

Rational subsets of groups

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2010 Mathematics Subject Classification: 20F10, 20E05, 68Q45, 68Q70; 2010-07-14 21:09

Key words: Free groups, inverse automata, Stallings automata, rational subsets.

Contents

1	Finitely generated groups	2
1.1	Free groups	3
2	Inverse automata and Stallings' construction	4
2.1	Inverse automata	4
2.2	Stallings' construction	5
2.3	Basic applications	7
2.4	Conjugacy	10
2.5	Further algebraic properties	11
2.6	Topological properties	14
2.7	Dynamical properties	16
3	Rational and recognizable subsets	17
3.1	Rational and recognizable subgroups	18
3.2	Benois' Theorem	18
3.3	Rational versus recognizable	21
3.4	Beyond free groups	22
3.5	Rational solution sets and rational constraints	24
References		25

*Work partially supported by SNF Grant No. yy-63821.xx

†Work partially supported by F.C.T. (Portugal) through C.M.U.P. and Project ASA (PTDC/MAT/65481/2006), with funds from the programs POCTI and POSI

Over the years, finite automata have been used effectively in the theory of infinite groups to represent rational subsets. This includes the important particular case of finitely generated subgroups (and the beautiful theory of Stallings automata for the free group case), but goes far beyond that: certain inductive procedures need a more general setting than mere subgroups and rational subsets constitute the natural generalization. The connections between automata theory and group theory are rich and deep, and many are portrayed in Sims' book [52].

This chapter is divided into three parts: in §1 we introduce basic concepts, terminology and notation for finitely generated groups, devoting special attention to free groups. These will also be used in Chapter 24.

§2 describes the use of finite inverse automata to study finitely generated subgroups of free groups. The automaton recognizes elements of a subgroup, represented as words in the ambient free group.

§3 considers, more generally, rational subsets of groups, when good closure and decidability properties of these subsets are satisfied.

1 Finitely generated groups

Let G be a group. Given $A \subseteq G$, let $\langle A \rangle = (A \cup A^{-1})^*$ denote the subgroup of G generated by A . We say that $H \leq G$ is *finitely generated* and write $H \leq_{f.g.} G$ if $H = \langle A \rangle$ for some finite subset A of H .

Given $H \leq G$, we denote by $[G : H]$ the *index* of H in G . If $[G : H]$ is finite, we write $H \leq_{f.i.} G$. It is well known that every finite index subgroup of a finitely generated group is finitely generated.

We denote by $\mathbb{1}$ the identity of G . An element $g \in G$ has *finite order* if $\langle g \rangle$ is finite. Elements $g, h \in G$ are *conjugate* if $h = x^{-1}gx$ for some $x \in G$. We use the notation $g^h = h^{-1}gh$ and $[g, h] = g^{-1}h^{-1}gh$ to denote, respectively, conjugates and commutators.

Given an alphabet A , we denote by A^{-1} a set of *formal inverses* of A , and write $\tilde{A} = A \cup A^{-1}$. We say that \tilde{A} is an *involutive alphabet*. We extend $^{-1} : A \rightarrow A^{-1} : a \mapsto a^{-1}$ to an involution on \tilde{A}^* through

$$(a^{-1})^{-1} = a, \quad (uv)^{-1} = v^{-1}u^{-1} \quad (a \in A, u, v \in \tilde{A}^*).$$

If $G = \langle A \rangle$, we have a canonical epimorphism $\rho : \tilde{A}^* \twoheadrightarrow G$, mapping $a^{\pm 1} \in \tilde{A}$ to $a^{\pm 1} \in G$. We present next some classical decidability problems:

Definition 1.1. Let $G = \langle A \rangle$ be a finitely generated group.

word problem: is there an algorithm that, upon receiving as input a word $u \in \tilde{A}^*$, determines whether or not $\rho(u) = \mathbb{1}$?

conjugacy problem: is there an algorithm that, upon receiving as input words $u, v \in \tilde{A}^*$, determines whether or not $\rho(u)$ and $\rho(v)$ are conjugate in G ?

membership problem for $\mathcal{K} \subseteq 2^G$: is there for every $X \in \mathcal{K}$ an algorithm that, upon receiving as input a word $u \in \tilde{A}^*$, determines whether or not $\rho(u) \in X$?

generalized word problem: is the membership problem for the class of finitely generated subgroups of G solvable?

