The higher topological complexity of subcomplexes of products of spheres and related polyhedral product space

Bárbara Mayela Gutiérrez Mejía

Joint work with Professors Jesús González and Sergey Yusvinzky, arXiv: 1501.07474

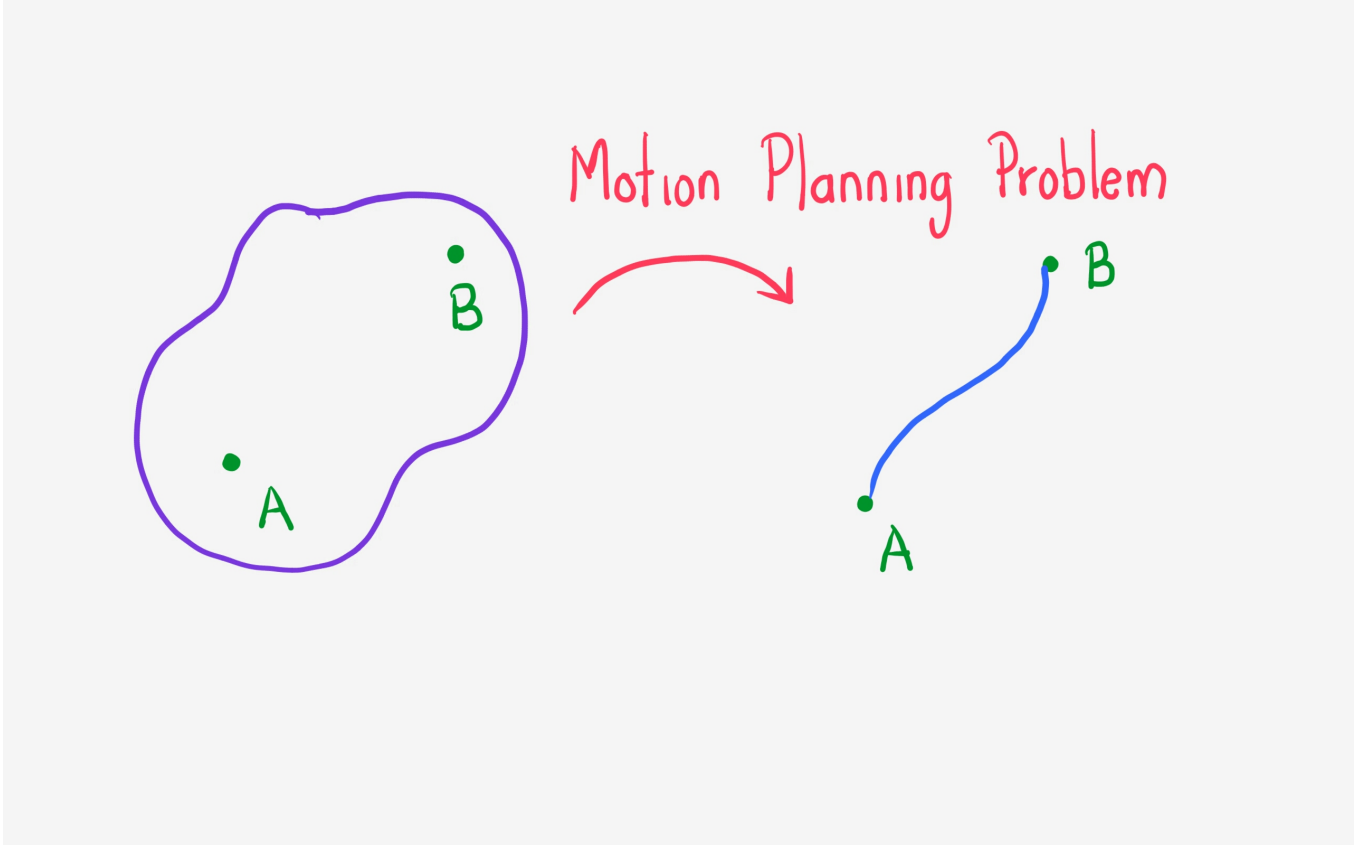
Mathematics Department
CINVESTAV, Mexico

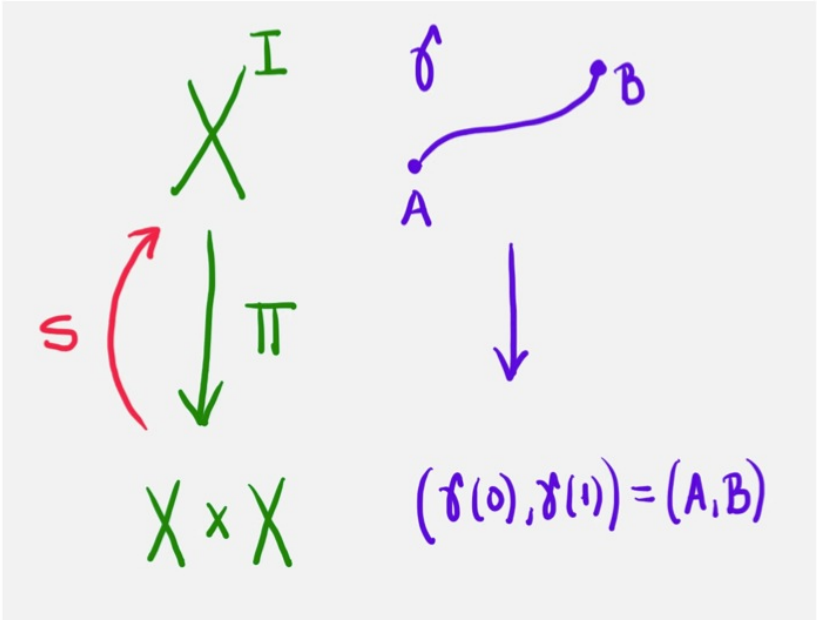
February 8, 2015

XXIst Oporto Meeting on Geometry, Topology and Physics

The motion planning problem

Let X be a path connected space. A motion planning algorithm in X takes as input a pair $(A, B) \in X \times X$ and returns (as an output) a continuous path in X that connects A with B.





More precisely, let X^I be the space of all continuous paths, $\gamma : [0, 1] \to X$ in X. Let $\pi : X^I \to X \times X$ be the end-points map given by $\pi(\gamma) = (\gamma(0), \gamma(1))$. Thus, a motion planning algorithm in X, is a continuous section of π, that is, a continuous map $s : X \times X \to X^I$, such that $\pi \circ s((x, y)) = (x, y)$ for all $(x, y) \in X \times X$.

