1 x_2)^\omega, \\ w_3 &= (x_1 x_2)^\omega x_3 \cdot x_1 \cdot x_3 (x_1 x_2)^\omega, \\ v_{n+1} &= (x_1 \cdots x_n)^\omega x_{n+1} \cdot v_n \cdot x_{n+1} (x_1 \cdots x_n)^\omega, \\ w_{n+1} &= (x_1 \cdots x_n)^\omega x_{n+1} \cdot w_n \cdot x_{n+1} (x_1 \cdots x_n)^\omega. \end{aligned}$$

- (a) *If S is a semigroup from DA generated by $n \geq 2$ idempotents, then S satisfies the pseudoidentity $v_{n+1} = w_{n+1}$.*
- (b) *The pseudovariety $J(n)$ fails the pseudoidentity $v_{n+1} = w_{n+1}$.*

Proof. (a) By hypothesis, there exists some continuous homomorphism $\pi: \bar{\Omega}_n DA \rightarrow S$ that maps the free generators x_i to idempotents.

We proceed by induction on n . Given $\varphi: \bar{\Omega}_{n+1} S \rightarrow S$, we must show that $\varphi(v_{n+1}) = \varphi(w_{n+1})$. Since π is onto and $\bar{\Omega}_n S$ is free profinite, φ factors through π , say as $\varphi = \pi \circ \psi$ for some continuous homomorphism $\psi: \bar{\Omega}_{n+1} S \rightarrow \bar{\Omega}_n DA$.

At the basis of the induction, let us consider first the case $n = 2$. If $\psi(x_1 x_2)$ has full content, then $\varphi((x_1 x_2)^\omega)$ belongs to the minimum ideal of S . Since this ideal is a rectangular band and both $\varphi(v_3)$ and $\varphi(w_3)$ are \mathcal{L} and \mathcal{R} -below $\varphi((x_1 x_2)^\omega)$, they are both equal to this idempotent. Otherwise, $\psi(x_1 x_2)$ only involves one of the free generators of $\bar{\Omega}_n DA$ and so $\varphi(x_1)$ is an idempotent, in which case the equality $\varphi(v_3) = \varphi(w_3)$ is trivially verified.

For the general case $n > 2$, similarly, if $\psi(x_1 \cdots x_n)$ has full content, then $\varphi(v_{n+1}) = \varphi(w_{n+1})$. Otherwise, let $T = \langle x_1, \dots, x_n \rangle$ so that $\varphi(T)$ is a semigroup of DA generated by at most $n - 1$ idempotents. By the induction hypothesis, it satisfies the pseudoidentity $v_n = w_n$, whence we obtain $\varphi(v_n) = \varphi(w_n)$. Taking into account the definition of v_{n+1} and w_{n+1} , we conclude that $\varphi(v_{n+1}) = \varphi(w_{n+1})$.

(b) Let $\varphi = \bar{\Omega}_{n+1} S \rightarrow \bar{\Omega}_n J$ be the continuous homomorphism that fixes x_1 and sends each other x_i to x_{i-1} . Consider the factorizations of v_{n+1} and w_{n+1} in idempotents and maximal explicit factors between them which results from the recursive definition of these implicit operations. Then a straightforward induction shows that both these factorizations and the result of applying φ to each factor (and eliminating the repetition of x_1 within each ω -power) are in canonical form. Hence, $\varphi(v_{n+1}) \neq \varphi(w_{n+1})$ by [1, Theorem 8.2.7]. Since $\bar{\Omega}_n J$ is residually finite, this shows that there is some member of $J(n)$ that fails $v_{n+1} = w_{n+1}$. \square

In view of the definitions, Lemma 6.5 yields the following result.

Proposition 6.6. *The inequality $\text{idgrank } V(n) > n$ holds for every pseudovariety V in the interval $[J, DA]$.* \square

Combining Propositions 6.6, 5.4, 5.2, 5.3 and 5.5 yields the following result.

Corollary 6.7. *The equality $\text{idgrank } V(n) = n + 1$ holds for every pseudovariety V in $\{J, R, L, DA\}$.* \square

We finish the paper with a brief comparison of the results obtained here for the equality $V = V_E$, with $V \in \{J, R, L\}$, and the results that follow from the work of Straubing [19] and Howie [9], and of Pin [15] and Laradji and Umar [10], respectively.

Straubing showed that an n -element \mathcal{J} -trivial monoid divides \mathcal{C}_{n+1} , and Pin proved that a finite \mathcal{R} -trivial monoid with cardinal n embeds into \mathcal{E}_n .

