Computing kernels of finite monoids

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Definitions

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Let S and T be monoids. A **relational morphism** of monoids $\tau : S \multimap T$ is a function from S into $\mathcal{P}(T)$, the power set of T, such that:

- for all $s \in S,\ \tau(s) \neq \emptyset$;
- for all $s_1, s_2 \in S,\ \tau(s_1)\tau(s_2) \subseteq \tau(s_1 s_2)$;
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A relational morphism $\tau : S \multimap T$ is, in particular, a relation in $S \times T$. Thus, composition of relational morphisms is naturally defined.

Homomorphisms, seen as relations, and inverses of onto homomorphisms are examples of relational morphisms.

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Given a pseudovariety H of groups, the H-**kernel** of a finite monoid S is the submonoid

$$\mathsf{K}_{\mathsf{H}}(S) = \bigcap \tau^{-1}(1),$$

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Since a relational morphism into a group belonging to a certain pseudovariety H_1 of groups is also a relational morphism into a group belonging to a pseudovariety H_2 containing it, the following fact follows.

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Fact 1.1

Let M be a finite monoid and let H_1 and H_2 be pseudovarieties of groups such that $\mathsf{H}_1 \subseteq \mathsf{H}_2$. Then $\mathsf{K}_{\mathsf{H}_2}(M) \subseteq \mathsf{K}_{\mathsf{H}_1}(M)$.

Some other easy consequences

Proposition 2.1 (˜ , 98)

Let G be a group and H a pseudovariety of groups. Then $\mathsf{K}_{\mathsf{H}}(G)$ is the smallest normal subgroup of G such that $G/\mathsf{K}_{\mathsf{H}}(G) \in \mathsf{H}$.

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Fact 2.3

If T is a subsemigroup of a finite semigroup S, then $\mathsf{K}_{\mathsf{H}}(T) \subseteq \mathsf{K}_{\mathsf{H}}(S)$.

Let e be an idempotent of a finite semigroup S. As for every relational morphism $\tau\colon S \multimap G$ into a group G we have $\tau(e)\tau(e) \subseteq \tau(e)$, we get that $\tau(e)$ is a subgroup of G.

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It follows that $e \in \tau^{-1}(1)$. If $x, y \in \tau^{-1}(1)$, then $1 \in \tau(x)\tau(y) \subseteq \tau(xy)$, therefore $xy \in \tau^{-1}(1)$, thus $\tau^{-1}(1)$ is a subsemigroup of S containing the idempotents. As the non-empty intersection of subsemigroups is a subsemigroup, we have the following fact.

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Let H *be a pseudovariety of groups and let* M *be a finite monoid. The relative kernel* $\mathsf{K}_{\mathsf{H}}(M)$ *is a submonoid of* M *containing the idempotents.* □

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Note that, for example, if we can determine a set X of generators of a monoid M such that $X \subseteq \mathsf{K}_{\mathsf{H}}(M)$, then we can conclude by Fact 2.4 that the $M = \langle X \rangle \subseteq \mathsf{K}_{\mathsf{H}}(M)$.

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Pin showed that the problem of computing $\mathsf{K}_{\mathsf{G}}(S)$ can be reduced to that of computing the closure (relative to the profinite topology) of a rational subset of the free group. This approach led to the solution given by Ribes Ribes and Zalesskiĭ.

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Algorithms to compute other relative kernels (e.g., kernels relative to pseudovarieties of *p*-groups and pseudovarieties of abelian groups) followed the idea of Pin.

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A different algorithm has been given by Steinberg.

The Mal'cev product, when the rightmost factor is a pseudovariety of groups, may be defined as follows: for a pseudovariety V of monoids and a pseudovariety H of groups, the **Mal'cev product** of V and H is the pseudovariety

$$\mathsf{V} \ⓜ \mathsf{H} = \{S \mid \mathsf{K}_{\mathsf{H}}(S) \in \mathsf{V}\}.$$

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Algorithms to compute relative kernels may lead to decidability results.

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Proposition 4.1 (Pin, 88)

Let $x \in M$. Then $x \in \mathsf{K}_\mathsf{G}(M)$ if and only if $1 \in \mathsf{Cl}_\mathsf{G}(\varphi^{-1}(x))$ (the closure is taken for the profinite group topology of A^*).