Proposition

There exists a continuous section s as above if and only if X is contractible.

Proposition

There exists a continuous section s as above if and only if X is contractible.

In 2003 Michael Farber proposed a topological model to study the instabilities of the motion planning problem in robotics.

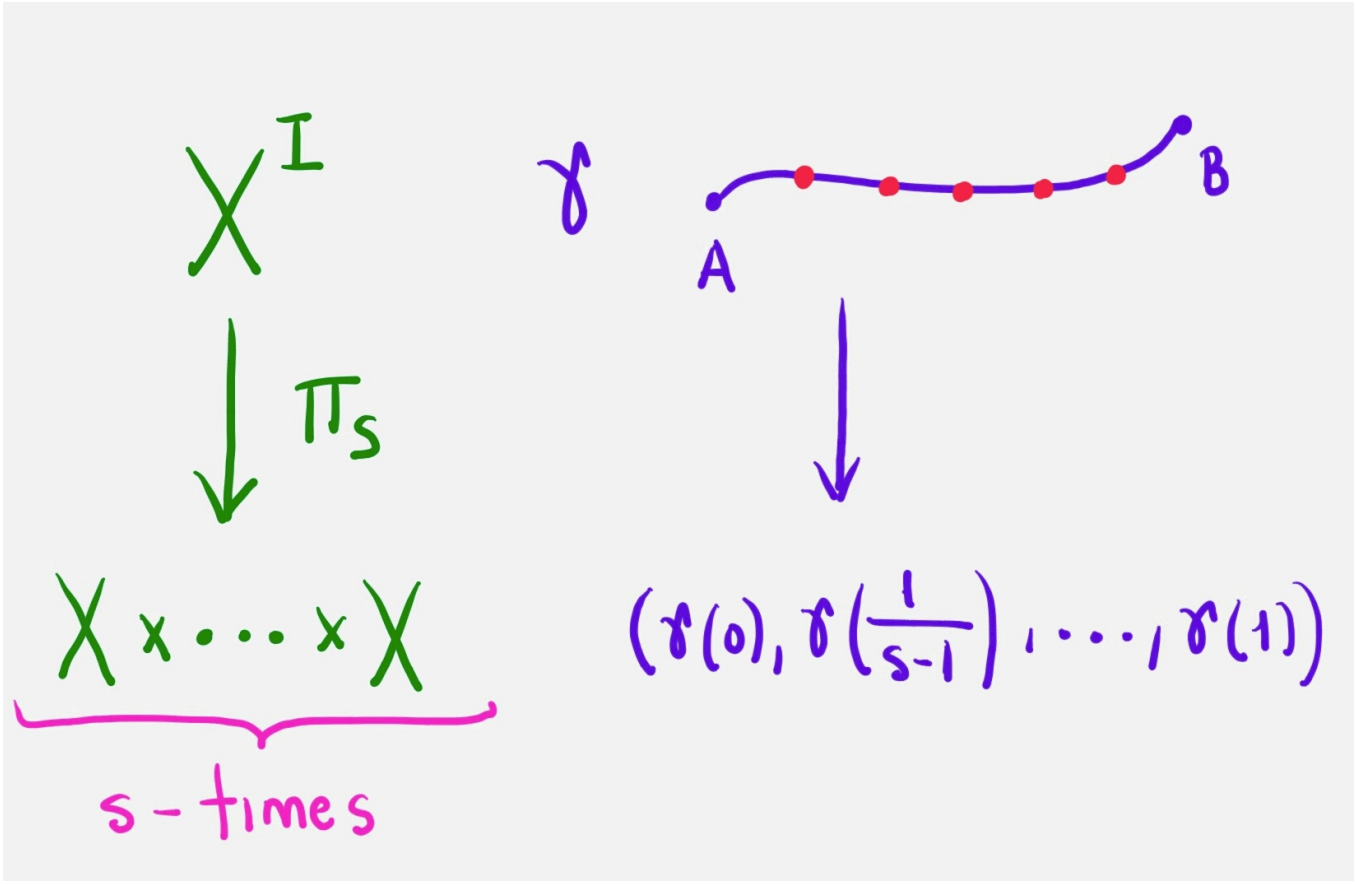
Proposition

There exists a continuous section s as above if and only if X is contractible.

In 2003 Michael Farber proposed a topological model to study the instabilities of the motion planning problem in robotics.

Definition

The topological complexity of a path connected space X, denoted by $\mathrm{TC}(X)$*, is the smallest number k for which there exists an open cover, $\mathcal{U}$, of $X \times X$, $\mathcal{U} = \{U_0, \ldots, U_k\}$ and maps $s_i : U_i \to PX$ such that $\pi \circ s_i((a, b)) = (a, b)$ for all $(a, b) \in U_i$, $i \in [0, k]$.*


X^I
π_s
$X \times \cdots \times X$
s - times
γ
A
B
$\left(\gamma(0), \gamma\left(\frac{1}{s-1}\right), \cdots, \gamma(1)\right)$

Definition

For a positive integer s, $s \geq 2$, the s-th topological complexity of X, $TC_s(X)$, is the reduced Schwarz genus of the fibration:

$$e_s = e_s^X : X^{J_s} \to X^s, \qquad e_s(f) = (f_1(1), \ldots, f_s(1)).$$

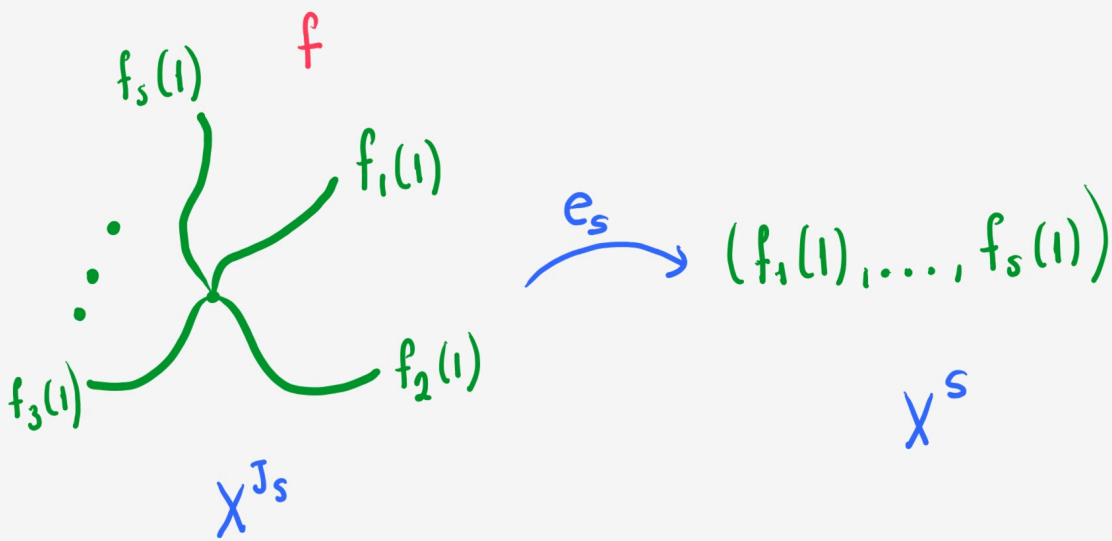
Here J_s denotes the wedge of s copies of the closed interval $[0,1]$, in all of which $0 \in [0,1]$ is the base point.