63 **order problem:** is there an algorithm that, upon receiving as input a word $u \in \tilde{A}^*$,
 64 determines whether $\rho(u)$ has finite or infinite order?
 65 **isomorphism problem for a class \mathcal{G} of groups:** is there an algorithm that, upon receiv-
 66 ing as input a description of groups $G, H \in \mathcal{G}$, decides whether or not $G \cong H$?
 67 Typically, \mathcal{G} may be a subclass of finitely presented groups (given by their presenta-
 68 tion), or automata groups (see Chapter 24) given by automata.

69 We can also require complexity bounds on the algorithms; more precisely, we may
 70 ask with which complexity bound an answer to the problem may be obtained, and also
 71 with which complexity bound a witness (a normal form for the word problem, an element
 72 conjugating $\rho(u)$ to $\rho(v)$ in case they are conjugate, an expression of u in the generators
 73 of X in the generalized word problem) may be constructed.

74 1.1 Free groups

75 **Definition 1.2.** Given an alphabet A , let \sim denote the congruence on \tilde{A}^* generated by
 76 the relation

$$\{(aa^{-1}, 1) \mid a \in \tilde{A}\}. \quad (1.1)$$

77 The quotient $F_A = \tilde{A}^*/\sim$ is the *free group on A* . We denote by $\theta : \tilde{A}^* \rightarrow F_A$ the
 78 canonical morphism $u \mapsto [u]_\sim$.

79 Free groups admit the following universal property: for every map $f : A \rightarrow G$, there
 80 is a unique group morphism $F_A \rightarrow G$ that extends f .

81 Alternatively, we can view (1.1) as a *confluent* length-reducing rewriting system on
 82 \tilde{A}^* , where each word $w \in \tilde{A}^*$ can be transformed into a unique *reduced* word \bar{w} with no
 83 factor of the form aa^{-1} , see [8]. As a consequence, the equivalence

$$u \sim v \quad \Leftrightarrow \quad \bar{u} = \bar{v} \quad (u, v \in \tilde{A}^*)$$

84 solves the word problem for F_A .

85 We shall use the notation $R_A = \overline{\tilde{A}^*}$. It is well known that F_A is isomorphic to R_A
 86 under the binary operation

$$u \star v = \overline{uv} \quad (u, v \in R_A).$$

87 We recall that the *length* $|g|$ of $g \in F_A$ is the length of the reduced form of g , also denoted
 88 by \bar{g} .

89 The letters of A provide a natural *basis* for F_A : they generate F_A and satisfy no non-
 90 trivial relations, that is, all reduced words on these generators represent distinct elements
 91 of F_A . A group is free if and only if it has a basis.

92 Throughout this chapter, we assume A to be a finite alphabet. It is well known that free
 93 groups F_A and F_B are isomorphic if and only if $\#A = \#B$. This leads to the concept of
 94 *rank* of a free group F : the *cardinal* of a basis of F , denoted by $\text{rk } F$. It is common to
 95 use the notation F_n to denote a free group of rank n .

96 We recall that a reduced word u is *cyclically reduced* if uu is also reduced. Any
 97 reduced word $u \in R_A$ admits a unique decomposition of the form $u = v w v^{-1}$ with w

98 cyclically reduced. A solution for the conjugacy problem follows easily from this: first
 99 reduce the words cyclically; then two cyclically reduced words in R_A are conjugate if and
 100 only if they are cyclic permutations of each other. On the other hand, the order problem
 101 admits a trivial solution: only the identity has finite order. Finally, the generalized word
 102 problem shall be discussed in the following subsection.

