Type II theorem and hyperdecidability of pseudovarieties of groups

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Abstract

The problem of computing the kernel of a finite monoid, popularized under the name "Rhodes type II conjecture", led Ash to the proof of a strong property (the hyperdecidability) of the pseudovariety **G** of all finite groups. This paper is a survey of recent results on the study of some related properties of pseudovarieties of groups.

1 Introduction

The kernel of a finite monoid M has been introduced by Rhodes aiming to treat questions related with the group-complexity of a finite monoid. The *kernel* is defined as the intersection of the subsets $\tau^{-1}(1)$ where τ is a relational morphism from M into a finite group G. This definition is clearly non constructive. However Rhodes conjectured that the kernel of a finite monoid is computable and, in fact, almost 20 years after, the following theorem, called *type II theorem*, appeared.

Theorem 1.1 (Ash [5], Ribes and Zalesskiĭ [19]) *The kernel of a finite monoid is computable.*

The proof by Ribes and Zalesskiĭ appears after works by Pin [16] and Pin and Reutenauer [18] translating the original problem into a problem on profinite groups. We refer to this proof as the *topological proof* and to the proof obtained by Ash as the *algebraic proof.* A paper by Henckell, Margolis,

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Pin and Rhodes [12], written soon after the algebraic proof, contains some of the history and consequences of the type II conjecture and an extensive literature on the theme.

Aiming to obtain methods for computing systematically semidirect products of pseudovarieties of semigroups, Almeida [2] introduced the notion of hyperdecidability of a pseudovariety of semigroups. The hyperdecidability of $\mathbf{G}$, the pseudovariety of all finite groups, was established by Ash when proving the type II theorem. See also [2, Theorem 7].

The motivation for the work presented here has been to obtain relative versions of results that hold for the pseudovariety $\mathbf{G}$ and are relevant in some proof of the type II conjecture. In order to do this, we relativize the definition of kernel of a finite monoid M to a pseudovariety $\mathbf{H}$ of groups,

$$K_{\mathbf{H}}(M) = \bigcap\{x \in M \mid x \in \tau^{-1}(1), \tau : M \multimap G \text{ relational morphism}, G \in \mathbf{H}\}$$

and we try to relativize the existing proofs.

The aim of this paper is to review what has been done on this subject. For notation and undefined terms, we refer the reader to [15] and [1]. In Sections 3 and 4 we recall the topological and algebraic proofs of the type II theorem and state relativized versions of the properties of $\mathbf{G}$ involved. In Section 6, we give an algorithm to compute the abelian kernel of a finite monoid.

2 Hyperdecidability

A (directed) *graph* $\Gamma = V(\Gamma) \dot{\cup} E(\Gamma)$ is the union of two disjoint sets: the set of *vertices* $V(\Gamma)$ and the set of *edges* $E(\Gamma)$, together with two functions $\alpha, \omega : E(\Gamma) \to V(\Gamma)$ which describe, respectively, the *beginning* and the *end* of each edge.

Let Γ be a finite graph and let X be a set. An X-*labeling* of Γ is a function $\epsilon : \Gamma \to X$. The graph Γ labeled by X is said to be an X-*graph* and we use the notation Γ_ϵ to point out that ϵ is the labeling.

Let M be a monoid and let $\epsilon : \Gamma \to M$ be an M-labeling of Γ. We say that ϵ is *consistent* if, for every edge $e \in E(\Gamma)$, we have $\epsilon(\alpha(e))\epsilon(e) = \epsilon(\omega(e))$. Let N be a monoid and $\tau : M \multimap N$ a relational morphism. Associated with ϵ and τ, we consider a $\mathcal{P}(N)$-labeling of Γ by labelling each element z of Γ by the subset $\tau(\epsilon(z)) \subseteq N$. Any N-labeling of Γ obtained by choosing, for each

$z \in \Gamma$, an element of the set $\tau(\epsilon(z)) \subseteq N$ is said to be τ-*related* with ϵ. We say that the M-graph Γ_ϵ is *inevitable for* τ if there is a consistent N-labeling of Γ, $\epsilon' : \Gamma \to N$, that is τ-related with ϵ.