Definition

For a positive integer s, $s \geq 2$, the s-th topological complexity of X, $TC_s(X)$, is the reduced Schwarz genus of the fibration:

$$e_s = e_s^X : X^{J_s} \to X^s, \qquad e_s(f) = (f_1(1), \ldots, f_s(1)).$$

Here J_s denotes the wedge of s copies of the closed interval $[0,1]$, in all of which $0 \in [0,1]$ is the base point.



- Thus, $\mathrm{TC}_s(X)+1$ is the smallest cardinality of open covers $\{U_i\}_i$ of X^s so that, on each U_i, e_s admits a section σ_i. In such a cover, U_i is called a *local domain*, the corresponding section σ_i is called a *local rule*, and the resulting family of pairs $\{(U_i,\sigma_i)\}$ is called a *motion planner*. The latter is said to be *optimal* if it has $\mathrm{TC}_s(X)+1$ local domains.

- Thus, $\mathrm{TC}_s(X) + 1$ is the smallest cardinality of open covers $\{U_i\}_i$ of X^s so that, on each U_i, e_s admits a section σ_i. In such a cover, U_i is called a *local domain*, the corresponding section σ_i is called a *local rule*, and the resulting family of pairs $\{(U_i, \sigma_i)\}$ is called a *motion planner*. The latter is said to be *optimal* if it has $\mathrm{TC}_s(X) + 1$ local domains.

- For practical purposes, the openness condition on local domains can be replaced (without altering the resulting numeric value of $\mathrm{TC}_S(X)$) by the requirement that local domains to be pairwise disjoint Euclidean neighborhood retracts (ENR).

- Thus, $\mathrm{TC}_s(X)+1$ is the smallest cardinality of open covers $\{U_i\}_i$ of X^s so that, on each U_i, e_s admits a section σ_i. In such a cover, U_i is called a *local domain*, the corresponding section σ_i is called a *local rule*, and the resulting family of pairs $\{(U_i,\sigma_i)\}$ is called a *motion planner*. The latter is said to be *optimal* if it has $\mathrm{TC}_s(X)+1$ local domains.

- For practical purposes, the openness condition on local domains can be replaced (without altering the resulting numeric value of $\mathrm{TC}_S(X)$) by the requirement that local domains to be pairwise disjoint Euclidean neighborhood retracts (ENR).

- Since e_s is the standard fibrational substitute of the diagonal inclusion $d_s = d_s^X : X \hookrightarrow X^s$, $\mathrm{TC}_s(X)$ coincides with the reduced Schwarz genus of d_s.

Proposition (Schwarz'07)

For a path connected space X,

$$\mathrm{zcl}_s\left(H^*(X;R)\right) \leq \mathrm{TC}_s(X) \leq \frac{s\ \mathrm{hdim}(X)}{\mathrm{conn}(X)+1}.$$

Proposition (Schwarz'07)

For a path connected space X,

$$\mathrm{zcl}_s\left(H^*(X;R)\right) \leq \mathrm{TC}_s(X) \leq \frac{s\ \mathrm{hdim}(X)}{\mathrm{conn}(X)+1}.$$

- In particular for every path connected X,

$$\mathrm{TC}_s(X) \leq s\,\mathrm{hdim}(X).$$

The spaces we work with arise as follows. For a positive integer k_i consider the minimal cellular structure on the k_i-dimensional sphere $S^{k_i} = e^0 \cup e^{k_i}$. Here e^0 is the base point, which is simply denoted by e. Take the product cell decomposition in

$$\mathbb{S}(k_1, \ldots, k_n) := S^{k_1} \times \cdots \times S^{k_n} = \bigsqcup_J e_J$$

whose cells e_J, indexed by subsets $J \subseteq [n] = \{1, \ldots, n\}$, are defined as

$$e_J = \left\{ (x_1, \ldots, x_n) \in \mathbb{S}(k_1, \ldots, k_n) \mid x_i = e^0 \text{ if and only if } i \notin J \right\}.$$

The spaces we work with arise as follows. For a positive integer k_i consider the minimal cellular structure on the k_i-dimensional sphere $S^{k_i} = e^0 \cup e^{k_i}$. Here e^0 is the base point, which is simply denoted by e. Take the product cell decomposition in

$$\mathbb{S}(k_1, \ldots, k_n) := S^{k_1} \times \cdots \times S^{k_n} = \bigsqcup_J e_J$$

whose cells e_J, indexed by subsets $J \subseteq [n] = \{1, \ldots, n\}$, are defined as

$$e_J = \left\{ (x_1, \ldots, x_n) \in \mathbb{S}(k_1, \ldots, k_n) \mid x_i = e^0 \text{ if and only if } i \notin J \right\}.$$

