A course on the braid groups

The *braid group* on n strands is the group $\mathcal{B}_n$ given by the presentation

$$\mathcal{B}_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1} \text{ for } 1 \le k \le n-2, \\ \text{and } \sigma_k\sigma_l = \sigma_l\sigma_k \text{ for } |k-l| \ge 2\rangle\,.$$

This group has many approaches from different branches of mathematics such as the low dimensional topology, the theory of groups, the algebraic geometry, and the mathematical physics.

The goal of this course is to present some aspects of the theory of braid groups. However, we will not always focus on the groups themselves, but often use them as a connecting thread to broach other subjects.

The first part of the course concerns the classical definition of the braid groups and their link with the configuration spaces and the discriminant.

The aim of the second part is the proof of Artin's presentation for $\mathcal{B}_n$ (the one given in the beginning of the abstract), but we will also study the standard (also called Coxeter type) presentation for the symmetric group. Somme homotopy theory and some group theory needed in the proof will be also explained.

Let M be an oriented and compact surface, possibly with boundary, and let $\mathcal{P}$ be a finite set of points in the interior of M. The *mapping class group* of M relatively to $\mathcal{P}$ is defined to be the group $\mathcal{M}(M,\mathcal{P})$ of isotopy classes of diffeomorphisms $F: M \to M$ which preserve the orientation, pointwise fix the boundary of M, and such that $F(\mathcal{P}) = \mathcal{P}$. In the third part we will present some basic examples of mapping class groups and some special elements belonging to them, and prove that the mapping class group of the disc relatively to n points is the braid group $\mathcal{B}_n$.

Bibliography. Some previous knowledge on fundamental groups and coverings may be useful to the listener. A good reference for this is [Mas]. The listener may also need to know the definitions of a free group and of a presentation for a group. The first 20 pages of [MKS] will suffice for that. Standard references for braid groups are [Bir], [Han] and [KuMu].

[**Bir**] **J.S. Birman**. *Braids, links, and mapping class groups.* Annals of Mathematics Studies, No. 82. Princeton University Press, Princeton, N.J., 1974.

[**Hans**] **V.L. Hansen**. *Braids and coverings: selected topics.* London Mathematical Society Student Texts, 18. Cambridge University Press, Cambridge, 1989.

[**KuMu**] **B.I. Kurpita, K. Murasugi**. *A study of braids.* Mathematics and its Applications, 484. Kluwer Academic Publishers, Dordrecht, 1999.

[**MKS**] **W. Magnus, A. Karrass, D. Solitar.** *Combinatorial group theory.* Reprint of the 1976 second edition. Dover Publications, Inc., Mineola, NY, 2004.

[**Mas**] **W.S. Massey.** *Algebraic topology: an introduction.* Reprint of the 1967 edition. Graduate Texts in Mathematics, Vol. 56. Springer-Verlag, New York-Heidelberg, 1977.