Let $\mathbf{V}$ be a pseudovariety of monoids. We say that Γ_ϵ is $\mathbf{V}$-*inevitable* if it is inevitable for all relational morphisms from M to monoids of $\mathbf{V}$. We say that $\mathbf{V}$ is *hyperdecidable* if there is an algorithm with **input** a labeled graph and **output** *Yes* or *No* according to whether the graph is or not $\mathbf{V}$-inevitable.

The M-graph Γ_ϵ is said to be *avoidable for* τ if Γ_ϵ is not inevitable for τ and is said to be $\mathbf{V}$-*avoidable* if it is not $\mathbf{V}$-inevitable.

Let $x \in M$. A labeled graph consisting of a single vertex labeled by 1 and a single edge labeled by x will be called a *loop labeled by* x. We observe that x belongs to $K_{\mathbf{H}}(M)$ if and only if the loop labeled by x is $\mathbf{H}$-inevitable.

From now on we will be interested in the hyperdecidability of pseudovarieties of groups.

Let A be a finite set, usually called an *alphabet*. An A-*monoid* is a monoid M together with a homomorphism $\varphi : A^* \to M$ such that $\varphi(A)$ generates M as a monoid, i.e., φ is surjective. An A-*group* is defined analogously. It is not difficult to see that, in order to test the $\mathbf{H}$-inevitability of an M-graph Γ_ϵ, where M is a finite A-monoid, it suffices to consider relational morphisms into A-groups of $\mathbf{H}$.

Let A be a finite alphabet and M, together with $\varphi : A^* \to M$, a finite A-monoid. We denote by $F_{\mathbf{H}}(A)$ the free group over the alphabet A in the variety of groups generated by the pseudovariety $\mathbf{H}$. It is considered endowed with the pro-$\mathbf{H}$ topology (i.e., the least topology rendering continuous all homomorphisms into members of $\mathbf{H}$, the elements of $\mathbf{H}$ being considered discrete), which gives it a structure of a Hausdorff topological group. If $\mathbf{H} = \mathbf{G}$, we use the notation $F(A)$ instead of $F_{\mathbf{G}}(A)$.

3 Topological proof

Let X be a topological group and let $\overline{Y}$ denote the closure of the subset $Y \subseteq X$. Consider the following algorithm, where S, T are subsets of X.

Algorithm 1

(1) $\overline{S} = S$, if S is finite;

(2) $\overline{S \cup T} = \overline{S} \cup \overline{T}$;

(3) $\overline{S \cdot T} = \overline{S} \cdot \overline{T}$;

(4) $\overline{S^*} = \langle S \rangle$.

The following result is due to Ribes and Zalesskiĭ [19].

Proposition 3.1 *The product of a finite number of finitely generated subgroups of $F(A)$ is closed.*

The following two results, whose proofs use Proposition 3.1, have been obtained by Pin and Reutenauer [18].

Proposition 3.2 *The closures of the rational subsets of $F(A)$ may be computed using Algorithm 1.*

Proposition 3.3 *The closed rational subsets of $F(A)$ are precisely those that may be obtained as finite unions of sets of the form $gG_1G_2\cdots G_r$, where $g \in F(A)$ and $G_1, \ldots, G_r$ are finitely generated subgroups of $F(A)$.*

Let A^* be endowed with the profinite group topology, i.e., the least topology rendering continuous all homomorphisms into finite groups. Proposition 3.2 allowed Pin [17] to obtain an effective algorithm to compute the closure of a rational language (given by a rational expression). This closure is again a rational language. The following proposition is also due to Pin [16].

Proposition 3.4 *An element $x \in M$ belongs to the kernel of M if and only if $1 \in \overline{\varphi^{-1}(x)}$.*

So, testing whether an element $x \in M$ is in the kernel of M can be done using the following steps:

Step 1 Compute a rational expression for $\varphi^{-1}(x)$;
Step 2 Compute the closure $\overline{\varphi^{-1}(x)}$;
Step 3 Test whether $1 \in \overline{\varphi^{-1}(x)}$.

Steps 1 and 3 may be done using standard algorithms. In fact, there are well known algorithms to compute rational expressions and to decide the membership of a word in a given rational set. As discussed before Proposition 3.4, there are algorithms to do Step 2.

We denote by $\mathcal{RZ}$ the class of pseudovarieties of groups satisfying a relativized version of Ribes and Zalesskiĭ's theorem, i.e., $\mathcal{RZ}$ is the class of pseudovarieties of groups $\mathbf{H}$ such that the product of a finite number of finitely generated subgroups of $F_{\mathbf{H}}(A)$ is closed.