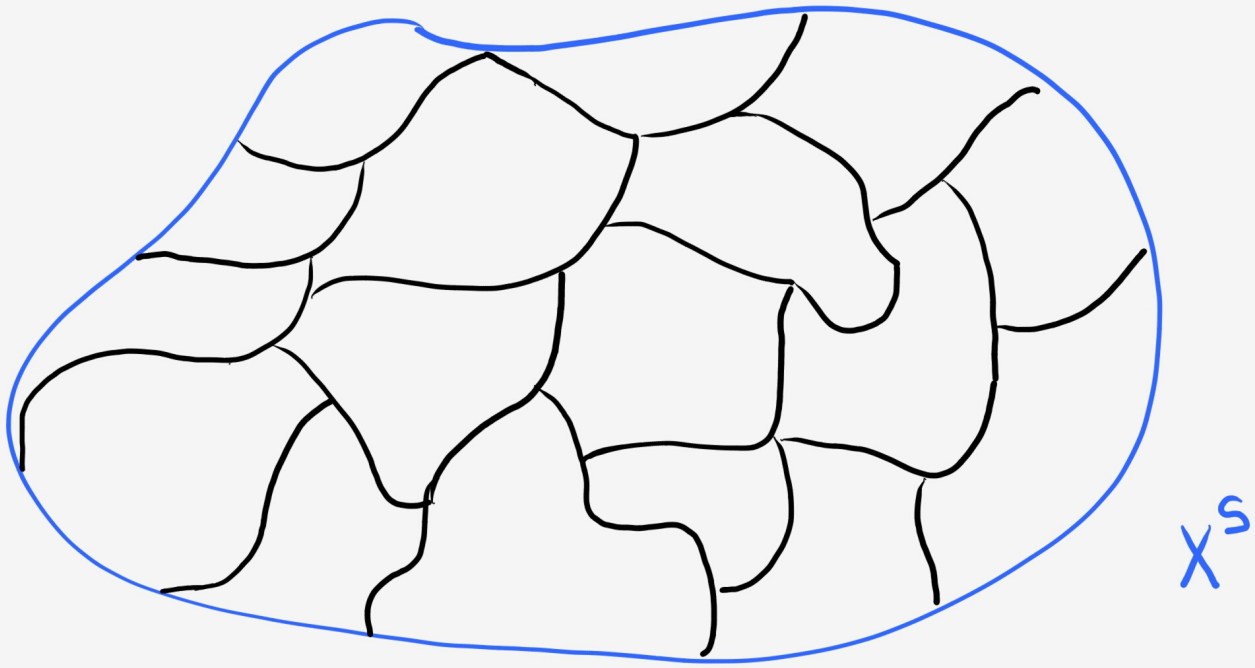
We are interested in subcomplexes, X, of $S^{k_1} \times \cdots \times S^{k_n}$ with the product cell decomposition.

Theorem (J. González, B. Gutiérrez, S. Yuzvinsky)

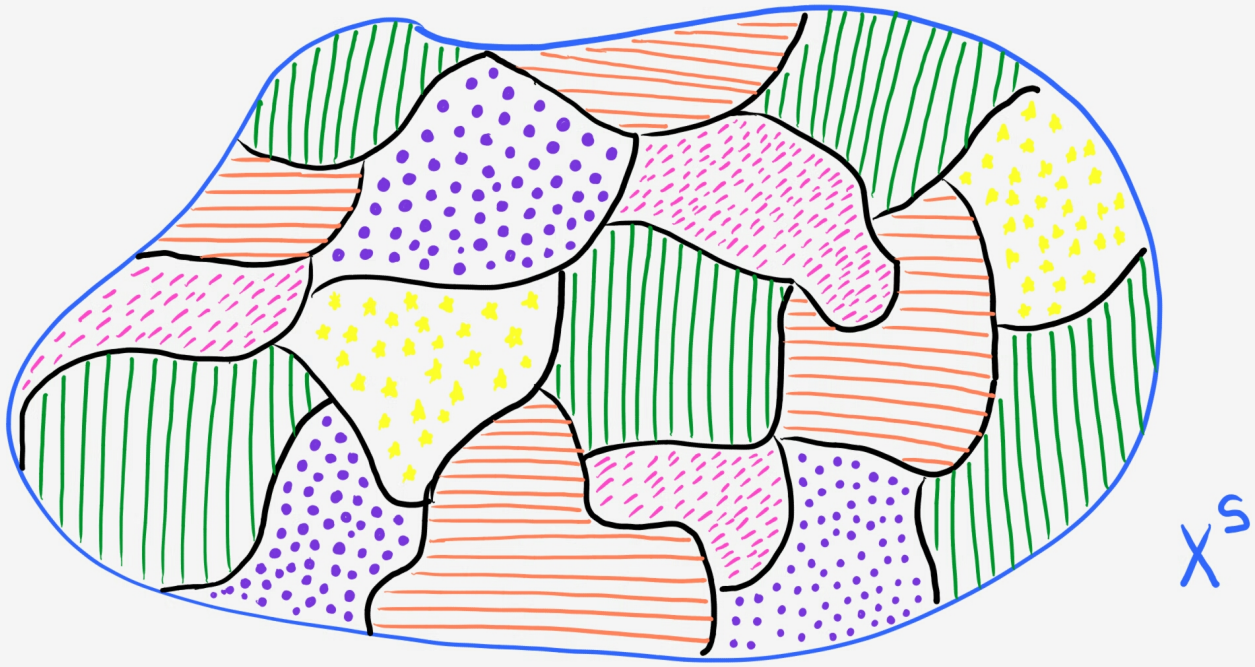
A subcomplex X of $\mathbb{S}(k_1, \ldots, k_n)$ has $\mathrm{TC}_s(X) = \mathrm{zcl}_s(H^*(X; \mathbb{Q}))$ provided all of the k_i have the same parity.

Strategy

Strategy



Strategy



Odd Case

Motion planner for an odd dimensional sphere, $\mathbb{S}(2d+1) = S^{2d+1}$.

- $s = 2$
  Local domains:

$$A_0 = \{(x, -x) \in S^{2d+1} \times S^{2d+1}\} \qquad A_1 = \{(x, y) \in S^{2d+1} \times S^{2d+1} \mid x \neq y\}$$