103 2 Inverse automata and Stallings' construction

104 The study of finitely generated subgroups of free groups entered a new era in the early
 105 eighties when Stallings made explicit and effective a construction [53] that can be traced
 106 back to the early part of the twentieth century in Schreier's coset graphs (see [52] and §24.1) ■
 107 and to Serre's work [45]. Stallings' seminal paper was built over *immersions of finite*
 108 *graphs*, but the alternative approach using finite inverse automata became much more
 109 popular over the years; for more on their link, see [25]. An extensive survey has been
 110 written by Kapovich and Miasnikov [19].

111 Stallings' construction for $H \leq_{f.g.} F_A$ consists in taking a finite set of generators for
 112 H in reduced form, building the so-called *flower automaton* and then proceeding to make
 113 this automaton deterministic through the operation known as *Stallings foldings*. This is
 114 clearly a terminating procedure, but the key fact is that the construction is independent
 115 from both the given finite generating set and the chosen folding sequence. A short simple
 116 automata-theoretic proof of this claim will be given. The finite inverse automaton $\mathcal{S}(H)$
 117 thus obtained is usually called the *Stallings automaton* of H . Over the years, Stallings au-
 118 tomata became the standard representation for finitely generated subgroups of free groups
 119 and are involved in many of the algorithmic results presently obtained.

120 Several of these algorithms are implemented in computer software, see e.g. CRAG [2], ■
 121 or the packages AUTOMATA and FGA in GAP [13].

122 2.1 Inverse automata

123 An automaton \mathcal{A} over an involutive alphabet \tilde{A} is *involutive* if, whenever (p, a, q) is an
 124 edge of \mathcal{A} , so is (q, a^{-1}, p) . Therefore it suffices to depict just the *positively labelled*
 125 edges (having label in A) in their graphical representation.

126 **Definition 2.1.** An involutive automaton is *inverse* if it is deterministic, trim and has a
 127 single terminal state.

128 If the latter happens to be the initial state, it is called the *basepoint*. It follows easily
 129 from the computation of the Nerode equivalence (see §10.2) that every inverse automaton
 130 is a minimal automaton.

131 Finite inverse automata capture the idea of an action (of a finite *inverse monoid*, their
 132 *transition monoid*) on a finite set (the vertex set) through partial bijections. ???define
 133 inverse monoid = monoid with axiom $\forall x \exists y : xyx = x$; put inverse monoid and transition
 134 monoid in index; refer to other chapter mentioning transition monoids



135 The next result is easily proven, but is quite useful.

136 **Proposition 2.1.** *Let \mathcal{A} be an inverse automaton and let $p \xrightarrow{uvw^{-1}w} q$ be a path in \mathcal{A} .*
 137 *Then there exists also a path $p \xrightarrow{uw} q$ in \mathcal{A} .*

138 Another important property relates languages to morphisms:

139 **Proposition 2.2.** *Given inverse automata \mathcal{A} and \mathcal{A}' , then $L(\mathcal{A}) \subseteq L(\mathcal{A}')$ if and only if*
 140 *there exists a morphism $\varphi : \mathcal{A} \rightarrow \mathcal{A}'$. Moreover, such a morphism is unique.*

141 ???morphisms defined elsewhere, and only surjective???



142 *Proof.* (\Rightarrow): Given a vertex q of \mathcal{A} , take a successful path

$$\rightarrow q_0 \xrightarrow{u} q \xrightarrow{v} t \rightarrow$$

143 in \mathcal{A} , for some $u, v \in \tilde{A}^*$. Since $L(\mathcal{A}) \subseteq L(\mathcal{A}')$, there exists a successful path

$$\rightarrow q'_0 \xrightarrow{u} q' \xrightarrow{v} t' \rightarrow$$

144 in \mathcal{A}' . We take $\varphi(q) = q'$.