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Commutative images of languages in A^* are used for the abelian kernel case, that is, the canonical homomorphism $\gamma : A^* \to \mathbb{Z}^n$ defined by $\gamma(a_i) = (0, \ldots, 0, 1, 0, \ldots, 0)$ (1 in position i), where a_i is the i^{th} element of A, is considered.

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Proposition 4.2 (˜ , 98)

Let $x \in M$. Then $x \in \mathsf{K_{Ab}}(M)$ if and only if $0 \in \mathsf{Cl_{Ab}}(\gamma(\varphi^{-1}(x)))$.

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A generalization, to all pseudovarieties of abelian groups, was obtained by Steinberg.

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To a supernatural number π one associates the pseudovariety H_π generated by the cyclic groups $\{\mathbb{Z}/n\mathbb{Z} \mid n \text{ divides } \pi\}$.

- $\mathsf{H}_{2^{+\infty}}$ is the pseudovariety of all 2-groups which are abelian;
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Proposition 4.3 (Steinberg, 99)

Let π be an infinite supernatural number and let $x \in M$. Then $x \in \mathsf{K}_{\mathsf{H}_\pi}(M)$ if and only if $0 \in \mathsf{Cl}_{\mathsf{H}_\pi}(\gamma(\varphi^{-1}(x)))$.

As a way to compute (a rational expression for) $\varphi^{-1}(x)$ one can consider the automaton $\Gamma(M, x)$ obtained from the right Cayley graph of M by taking the neutral element as the initial state and x as final state. Note that the language of $\Gamma(M, x)$ is precisely $\varphi^{-1}(x)$.