Odd Case

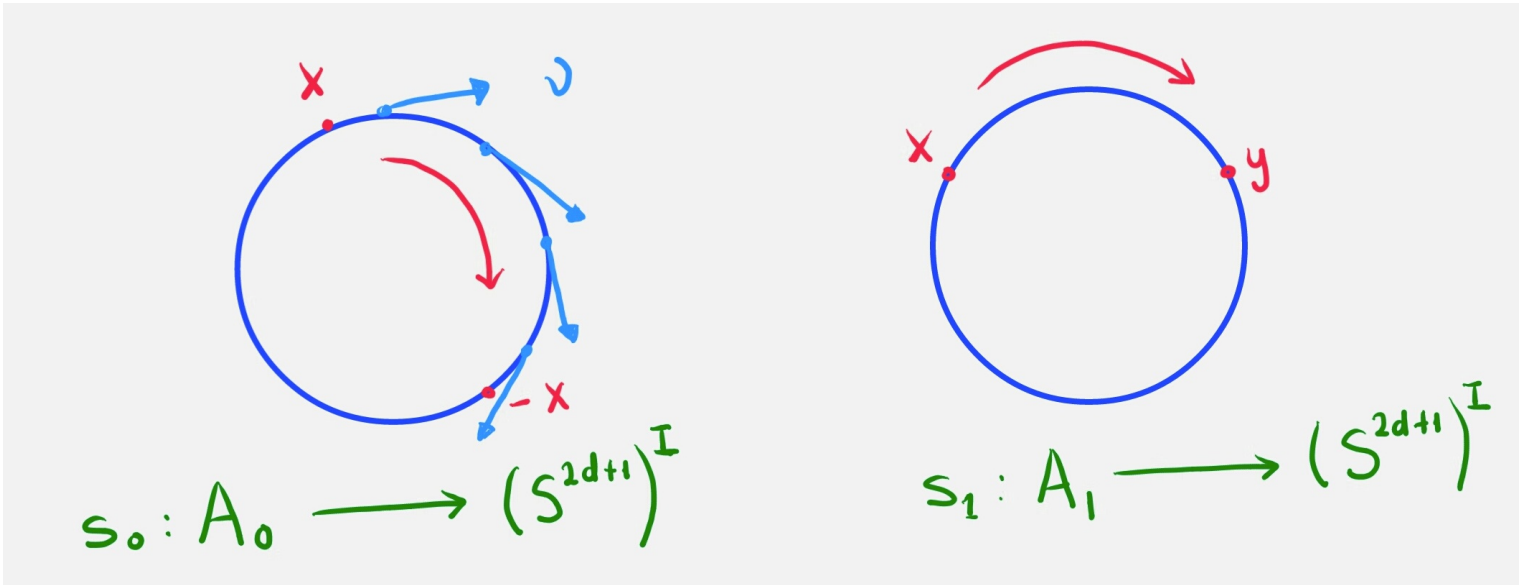
Motion planner for an odd dimensional sphere, $\mathbb{S}(2d+1) = S^{2d+1}$.

- $s = 2$

  Local domains:

  $$A_0 = \{(x, -x) \in S^{2d+1} \times S^{2d+1}\} \qquad A_1 = \{(x, y) \in S^{2d+1} \times S^{2d+1} \mid x \neq y\}$$

  Local rules:



- We think of an element $(b_1, b_2, \ldots, b_s) \in X^s$, with $b_j = (b_{1j}, \ldots, b_{nj}) \in X \subseteq \mathbb{S}(k_1, \ldots, k_n)$, as a matrix of size $n \times s$ whose entry b_{ij} belongs to S^{k_i} for all $(i, j) \in [n] \times [s]$.

- We think of an element $(b_1, b_2, \ldots, b_s) \in X^s$, with $b_j = (b_{1j}, \ldots, b_{nj}) \in X \subseteq \mathbb{S}(k_1, \ldots, k_n)$, as a matrix of size $n \times s$ whose entry b_{ij} belongs to S^{k_i} for all $(i, j) \in [n] \times [s]$.
- Let
$$\mathcal{P} = \{(P_1, \ldots, P_n) \mid P_i \text{ is a partition of } [s] \text{ for each } i \in [n]\}$$
be the set of n-tuples of partitions of the interval $[s]$.

- We think of an element $(b_1, b_2, \ldots, b_s) \in X^s$, with $b_j = (b_{1j}, \ldots, b_{nj}) \in X \subseteq \mathbb{S}(k_1, \ldots, k_n)$, as a matrix of size $n \times s$ whose entry b_{ij} belongs to S^{k_i} for all $(i, j) \in [n] \times [s]$.
- Let
$$\mathcal{P} = \{(P_1, \ldots, P_n) \mid P_i \text{ is a partition of } [s] \text{ for each } i \in [n]\}$$
be the set of n-tuples of partitions of the interval $[s]$.
- The norm of each such $P = (P_1, \ldots, P_n) \in \mathcal{P}$ is defined as
$$|P| := \sum_{i=1}^{n} (n(P_i) - 1) = \sum_{i=1}^{n} |P_i| - n,$$

- We think of an element $(b_1, b_2, \ldots, b_s) \in X^s$, with $b_j = (b_{1j}, \ldots, b_{nj}) \in X \subseteq \mathbb{S}(k_1, \ldots, k_n)$, as a matrix of size $n \times s$ whose entry b_{ij} belongs to S^{k_i} for all $(i, j) \in [n] \times [s]$.
- Let
$$\mathcal{P} = \{(P_1, \ldots, P_n) \mid P_i \text{ is a partition of } [s] \text{ for each } i \in [n]\}$$
be the set of n-tuples of partitions of the interval $[s]$.
- The norm of each such $P = (P_1, \ldots, P_n) \in \mathcal{P}$ is defined as
$$|P| := \sum_{i=1}^{n} (n(P_i) - 1) = \sum_{i=1}^{n} |P_i| - n,$$
- We let
$$X_P^s = \left\{ (b_1, b_2, \ldots, b_s) \in X^s \;\middle|\; \begin{array}{l} \text{for each } i \in [n],\ b_{ik} = \pm b_{i\ell} \text{ if and only if} \\ \text{both } k \text{ and } \ell \text{ belong to the same part of } P_i \end{array} \right\},$$

Let,

$$\mathcal{K}_X = \{J \subseteq [n] \mid e_J \text{ is a cell of } X\}.$$

Let,

$$\mathcal{K}_X = \{J \subseteq [n] \ \mid \ e_J \text{ is a cell of } X\}.$$

And define,

$$\mathrm{N}^s(X) := \mathrm{N}^s(\mathcal{K}_X) := \max\left\{ \sum_{\ell=1}^{s} |J_\ell| - \left| \bigcap_{\ell=1}^{s} J_\ell \right| \ \mid \ J_j \text{ is a simplex of } \mathcal{K}_X \text{ for all } j \in [s] \right\}.$$

Let,

$$\mathcal{K}_X = \{J \subseteq [n] \ \mid \ e_J \text{ is a cell of } X\}.$$

And define,

$$\mathrm{N}^s(X) := \mathrm{N}^s(\mathcal{K}_X) := \max\left\{ \sum_{\ell=1}^{s} |J_\ell| - \left| \bigcap_{\ell=1}^{s} J_\ell \right| \ \mid \ J_j \text{ is a simplex of } \mathcal{K}_X \text{ for all } j \in [s] \right\}.$$