145 To show that φ is well defined, suppose that

$$\rightarrow q_0 \xrightarrow{u'} q \xrightarrow{v'} t \rightarrow$$

146 is an alternative successful path in \mathcal{A} . Since $u'v' \in L(\mathcal{A}) \subseteq L(\mathcal{A}')$, there exists a success-
 147 ful path

$$\rightarrow q'_0 \xrightarrow{u'} q'' \xrightarrow{v'} t' \rightarrow$$

148 in \mathcal{A}' and it follows that $q' = q''$ since \mathcal{A}' is inverse. Thus φ is well defined.

149 It is now routine to check that φ is a morphism from \mathcal{A} to \mathcal{A}' and its uniqueness.

150 (\Leftarrow): Immediate from the definition of morphism. \square

151 2.2 Stallings' construction

152 Let X be a finite subset of R_A . We build an involutive automaton $\mathcal{F}(X)$ by fixing a
 153 basepoint q_0 and gluing to it a *petal* labelled by every word in X as follows: if $x =$
 154 $a_1 \dots a_k \in X$, with $a_i \in \tilde{A}$, the petal consists of a closed path of the form

$$q_0 \xrightarrow{a_1} \bullet \xrightarrow{a_2} \dots \xrightarrow{a_k} q_0$$

155 and the respective inverse edges. All such intermediate vertices \bullet are assumed to be
 156 distinct in the automaton. For obvious reasons, $\mathcal{F}(X)$ is called the *flower automaton* of
 157 X .

158 $\mathcal{F}(X)$ is almost an inverse automaton – except that it need not be deterministic. We
 159 can fix it by performing a sequence of so-called *Stallings foldings*. Assume that \mathcal{A} is a
 160 trim involutive automaton with a basepoint, possessing two distinct edges of the form

$$p \xrightarrow{a} q, \quad p \xrightarrow{a} r \tag{2.1}$$

161 for $a \in \tilde{A}$. The *folding* is performed by identifying these two edges, as well as the two
 162 respective inverse edges. In particular, the vertices q and r are also identified (if they were
 163 distinct).

164 The number of edges is certain to decrease through foldings. Therefore, if we perform
 165 enough of them, we are sure to turn $\mathcal{F}(X)$ into a finite inverse automaton.

166 **Definition 2.2.** The *Stallings automaton* of X is the finite inverse automaton $\mathcal{S}(X)$ ob-
 167 tained through folding $\mathcal{F}(X)$.

168 We shall see that $\mathcal{S}(X)$ depends only on the finitely generated subgroup $\langle X \rangle$ of F_A
 169 generated by X , being in particular independent from the choice of foldings taken to reach
 170 it.

171 Since inverse automata are minimal, it suffices to characterize their language in terms
 172 of H to prove uniqueness (up to isomorphism):

Proposition 2.3. Fix $H \leq_{f.g.} F_A$ and let $X \subseteq R_A$ be a finite generating set for H . Then

$$L(\mathcal{S}(X)) = \bigcap \{L \subseteq \tilde{A}^* \mid L \text{ is recognized by a finite inverse automaton} \\ \text{with a basepoint and } \overline{H} \subseteq L\}.$$

173 *Proof.* (\supseteq): Clearly, $\mathcal{S}(X)$ is a finite inverse automaton with a basepoint. Since $X \cup$
 174 $X^{-1} \subseteq L(\mathcal{F}(X)) \subseteq L(\mathcal{S}(X))$, it follows easily from Proposition 2.1 that

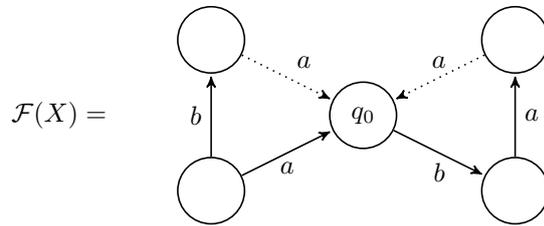
$$\overline{H} \subseteq L(\mathcal{S}(X)). \quad (2.2)$$