On the other hand, several works deal with the ranks and idempotent ranks of various finite transformation semigroups. Recall that the *rank* of a finite semigroup is the minimum number of generators, and the *idempotent rank* of an idempotent-generated finite semigroup is the minimum number of idempotent generators. Gomes and Howie [7] showed that the rank and idempotent rank of the subsemigroup of \mathcal{T}_n consisting of all

full transformations with range less than n are both equal to $n(n-1)/2$. The rank and idempotent rank of the subsemigroup of all contractive finite full transformations are both equal to $n+1$, as shown by Umar [20]. Finally, Laradji and Umar [10] proved that the rank and idempotent rank of the subsemigroup of all contractive and order-preserving finite full transformations are both equal to n . We refer the reader to [10, 21] for detailed information on this topic.

Therefore, in the results quoted in §3 that follow from the works of the authors cited above, the idempotent rank of the idempotent-generated semigroup is related to the cardinality of the embedded semigroup. In this section together with §5, we proved that any finite semigroup of J , R , L or DA with rank n divides an idempotent-generated semigroup of the same type with idempotent rank at most $n+1$ (see Corollary 6.7). So, here the control on the number of generators is related with the number of generators of the embedded semigroup, which may be much smaller than its cardinality.

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References

1. J. ALMEIDA, *Finite semigroups and universal algebra* (World Scientific, 1994).
2. J. ALMEIDA, Profinite semigroups and applications, in *Structural theory of automata, semigroups, and universal algebra* (ed. V. B. Kudryavtsev and I. G. Rosenberg), NATO Science Series II: Mathematics, Physics and Chemistry, Volume 207, pp. 1–45 (Springer, 2005).
3. J. ALMEIDA AND P. WEIL, Free profinite R -trivial monoids, *Int. J. Alg. Comput.* **7** (1997), 625–671.
4. K. AUINGER, T. E. HALL, N. R. REILLY AND S. ZHANG, Congruences on the lattice of pseudovarieties of finite semigroups, *Int. J. Alg. Comput.* **7** (1997), 433–455.
5. C. BENZAKEN AND H. C. MAYR, Notion de demi-bande, demi-bandes de type deux, *Semigroup Forum* **10** (1975), 115–128.
6. S. EILENBERG, *Automata, languages and machines*, Volume B (Academic Press, New York, 1976).
7. G. GOMES AND J. HOWIE, On the ranks of certain finite semigroups of transformations, *Math. Proc. Camb. Phil. Soc.* **101** (1987), 395–403.
8. J. M. HOWIE, The subsemigroup generated by the idempotents of a full transformation semigroup, *J. Lond. Math. Soc.* **41** (1966), 707–716.
9. J. M. HOWIE, Products of idempotents in certain semigroups of order-preserving transformations, *Proc. Edinb. Math. Soc.* **17** (1971), 223–236.
10. A. LARADJI AND A. UMAR, On certain finite semigroups of order-decreasing transformations, I, *Semigroup Forum* **69** (2004), 184–200.
11. A. MOURA, E -local pseudovarieties, Technical Report CMUP 2009-45, University of Porto (2009) (available at <http://cmup.fc.up.pt/cmup/amoura/AMoura.Elocalpv.pdf>).

12. A. MOURA, Representations of the free profinite object over DA , *Int. J. Alg. Comput.* in press (DOI:10.1142/S0218196711006467).
13. F. PASTIJN, Embedding semigroups in semibands, *Semigroup Forum* **14** (1977), 247–263.
14. M. PETRICH, Embedding semigroups into idempotent generated ones, *Monatsh. Math.* **141** (2004), 315–322.
15. J.-E. PIN, *Varieties of formal languages* (Plenum, London, 1986).
16. N. R. REILLY AND S. ZHANG, Operators on the lattice of pseudovarieties of finite semigroups, *Semigroup Forum* **57** (1998), 208–239.
17. J. REITERMAN, The Birkhoff theorem for finite algebras, *Alg. Univers.* **14** (1982), 1–10.
18. J. G. ROSENSTEIN, *Linear orderings* (Academic Press, New York, 1982).
19. H. STRAUBING, On finite \mathcal{J} -trivial monoids, *Semigroup Forum* **19** (1980), 107–110.
20. A. UMAR, On the semigroups of order-decreasing finite full transformations, *Proc. R. Soc. Edinb. A* **120** (1992), 129–142.
21. A. UMAR, On the ranks of certain finite semigroups of order-decreasing transformations, *Portugaliae Math.* **53** (1996), 23–32.