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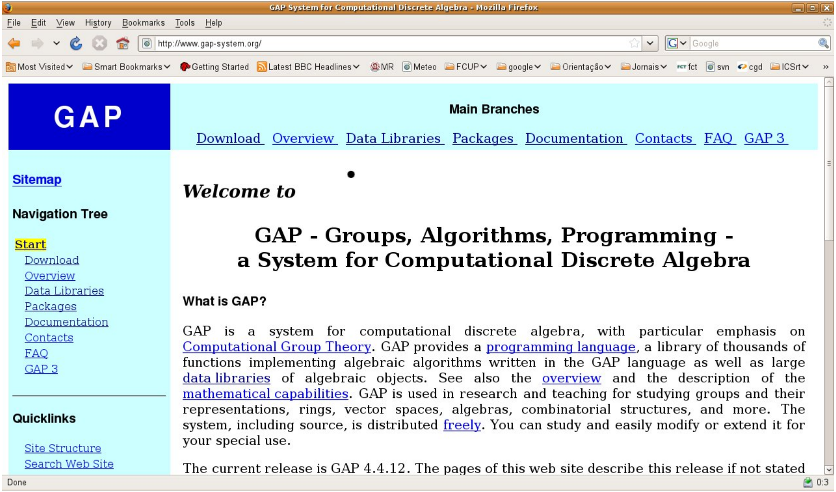
The need to visualize the results motivated the GAP package "sgpviz".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GAP - Groups, Algorithms, Programming - a System for Computational Discrete Algebra
What is GAP?
GAP is a system for computational discrete algebra, with particular emphasis on Computational Group Theory. GAP provides a programming language, a library of thousands of functions implementing algebraic algorithms written in the GAP language as well as large data libraries of algebraic objects. See also the overview and the description of the mathematical capabilities. GAP is used in research and teaching for studying groups and their representations, rings, vector spaces, algebras, combinatorial structures, and more. The system, including source, is distributed freely. You can study and easily modify or extend it for your special use.
The current release is GAP 4.4.12. The pages of this web site describe this release if not stated
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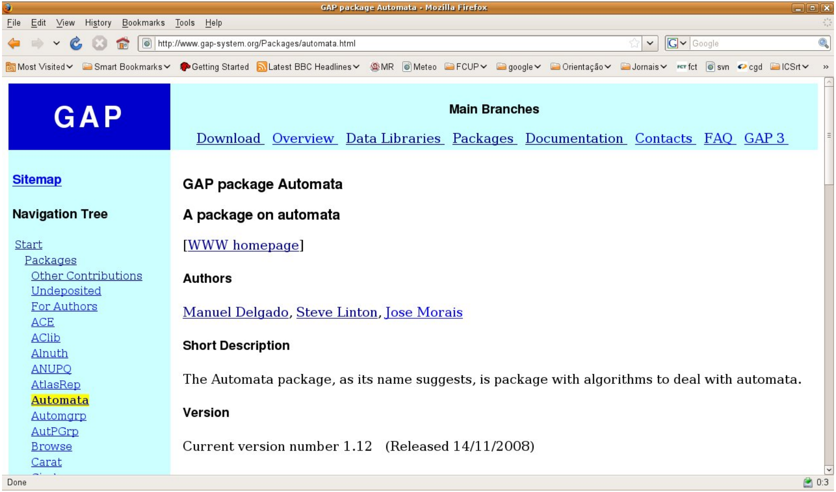
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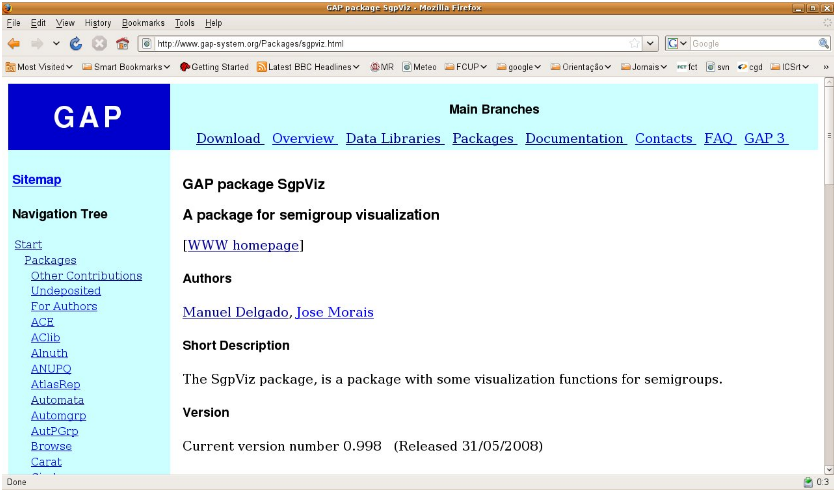
GAP package Automata
A package on automata
[WWW homepage]
Authors
Manuel Delgado, Steve Linton, Jose Morais
Short Description
The Automata package, as its name suggests, is package with algorithms to deal with automata.
Version
Current version number 1.12 (Released 14/11/2008)
Done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Automata
Automgrp
AutPGrp
Browse
Carat
GAP package SgpViz
A package for semigroup visualization
[WWW homepage]
Authors
Manuel Delgado, Jose Morais
Short Description
The SgpViz package, is a package with some visualization functions for semigroups.
Version
Current version number 0.998 (Released 31/05/2008)
Done