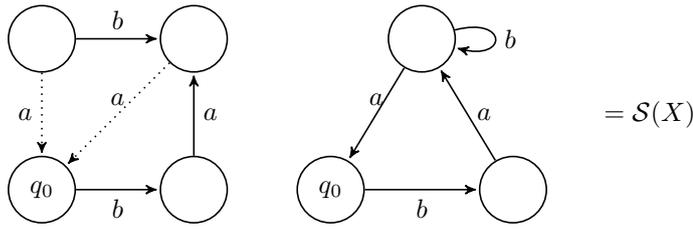
175 (\subseteq): Let $L \subseteq \tilde{A}^*$ be recognized by a finite inverse automaton \mathcal{A} with a basepoint,
 176 with $\overline{H} \subseteq L$. Since $X \subseteq \overline{H}$, we have an automaton morphism from $\mathcal{F}(X)$ to \mathcal{A} , hence
 177 $L(\mathcal{F}(X)) \subseteq L$. To prove that $L(\mathcal{S}(X)) \subseteq L$, it suffices to show that inclusion in L is
 178 preserved through foldings.

179 Indeed, assume that $L(\mathcal{B}) \subseteq L$ and \mathcal{B}' is obtained from \mathcal{B} by folding the two edges
 180 in (2.1). It is immediate that every successful path $q_0 \xrightarrow{u} t$ in \mathcal{B}' can be lifted to a success-
 181 ful path $q_0 \xrightarrow{v} t$ in \mathcal{B} by successively inserting the word $a^{-1}a$ into u . Now $v \in L = L(\mathcal{A})$
 182 implies $u \in L$ in view of Proposition 2.1. \square

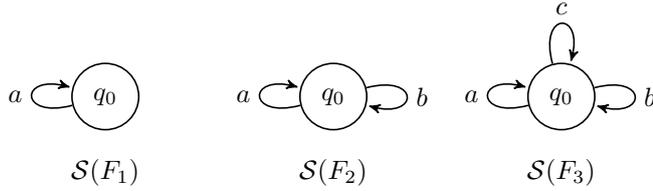
183 Now, given $H \leq F_A$ finitely generated, we take a finite set X of generators. Without
 184 loss of generality, we may assume that X consists of reduced words, and we may define
 185 $\mathcal{S}(H) = \mathcal{S}(X)$ to be the *Stallings automaton* of H .

186 **Example 2.1.** The Stallings' construction for $X = \{a^{-1}ba, ba^2\}$, where the next edges
 187 to be identified are depicted by dotted lines, is





A simple, yet important example is given by applying the construction to F_n itself, when we obtain the so-called *bouquet* of n circles:



188 In terms of complexity, the best known algorithm for the construction of $\mathcal{S}(X)$ is due
 189 to Touikan, which reached in [55] a time complexity of $O(n \log^* n)$, where n is the sum
 190 of the lengths of the elements of X .

2.3 Basic applications

192 The most fundamental application of Stallings' construction is an elegant and efficient
 193 solution to the generalized word problem:

194 **Theorem 2.4.** *The generalized word problem in F_A is solvable.*

195 We will see many groups in Chapter 24 that have solvable word problem; however,
 196 few of them have solvable generalized word problem. The proof of 2.4 relies on the

197 **Proposition 2.5.** *Consider $H \leq_{f.g.} F_A$ and $u \in F_A$. Then $u \in H$ if and only if $\bar{u} \in$
 198 $L(\mathcal{S}(H))$.*

199 *Proof.* (\Rightarrow): Follows from (2.2).