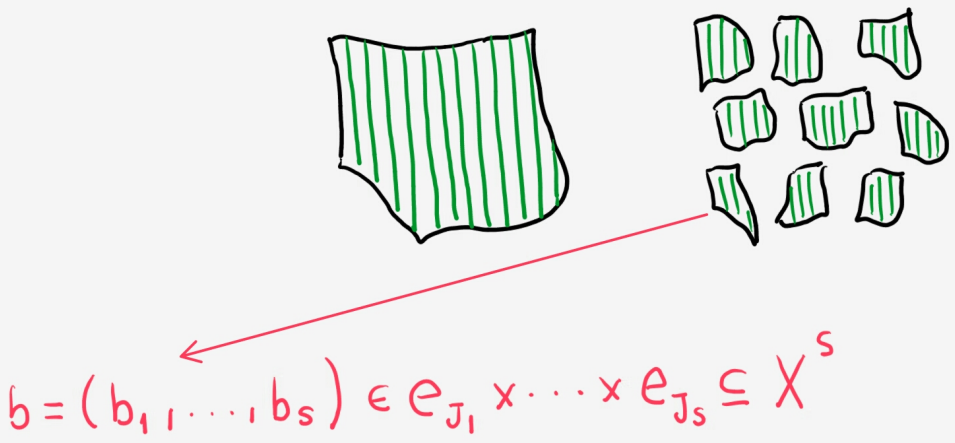
Let

$$D_j := \bigcup X_P^s, \quad j \in [\mathrm{N}^s(X)]_0,$$

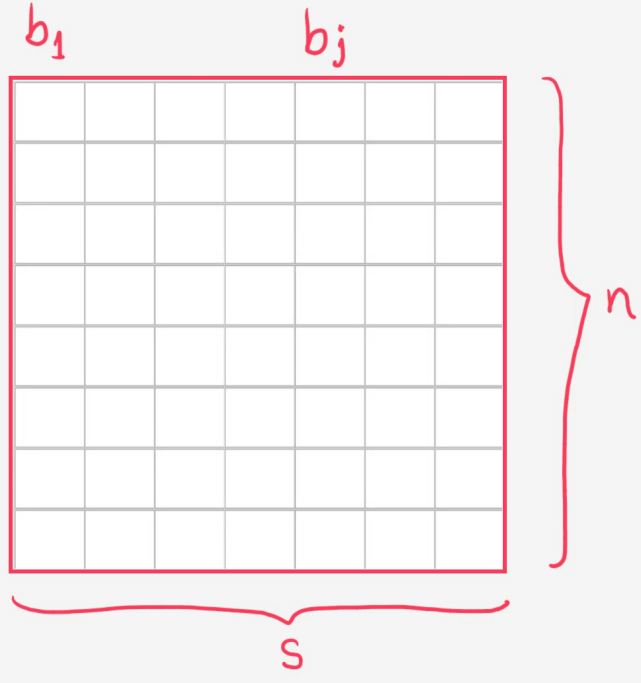
where the union runs over those $P \in \mathcal{P}$ with $|P| = j$.

Local rules

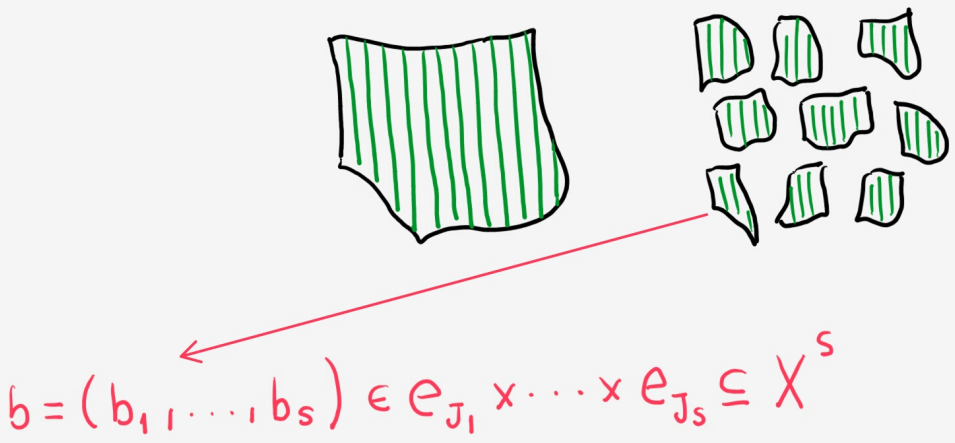
Local rules



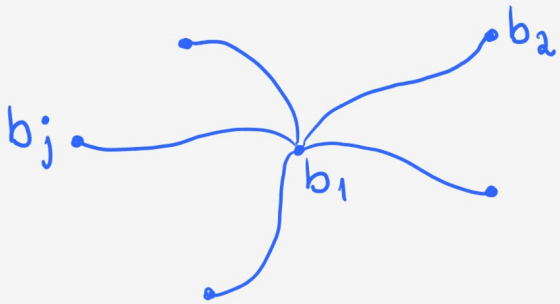
$$b = (b_1, \ldots, b_s) \in e_{J_1} \times \cdots \times e_{J_s} \subseteq X^s$$