Our first result shows that the relativized versions of Propositions 3.1, 3.2 and 3.3 are equivalent.

Theorem 3.5 *Let $\mathbf{H}$ be a pseudovariety of groups. The following conditions are equivalent.*

(i) $\mathbf{H} \in \mathcal{RZ}$*;*

(ii) the closures of the rational subsets of $F_{\mathbf{H}}(A)$ may be computed using Algorithm 1;

(iii) the closed rational subsets of $F_{\mathbf{H}}(A)$ are precisely those that may be obtained as finite unions of sets of the form $gG_1G_2\cdots G_r$, where $g \in F_{\mathbf{H}}(A)$ and $G_1, \ldots, G_r$ are finitely generated subgroups of $F_{\mathbf{H}}(A)$.

Proof. The proofs by Pin and Reutenauer of Propositions 3.2 and 3.3 work *ipsis verbis* to prove $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (iii)$, respectively.

In order to prove $(ii) \Rightarrow (i)$, let $H_1, \ldots, H_n$ be finitely generated subgroups (and therefore rational subsets) of $F_{\mathbf{H}}(A)$. Then, by hypothesis, $\overline{H_i} = \overline{H_i^*} = \langle H_i \rangle = H_i$, for any $i \in \{1, \ldots, n\}$ and, using once again the hypothesis, $\overline{H_1 \cdots H_n} = \overline{H_1} \cdots \overline{H_n} = H_1 \cdots H_n$, thus $\mathbf{H} \in \mathcal{RZ}$.

In order to prove $(iii) \Rightarrow (i)$, let $H_1, \ldots, H_n$ be finitely generated subgroups of $F_{\mathbf{H}}(A)$. By hypothesis $H_1 \cdots H_n$ is a closed rational subset of $F_{\mathbf{H}}(A)$, thus $\mathbf{H} \in \mathcal{RZ}$. $\square$

4 Algebraic proof

Let $A^{-1} = \{a^{-1} \mid a \in A\}$ be a disjoint copy of A. A word $w = a_1^{\epsilon_1} \cdots a_r^{\epsilon_r} \in (A \dot\cup A^{-1})^*$, $\epsilon_i \in \{1, -1\}$, $1 \leq i \leq r$, is said to be *adequate for* $x \in M$ if x has a factorization $x = x_1 \cdots x_r$ such that, for each i, if $\epsilon_i = 1$ then $\varphi(a_i) = x_i$ and if $\epsilon_i = -1$ then $x_i \varphi(a_i) x_i = x_i$. We denote by K_x the set of adequate words for x. The canonical homomorphism from $(A \dot\cup A^{-1})^*$ onto $F_{\mathbf{H}}(A)$ is denoted by $[\cdot]_{\mathbf{H}}$. We observe that $\varphi^{-1}(x)$ may be seen as a subset of $(A \dot\cup A^{-1})^*$. For pseudovarieties of $\mathcal{RZ}$ the set $[K_x]_{\mathbf{H}}$ may be expressed in another way, as stated in the following proposition proved in [7].

Proposition 4.1 *Let $\mathbf{H} \in \mathcal{RZ}$ be a pseudovariety of groups. Then $[K_x]_{\mathbf{H}} = \overline{[\varphi^{-1}(x)]_{\mathbf{H}}}$.*

Let $\theta_{(M,\mathbf{H})} : M \to \mathcal{P}(F_{\mathbf{H}}(A))$ be the relational morphism (see [7]) defined by $\theta_{(M,\mathbf{H})}(x) = [K_x]_{\mathbf{H}}$. It is called the $(M, \mathbf{H})$-*canonical* relational morphism. In the sequel we will write only θ for $\theta_{(M,\mathbf{H})}$, if M and $\mathbf{H}$ are understood.

After results obtained by Almeida [2] (see also [3]), Ash's theorem [5, Theorem 2.1] may be stated as follows:

Proposition 4.2 *Let ϵ be a labeling of the finite graph Γ by a finite monoid M. Then, the graph Γ_ϵ is $\mathbf{G}$-inevitable if and only if it is inevitable for the $(M, \mathbf{G})$-canonical relational morphism.*

We denote by $\mathcal{A}$ the class of pseudovarieties of groups $\mathbf{H}$ such that, given a labeling ϵ of the finite graph Γ by a finite monoid M, the graph Γ_ϵ is $\mathbf{H}$-inevitable if and only if it is inevitable for the $(M, \mathbf{H})$-canonical relational morphism. Another class, $\mathcal{A}'$, may be defined considering labeled loops instead of considering all labeled finite graphs.