200 (\Leftarrow): It follows easily from the last paragraph of the proof of Proposition 2.3 that, if
 201 \mathcal{B}' is obtained from \mathcal{B} by performing Stallings foldings, then $L(\mathcal{B}') = L(\mathcal{B})$. Hence, if
 202 $H = \langle X \rangle$, we get

$$\overline{L(\mathcal{S}(H))} = \overline{L(\mathcal{F}(X))} = \overline{(X \cup X^{-1})^*} = \bar{H}$$

203 and the implication follows. □

204 It follows from our previous remark that the complexity of the generalized word prob-
 205 lem is $O(n \log^* n + m)$, where n is the sum of the lengths of the elements of X and
 206 m is the length of the input word. In particular, once the subgroup X has been fixed,
 207 complexity is linear in m .

208 **Example 2.2.** We may use the Stallings automaton constructed in Example 2.1 to check
 209 that $baba^{-1}b^{-1} \in H = \langle a^{-1}ba, ba^2 \rangle$ but $ab \notin H$.

210 Stallings automata provide also an effective construction for bases of finitely generated
 211 subgroups. Consider $H \leq_{f.g.} F_A$, and let m be the number of vertices of $\mathcal{S}(H)$. A
 212 *spanning tree* T for $\mathcal{S}(H)$ consists of $m - 1$ edges and their inverses which, together,
 213 connect all the vertices of $\mathcal{S}(H)$. Given a vertex p of $\mathcal{S}(H)$, we denote by g_p the T -
 214 *geodesic* connecting the basepoint q_0 to p , that is, $q_0 \xrightarrow{g_p} p$ is the shortest path contained
 215 in T connecting q_0 to p .

216 **Proposition 2.6.** *Let $H \leq_{f.g.} F_A$ and let T be a spanning tree for $\mathcal{S}(H)$. Let E_+ be the*
 217 *set of positively labelled edges of $\mathcal{S}(H)$. Then H is free with basis*

$$Y = \{g_p a g_q^{-1} \mid (p, a, q) \in E_+ \setminus T\}.$$

218 *Proof.* It follows from Proposition 2.5 that $L(\mathcal{S}(H)) \subseteq H$, hence $Y \subseteq H$. To show that
 219 $H = \langle Y \rangle$, take $h = a_1 \dots a_k \in H$ in reduced form ($a_i \in \tilde{A}$). By Proposition 2.5, there
 220 exists a successful path

$$q_0 \xrightarrow{a_1} q_1 \xrightarrow{a_2} \dots \xrightarrow{a_k} q_k = q_0$$

221 in $\mathcal{S}(H)$. For $i = 1, \dots, k$, we have either $g_{q_{i-1}} a_i g_{q_i}^{-1} \in Y \cup Y^{-1}$ or $\overline{g_{q_{i-1}} a_i g_{q_i}^{-1}} = 1$,
 222 the latter occurring if $(q_{i-1}, a_i, q_i) \in T$. In any case, we get

$$h = a_1 \dots a_k = \overline{(g_{q_0} a_1 g_{q_1}^{-1})(g_{q_1} a_2 g_{q_2}^{-1}) \dots (g_{q_{k-1}} a_k g_{q_0}^{-1})} \in \langle Y \rangle$$

223 and so $H = \langle Y \rangle$.

224 It remains to show that the elements of Y satisfy no nontrivial relations. Let y_1, \dots, y_k
 225 $\in Y \cup Y^{-1}$ with $y_i \neq y_{i-1}^{-1}$ for $i = 2, \dots, k$. Write $y_i = g_{p_i} a_i g_{r_i}^{-1}$, where $a_i \in \tilde{A}$ labels
 226 the edge not in T . It follows easily from $y_i \neq y_{i-1}^{-1}$ and the definition of spanning tree
 227 that

$$\overline{y_1 \dots y_k} = \overline{g_{p_1} a_1 g_{r_1}^{-1} g_{p_2} a_2 \dots a_{k-1} g_{r_{k-1}}^{-1} g_{p_k} a_k g_{r_k}},$$