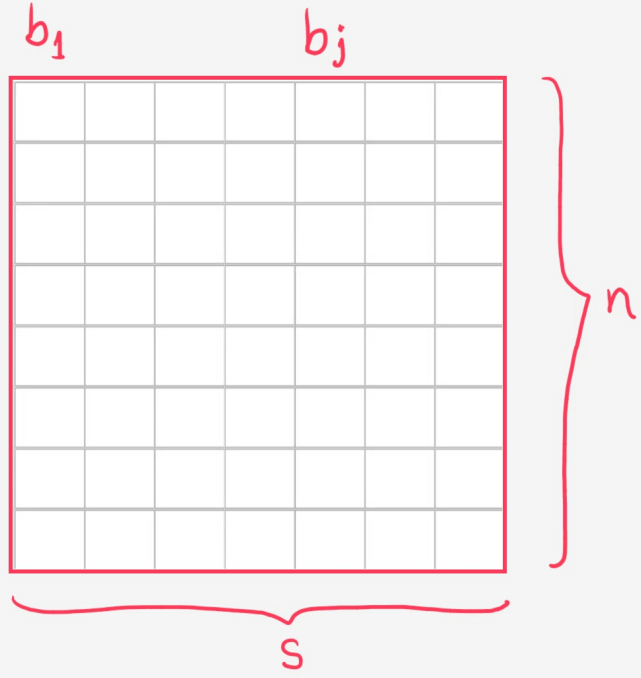


Local rules

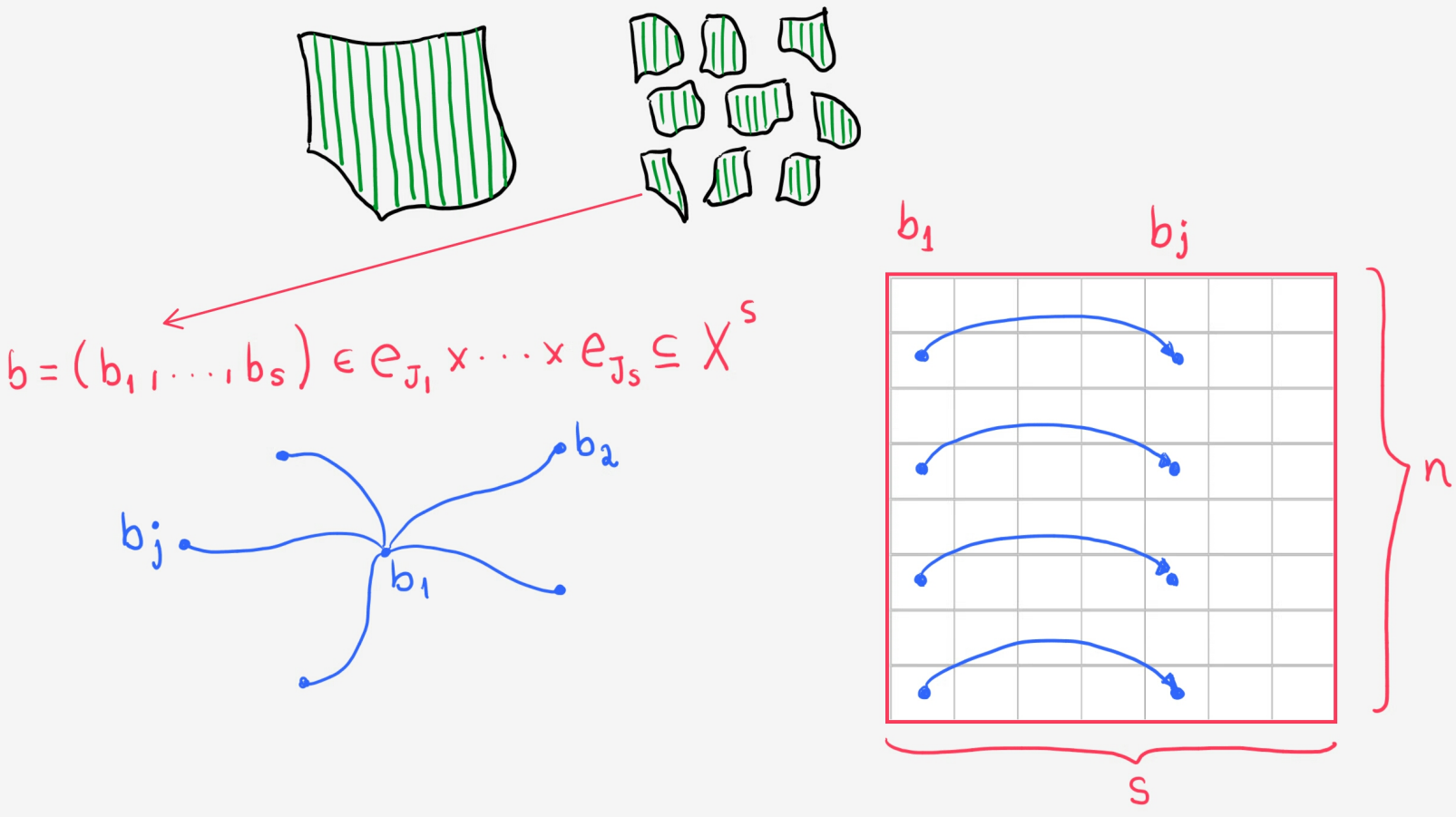


$$b = (b_1, \ldots, b_s) \in e_{J_1} \times \cdots \times e_{J_s} \subseteq X^s$$