From Proposition 4.1 and an analogue of Proposition 3.4 stated in [6] one deduces the following proposition.

Theorem 4.3 *The following inclusion holds: $\mathcal{RZ} \subseteq \mathcal{A}'$.*

The algorithm establishing the hyperdecidability of $\mathbf{G}$ is given in Proposition 3.1 of Ash's paper. In fact, this proposition allows the construction (up to obvious identifications) of all connected $\mathbf{G}$-inevitable M-graphs with at most n vertices, for any given positive integer n.

5 Systems with constraints in relatively free groups

In this section, we give a characterization of the classes $\mathcal{A}$ and $\mathcal{RZ}$ in terms of systems of equations with constraints in relatively free groups.

The proofs of 5.2 and 5.3 may be found in [8]. The ideas inspiring them, as well as Lemma 5.1, come from a joint work with Almeida [3], where the property stated in 5.3 (ii) has been proved for the pseudovariety $\mathbf{G}$. Herwig and Lascar [13] have also proved the same property for $\mathbf{G}$, using methods from model theory that seem to be very different from ours.

To a finite graph Γ we associate a system of equations

$$x_{\alpha(e)} x_e = x_{\omega(e)} \quad (e \in E(\Gamma)). \tag{1}$$

A relational morphism $\tau : M \multimap N$ from the A-monoid M into the monoid N and a labeling ϵ of Γ by M provide the constraints for the solutions of (1) in the monoid N:

$$x_z \in \tau(\epsilon(z)) \quad (z \in \Gamma). \tag{2}$$

Lemma 5.1 *The system (1) has a solution under the constraints (2) if and only if* Γ_ϵ *is inevitable for* τ.

Theorem 5.2 *Let* $\mathbf{H}$ *be a pseudovariety of groups. The following conditions are equivalent.*

(i) $\mathbf{H} \in \mathcal{A}$*;*

(ii) for each finite labeled graph Γ_ϵ*, if the system (1) has no solution under the constraints* $x_z \in \theta(\epsilon(z))$ $(z \in \Gamma)$*, then there is a normal subgroup* K *of finite index of* $F_{\mathbf{H}}(A)$ *such that the same system also has no solution under the constraints* $x_z \in \theta(\epsilon(z)) \cdot K$ $(z \in \Gamma)$.

Theorem 5.3 *Let* $\mathbf{H}$ *be a pseudovariety of groups. The following conditions are equivalent.*

(i) $\mathbf{H} \in \mathcal{RZ}$*;*

(ii) for each finite graph Γ *and, for each* $z \in \Gamma$, $H_{1,z}, \ldots, H_{n_z,z}$ *finitely generated subgroups of* $F_{\mathbf{H}}(A)$ *and* $g_z \in F_{\mathbf{H}}(A)$*, if the system (1) has no solution under the constraints* $x_z \in g_z H_{1,z} \cdots H_{n_z,z}$*, then, for each* $i \in \{1, \ldots, n_z\}$*, there is a subgroup* $K_{i,z}$ *of* $F_{\mathbf{H}}(A)$ *of finite index containing* $H_{i,z}$ *such that the system (1) also has no solution under the constraints* $x_z \in g_z K_{1,z} \cdots K_{n_z,z}$.

6 The abelian kernel of a monoid

It is well known that every subgroup of the free abelian group $F_{\mathbf{Ab}}(A)$ is closed for the profinite (i.e., pro-$\mathbf{G}$) topology (see [6]), so $\mathbf{Ab} \in \mathcal{RZ}$, and, by Proposition 4.3, $\mathbf{Ab} \in \mathcal{A}'$. Therefore, in order to test whether an element of a finite monoid is in the abelian kernel, it suffices to test whether the loop labeled by x is inevitable for the $(M, \mathbf{Ab})$-canonical relational morphism.