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[2, 1, 4, _, 3]	[2, 1, 5, _, 3]	[2, 1, 5, _, 4]	[3, 1, 5, _, 4]	[3, 2, 5, _, 4]
[2, 3, 4, _, 1]	[2, 3, 5, _, 1]	[2, 4, 5, _, 1]	[3, 4, 5, _, 1]	[3, 4, 5, _, 2]
[3, 2, 1, _, 4]	[3, 2, 1, _, 5]	[4, 2, 1, _, 5]	[4, 3, 1, _, 5]	[4, 3, 2, _, 5]
[3, 4, 1, _, 2]	[3, 5, 1, _, 2]	[4, 5, 1, _, 2]	[4, 5, 1, _, 3]	[4, 5, 2, _, 3]
[4, 1, 2, _, 3]	[5, 1, 2, _, 3]	[5, 1, 2, _, 4]	[5, 1, 3, _, 4]	[5, 2, 3, _, 4]
[4, 3, 2, _, 1]	[5, 3, 2, _, 1]	[5, 4, 2, _, 1]	[5, 4, 3, _, 1]	[5, 4, 3, _, 2]
[1, 2, _, 3, 4]	[1, 2, _, 3, 5]	*[1, 2, _, 4, 5]	[1, 3, _, 4, 5]	[2, 3, _, 4, 5]
[1, 4, _, 3, 2]	[1, 5, _, 3, 2]	[1, 5, _, 4, 2]	[1, 5, _, 4, 3]	[2, 5, _, 4, 3]
[2, 1, _, 4, 3]	[2, 1, _, 5, 3]	[2, 1, _, 5, 4]	[3, 1, _, 5, 4]	[3, 2, _, 5, 4]
[2, 3, _, 4, 1]	[2, 3, _, 5, 1]	[2, 4, _, 5, 1]	[3, 4, _, 5, 1]	[3, 4, _, 5, 2]
[3, 2, _, 1, 4]	[3, 2, _, 1, 5]	[4, 2, _, 1, 5]	[4, 3, _, 1, 5]	[4, 3, _, 2, 5]
[3, 4, _, 1, 2]	[3, 5, _, 1, 2]	[4, 5, _, 1, 2]	[4, 5, _, 1, 3]	[4, 5, _, 2, 3]
[4, 1, _, 2, 3]	[5, 1, _, 2, 3]	[5, 1, _, 2, 4]	[5, 1, _, 3, 4]	[5, 2, _, 3, 4]
[4, 3, _, 2, 1]	[5, 3, _, 2, 1]	[5, 4, _, 2, 1]	[5, 4, _, 3, 1]	[5, 4, _, 3, 2]
[1, _, 2, 3, 4]	[1, _, 2, 3, 5]	[1, _, 2, 4, 5]	*[1, _, 3, 4, 5]	[2, _, 3, 4, 5]
[1, _, 4, 3, 2]	[1, _, 5, 3, 2]	[1, _, 5, 4, 2]	[1, _, 5, 4, 3]	[2, _, 5, 4, 3]
[2, _, 1, 4, 3]	[2, _, 1, 5, 3]	[2, _, 1, 5, 4]	[3, _, 1, 5, 4]	[3, _, 2, 5, 4]
[2, _, 3, 4, 1]	[2, _, 3, 5, 1]	[2, _, 4, 5, 1]	[3, _, 4, 5, 1]	[3, _, 4, 5, 2]
[3, _, 2, 1, 4]	[3, _, 2, 1, 5]	[4, _, 2, 1, 5]	[4, _, 3, 1, 5]	[4, _, 3, 2, 5]
[3, _, 4, 1, 2]	[3, _, 5, 1, 2]	[4, _, 5, 1, 2]	[4, _, 5, 1, 3]	[4, _, 5, 2, 3]
[4, _, 1, 2, 3]	[5, _, 1, 2, 3]	[5, _, 1, 2, 4]	[5, _, 1, 3, 4]	[5, _, 2, 3, 4]
[4, _, 3, 2, 1]	[5, _, 3, 2, 1]	[5, _, 4, 2, 1]	[5, _, 4, 3, 1]	[5, _, 4, 3, 2]
[_, 1, 2, 3, 4]	[_, 1, 2, 3, 5]	[_, 1, 2, 4, 5]	[_, 1, 3, 4, 5]	*[_, 2, 3, 4, 5]
[_, 1, 4, 3, 2]	[_, 1, 5, 3, 2]	[_, 1, 5, 4, 2]	[_, 1, 5, 4, 3]	[_, 2, 5, 4, 3]
[_, 2, 1, 4, 3]	[_, 2, 1, 5, 3]	[_, 2, 1, 5, 4]	[_, 3, 1, 5, 4]	[_, 3, 2, 5, 4]
[_, 2, 3, 4, 1]	[_, 2, 3, 5, 1]	[_, 2, 4, 5, 1]	[_, 3, 4, 5, 1]	[_, 3, 4, 5, 2]
[_, 3, 2, 1, 4]	[_, 3, 2, 1, 5]	[_, 4, 2, 1, 5]	[_, 4, 3, 1, 5]	[_, 4, 3, 2, 5]
[_, 3, 4, 1, 2]	[_, 3, 5, 1, 2]	[_, 4, 5, 1, 2]	[_, 4, 5, 1, 3]	[_, 4, 5, 2, 3]
[_, 4, 1, 2, 3]	[_, 5, 1, 2, 3]	[_, 5, 1, 2, 4]	[_, 5, 1, 3, 4]	[_, 5, 2, 3, 4]
[_, 4, 3, 2, 1]	[_, 5, 3, 2, 1]	[_, 5, 4, 2, 1]	[_, 5, 4, 3, 1]	[_, 5, 4, 3, 2]

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Jointly with Cordeiro and Fernandes for finite superatural numbers and with Cordeiro for the general case, we obtained:

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Lemma 5.2

Let $f : G \to H$ be a homomorphism from G into a finite p-group H and let $x \in G$. If $p \nmid \operatorname{ord} x$, then $f(x) = 1$.