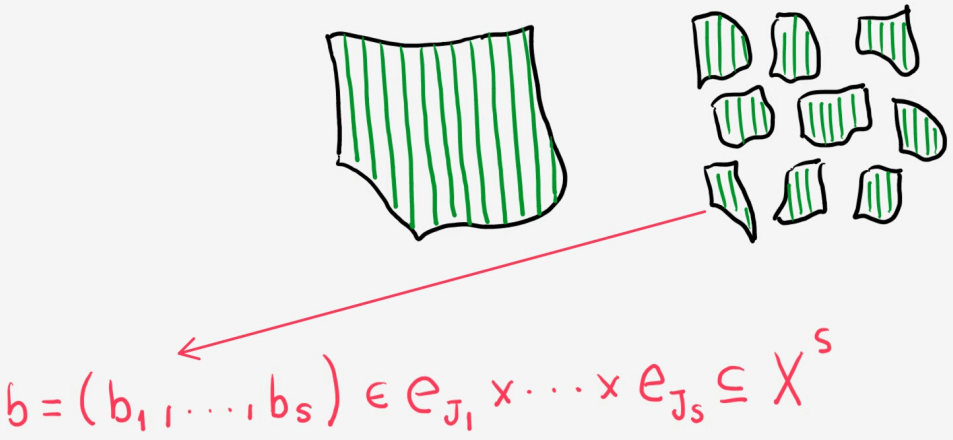




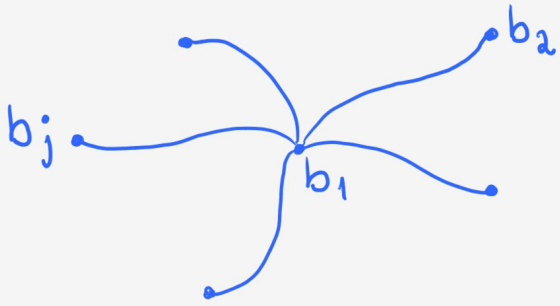
Local rules

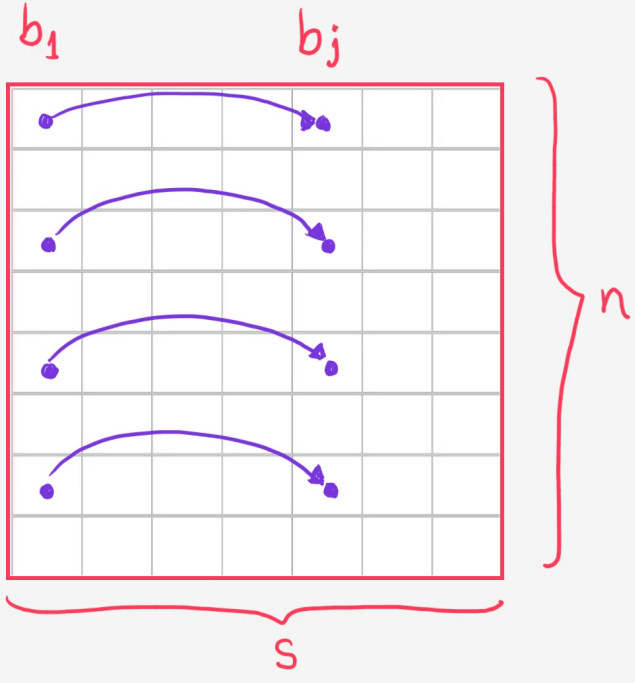

$b = (b_1, \ldots, b_s) \in e_{J_1} \times \cdots \times e_{J_s} \subseteq X^s$
b_j
b_2
b_1
b_1
b_j
n
s

Local rules



$$b = (b_1, \ldots, b_s) \in e_{J_1} \times \cdots \times e_{J_s} \subseteq X^s$$





Theorem (J. González, B. Gutiérrez, S. Yuzvinsky)

Assume all of the k_i are odd. A subcomplex X of the minimal CW cell structure on $\mathbb{S}(k_1, \cdots, k_n)$ has

$$\mathrm{TC}_s(X) = \mathrm{N}^s(X).$$

Examples

- Let X be a subcomplex of $\mathbb{S}(k_1, \ldots, k_s)$, where all k_i's are odd. Let $\omega := \omega(X)$ the number of maximal cells of X. Then, for $s \geq \omega$

$$\mathrm{TC}_s(X) = d(s - \omega) + \mathrm{TC}_\omega(X),$$

where $d - 1 = \dim \mathcal{K}_X$.

Examples

- Let X be a subcomplex of $\mathbb{S}(k_1, \ldots, k_s)$, where all k_i's are odd. Let $\omega := \omega(X)$ the number of maximal cells of X. Then, for $s \geq \omega$

$$\mathrm{TC}_s(X) = d(s - \omega) + \mathrm{TC}_\omega(X),$$

  where $d - 1 = \dim \mathcal{K}_X$.
- Let $\mathbb{S}(k_1, \ldots, k_n)^{(d)}$ stand for the subcomplex of $\mathbb{S}(k_1, \ldots, k_n)$, which has all possible cells associated to subsets of cardinality d. Then,

$$\mathrm{TC}_s(\mathbb{S}(k_1, \ldots, k_n)^{(d)}) = \min\{sd, (s-1)n\}.$$

Examples

- Let X be a subcomplex of $\mathbb{S}(k_1, \ldots, k_s)$, where all k_i's are odd. Let $\omega := \omega(X)$ the number of maximal cells of X. Then, for $s \geq \omega$

$$\mathrm{TC}_s(X) = d(s - \omega) + \mathrm{TC}_\omega(X),$$

  where $d - 1 = \dim \mathcal{K}_X$.
- Let $\mathbb{S}(k_1, \ldots, k_n)^{(d)}$ stand for the subcomplex of $\mathbb{S}(k_1, \ldots, k_n)$, which has all possible cells associated to subsets of cardinality d. Then,

$$\mathrm{TC}_s(\mathbb{S}(k_1, \ldots, k_n)^{(d)}) = \min\{sd, (s-1)n\}.$$

- Let X and Y subcomplexes of $\mathbb{S}(k_1 \ldots, k_n)$ and $\mathbb{S}(\ell_1, \ldots, \ell_m)$ respectively, where k_i and ℓ_j are odd for all $i \in [n]$ and $j \in [m]$. Then,

$$\mathrm{TC}_s(X \times Y) = \mathrm{TC}_s(X) + \mathrm{TC}_s(Y).$$

Examples

- Let X be a subcomplex of $\mathbb{S}(k_1, \ldots, k_s)$, where all k_i's are odd. Let $\omega := \omega(X)$ the number of maximal cells of X. Then, for $s \geq \omega$

$$\mathrm{TC}_s(X) = d(s - \omega) + \mathrm{TC}_\omega(X),$$

  where $d - 1 = \dim \mathcal{K}_X$.
- Let $\mathbb{S}(k_1, \ldots, k_n)^{(d)}$ stand for the subcomplex of $\mathbb{S}(k_1, \ldots, k_n)$, which has all possible cells associated to subsets of cardinality d. Then,

$$\mathrm{TC}_s(\mathbb{S}(k_1, \ldots, k_n)^{(d)}) = \min\{sd, (s-1)n\}.$$

- Let X and Y subcomplexes of $\mathbb{S}(k_1 \ldots, k_n)$ and $\mathbb{S}(\ell_1, \ldots, \ell_m)$ respectively, where k_i and ℓ_j are odd for all $i \in [n]$ and $j \in [m]$. Then,

$$\mathrm{TC}_s(X \times Y) = \mathrm{TC}_s(X) + \mathrm{TC}_s(Y).$$

- Let X and Y subcomplexes of $\mathbb{S}(k_1 \ldots, k_n)$ and $\mathbb{S}(\ell_1, \ldots, \ell_m)$ respectively, where k_i and ℓ_j are odd for all $i \in [n]$ and $j \in [m]$. If $\mathrm{cat}(X) \geq \mathrm{cat}(Y)$, then

$$\mathrm{TC}_s(X \vee Y) = \max\{\mathrm{TC}_s(X),\ \mathrm{TC}_s(Y),\ \mathrm{cat}(X^{s-1}) + \mathrm{cat}(Y)\}$$

Even Case

Theorem (J. González, B. Gutiérrez, S. Yuzvinsky)

Assume all of the k_i are even. A subcomplex X of the minimal CW structure on $\mathbb{S}(k_1, \ldots, k_n)$ has

$$\mathrm{TC}_s(X) = s(1 + \dim(\mathcal{K}_X)).$$