We denote by $|w|_{a_i}$ the number of occurrences of the letter a_i in $w \in A^*$. The free abelian group $F_{\mathbf{Ab}}(A)$ on $|A|$ generators is isomorphic to $\mathbb{Z}^n$, where $n = |A|$, and we may construct the isomorphism in such a way that, for $w \in A^*$, $[w]_{\mathbf{Ab}} = (|w|_{a_1}, \ldots, |w|_{a_n})$.

We give now an algorithm with **input** a finite A-monoid M (given, for example, by its multiplication table) and a loop labeled by an element of the monoid, and **output** *Yes*, if the labeled loop is **Ab**-inevitable, and *No*, if it is **Ab**-avoidable.

Algorithm 2

Step 1 Construct the Cayley graph Γ_M of the A-monoid M;
Step 2 For $x \in M$ consider the automaton $\mathcal{A}(x)$ obtained from the Cayley graph Γ_M by taking 1 for initial state and x for final state, and compute a rational expression for the language recognized by $\mathcal{A}(x)$;
Step 3 Compute the commutative image of the rational expression obtained in Step 2;
Step 4 Compute the closure of the subset obtained in Step 3.

There are algorithms to do Step 1 (see [10]). In Step 2, the language $\varphi^{-1}(x)$ is precisely the language recognized by the automaton $\mathcal{A}(x)$ and a rational expression for this language may be obtained using Kleene's algorithm. In [6], an algorithm to do Step 3 is given, the set $[\varphi^{-1}(x)]_{\mathbf{Ab}}$ being obtained as a finite union of subsets of the form $a+b_1\mathbb{N}+\cdots+b_r\mathbb{N}$. Step 4 is performed by substituting each occurrence of $\mathbb{N}$ by $\mathbb{Z}$ (see Proposition 4.1).

As $\mathbf{Ab} \in \mathcal{A}'$, to determine whether Γ_ϵ is **Ab**-inevitable is equivalent (cf. Lemma 5.1) to determine whether the system of equations

$$x_{\alpha(e)} + x_e = x_{\omega(e)} \quad (e \in E(\Gamma))$$

has any solution under the constraints $x_z \in \overline{[\varphi^{-1}(\epsilon(z))]_{\mathbf{Ab}}}$ $(z \in \Gamma)$. These constraints may, using the preceding steps, be written in an effective way as disjunctions of a finite number of constraints of the form

$$x_z \in a_z + b_{1,z}\mathbb{Z} + \cdots + b_{n_z,z}\mathbb{Z}, \quad a_z, b_{1,z}, \ldots, b_{n_z,z} \in \mathbb{Z}^n.$$

Consequently, the problem of existence of an algorithm to test the **Ab**-inevitability of a graph is reduced to that of the existence of an algorithm to solve the system

$$x_{\alpha(e)} + x_e = x_{\omega(e)} \quad (e \in E(\Gamma))$$

under the constraints

$$x_z \in a_z + b_{1,z}\mathbb{Z} + \cdots + b_{n_z,z}\mathbb{Z}, \quad a_z, b_{1,z}, \ldots, b_{n_z,z} \in \mathbb{Z}^n.$$

This system has a solution if and only if the linear diophantine system obtained by substitution of each x_z subject to the constraint $x_z \in a_z+b_{1,z}\mathbb{Z}+$

$\cdots + b_{n_z,z}\mathbb{Z}$ by $a_z + b_{1,z}y_{1,z} + \cdots + b_{n_z,z}y_{n_z,z}$, with indeterminates $y_{1,z}, \ldots, y_{n_z,z}$, has a solution.

There are well known algorithms to solve linear diophantine systems. See, for example, [9]. So, in particular, there are algorithms to test whether they have a solution. We then have an algorithm to compute the abelian kernel of a monoid.

This algorithm is not efficient, because the length of the rational expressions obtained using Kleene's Algorithm grows exponentially (see [8]). We do not know if this step may be modified in order to obtain a polynomial algorithm.

7 Final remarks

Several questions that arise naturally have not yet found an answer. We indicate two of them.

Question 1. Does the inclusion $\mathcal{RZ} \subseteq \mathcal{A}$ hold?

Let $\mathcal{A}''$ be the class of pseudovarieties of groups whose definition is obtained from that of $\mathcal{A}$ replacing "finite monoids" by "finite inverse monoids". Ash [5] proves that $\mathbf{G}$ belongs to both classes. Observing (see [7]) that part of Ash's proof works as well for any other pseudovariety of groups, one may conclude that the obvious inclusion $\mathcal{A}'' \subseteq \mathcal{A}$ is in fact an equality. So, in order to solve the question, one could attempt to prove the inclusion $\mathcal{RZ} \subseteq \mathcal{A}''$.