Proof.

Let $n = \operatorname{ord} f(x)$. Since f is a homomorphism, we have that $n \mid \operatorname{ord} x$, and therefore $p \nmid n$. But, as $f(x)$ belongs to a p-group, p must divide n, unless $n = 1$. Thus $\operatorname{ord} f(x) = 1$, which implies $f(x) = 1$. □

Theorem 5.3

$$K_{G_p}(G) = N_{\overline{p}}.$$

Proof.

Let $x \in G$ and suppose that $p \nmid \operatorname{ord} x$. Since to compute a relative kernel of a group it suffices to consider homomorphisms, it follows from the above lemma that $x \in K_{G_p}(G)$. Therefore, $K_{G_p}(G) \supseteq N_{\overline{p}}$.
For the converse, it suffices to note that the quotient $G/N_{\overline{p}}$ is a p-group and to use Proposition 2.1. Let $x \in G/N_{\overline{p}}$. If $p \nmid \operatorname{ord} x$, then $xN_{\overline{p}} = N_{\overline{p}}$. Suppose that $\operatorname{ord} x = p^\alpha k$, where α is the greatest power of p such that $p^\alpha \mid \operatorname{ord} x$. By observing that $x^{p^\alpha} \in N_{\overline{p}}$, we conclude that the order of $xN_{\overline{p}}$ divides p^α, thus is a power of p. □

We define recursively $\mathsf{K}_{\mathsf{H}}^{n}(S)$ as follows:

- $\mathsf{K}_{\mathsf{H}}^{0}(S) = S$;
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Since S is finite and the operator K_{H} is non-increasing, it follows that the sequence $\mathsf{K}^n_{\mathsf{H}}(S)$ is eventually constant; we denote this constant value by

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Observe that $\mathsf{K}^\omega_{\mathsf{H}}(S)$ is the largest subsemigroup of S fixed by K_{H}.

For a pseudovariety V and $n \geq 0$, we define the operator $(-)^n \textcircled{m}\, \mathsf{H}$ recursively as follows:

- $\mathsf{V}\,^0\textcircled{m}\, \mathsf{H} = \mathsf{V}$;
- $\mathsf{V}\,^{n+1}\textcircled{m}\, \mathsf{H} = (\mathsf{V}\,^n\textcircled{m}\, \mathsf{H})\, \textcircled{m}\, \mathsf{H}$;
- $\mathsf{V}\,^\omega\textcircled{m}\mathsf{H} = \cup_{n\geq 0} \mathsf{V}\,^n\textcircled{m}\, \mathsf{H}$.

It is easy to see

$\mathsf{V}\ {}^{n}\!\textcircled{m}\ \mathsf{H} = \{S \mid \mathsf{K}^{n}_{\mathsf{H}}(S) \in \mathsf{V}\}$ and $\mathsf{V}\ {}^{\omega}\!\textcircled{m}\mathsf{H} = \{S \mid \mathsf{K}^{\omega}_{\mathsf{H}}(S) \in \mathsf{V}\}$.

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A much more general result has then been obtained in a joint work with Fernandes, Margolis and Steinberg (2004). It states that:

for a non-trivial pseudovariety H of groups, a semigroup with an aperiodic idempotent-generated subsemigroup is H-solvable if and only if it subgroups are H-solvable.

We proved, in particular, that

$$\mathsf{EA} = \mathsf{A} \;{}^{\omega}\!\textcircled{m}\, \mathsf{G}$$

where we denote by EA the pseudovariety consisting of all monoids whose idempotents generate an aperiodic submonoid

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By using a modification of the technique, it has been shown in a joint work with Steinberg that:

a semigroup S is H-solvable if and only if, for each idempotent $e \in S$, there is a subnormal series with smallest element the maximal subgroup at e of the idempotent-generated subsemigroup of S and largest element the maximal subgroup of S at e such that the successive quotients belong to H.