Question 2. Is the pseudovariety $\mathbf{Ab}$ hyperdecidable?

We observe that a positive answer to Question 1 would give immediately a positive answer to Question 2. In fact, small changes in the algorithm presented in Section 6 would give an algorithm to test the $\mathbf{Ab}$-inevitability of a labeled graph. Another way would be to look at a recent work of Almeida and Steinberg [4]. In fact, the pseudovarieties in $\mathcal{A}$ are *κ-reducible* (using the terminology due to Almeida and Steinberg) and those are hyperdecidable [4].

Kleene's algorithm, used to perform Step 2 in Algorithm 2, is implemented in some computer programs, for example, "AMoRE" [14]. In the framework of the Project GAP at the LIAFA (Univ. Paris 7), ongoing work of the author implements, using the GAP programming language [11], the algorithm presented in this paper to test whether a given labeled finite graph is $\mathbf{Ab}$-inevitable.

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References

[1] J. Almeida, "Finite Semigroups and Universal Algebra", World Scientific, Singapore, 1994.

[2] ———, *Hyperdecidable pseudovarieties and the calculation of semidirect products*, Int. J. Algebra and Computation. To appear.

[3] J. Almeida *and* M. Delgado, *Sur certains systèmes d'équations avec contraintes dans un groupe libre*, Portugaliæ Mathematica. To appear.

[4] J. Almeida *and* B. Steinberg, *On the decidability of iterated semidirect products with applications to complexity*, Proc. London Math. Society. To appear.

[5] C. Ash, *Inevitable graphs: a proof of the type II conjecture and some related decision procedures*, Int. J. Algebra and Computation **1** (1991) 127–146.

[6] M. Delgado, *Abelian pointlikes of a monoid*, Semigroup Forum **56** (1998) 339–361.

[7] ———, *On the hyperdecidability of pseudovarieties of groups*, Int. J. Algebra and Computation. To appear.

[8] ———, "Teorema do Tipo II e Hiperdecidibilidade de Pseudovariedades de Grupos", Doctoral thesis, University of Porto, 1997.

[9] E. Domenjoud and A. P. Tomás, *From Elliott-MacMahon to an Algorithm for General Linear Constraints on Naturals*, in Proc. of 1st International Conf. on Principles and Practice of Constraint Programming (CP'95), Cassis (France), volume 976 of LNCS, pages 18-35, Springer-Verlag, 1995.

[10] V. Froidure *and* J. E. Pin, *Algorithms for computing finite semigroups*, in Foundations of Computational Mathematics, F. Cucker and M. Shub eds., Berlin, Lecture Notes in Computer Science, Springer Verlag, (1997), 112–126.

[11] Martin Schönert et al, *GAP – Groups, Algorithms, and Programming*, Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, fifth edition, 1995.

[12] K. Henckell, S. Margolis, J.-E. Pin, *and* J. Rhodes, *Ash's type II theorem, profinite topology and Malcev products. Part I*, Int. J. Algebra and Computation **1** (1991) 411–436.

[13] B. Herwig *and* D. Lascar, *Extending partial automorphisms and the profinite topology on the free groups*, Transactions of the American Math. Soc. To appear.

[14] O. Matz, A. Miller, A. Pothoff, W. Thomas, *and* E. Valkema, *Report on the program AMoRE*, Tech. Rep. 9507, Christian Albrechts Universität, Kiel, 1995.

[15] J. E. Pin, "Varieties of Formal Languages", Plenum, London, 1986.

[16] ———, *A topological approach to a conjecture of Rhodes*, Bull. Austral. Math. Soc. **38** (1988) 421–431.

[17] ———, *Topologies for the free monoid*, J. Algebra **137** (1991) 297–237.

[18] J.-E. Pin *and* C. Reutenauer, *A conjecture on the Hall topology for the free group*, Bull. London Math. Soc. **23** (1991) 356–362.

[19] L. Ribes *and* P. A. Zalesskiĭ, *On the profinite topology on a free group*, Bull. London Math. Soc. **25** (1993) 37–43.

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