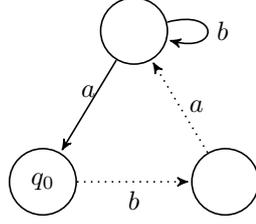
228 a nonempty reduced word if $k \geq 1$. Therefore Y is a basis of H as claimed. \square

229 In the process, we obtain also a proof of the Nielsen-Schreier Theorem, in the case of
 230 finitely generated subgroups. A simple topological proof may be found in [35]:

231 **Theorem 2.7** (Nielsen-Schreier). *Every subgroup of a free group is itself free.*

232 **Example 2.3.** We use the Stallings automaton constructed in Example 2.1 to construct a
 233 basis of $H = \langle a^{-1}ba, ba^2 \rangle$.

If we take the spanning tree T defined by the dotted lines in



234 then $\#E_+ \setminus T = 2$ and the corresponding basis is $\{ba^2, baba^{-1}b^{-1}\}$. Another choice of
 235 spanning tree actually proves that the original generating set is also a basis.

236 We remark that Proposition 2.6 can be extended to the case of infinitely generated
 237 subgroups, proving the general case of Theorem 2.7. However, in this case there is no ef-
 238 fective construction such as Stallings', and the (infinite) inverse automaton $\mathcal{S}(H)$ remains
 239 a theoretical object, using appropriate cosets as vertices.

240 Another classical application of Stallings' construction regards the identification of
 241 finite index subgroups.

242 **Proposition 2.8.** Consider $H \leq_{f.g.} F_A$.

- 243 (i) H is a finite index subgroup of F_A if and only if $\mathcal{S}(H)$ is a complete automaton.
 244 (ii) If H is a finite index subgroup of F_A , then its index is the number of vertices of
 245 $\mathcal{S}(H)$.

246 *Proof.* (i) (\Rightarrow): Suppose that $\mathcal{S}(H)$ is not complete. Then there exist some vertex q and
 247 some $a \in \tilde{A}$ such that $q \cdot a$ is undefined. Let g be a geodesic connecting the basepoint q_0
 248 to q in $\mathcal{S}(H)$. We claim that

$$Hga^m \neq Hga^n \quad \text{if } m - n > |g|. \quad (2.3)$$

249 Indeed, $Hga^m = Hga^n$ implies $ga^{m-n}g^{-1} \in H$ and so $\overline{ga^{m-n}g^{-1}} \in L(\mathcal{S}(H))$ by
 250 Proposition 2.5. Since ga is reduced due to $\mathcal{S}(H)$ being inverse, it follows from $m - n >$
 251 $|g|$ that $\overline{gaa^{m-n-1}g^{-1}} = \overline{ga^{m-n}g^{-1}} \in L(\mathcal{S}(H))$: indeed, g^{-1} is not long enough to
 252 erase all the a 's. Since $\mathcal{S}(H)$ is deterministic, $q \cdot a$ must be defined, a contradiction.
 253 Therefore (2.3) holds and so H has infinite index.

254 (\Leftarrow): Let Q be the vertex set of $\mathcal{S}(H)$ and fix a geodesic $q_0 \xrightarrow{g_q} q$ for each $q \in Q$.
 255 Take $u \in F_A$. Since $\mathcal{S}(H)$ is complete, we have a path $q_0 \xrightarrow{u} q$ for some $q \in Q$. Hence
 256 $ug_q^{-1} \in H$ and so $u = ug_q^{-1}g_q \in Hg_q$. Therefore $F_A = \bigcup_{q \in Q} Hg_q$ and so $H \leq_{f.i.} F_A$.

257 (ii) In view of $F_A = \bigcup_{q \in Q} Hg_q$, it suffices to show that the cosets Hg_q are all distinct.
 258 Indeed, assume that $Hg_p = Hg_q$ for some vertices $p, q \in Q$. Then $g_p g_q^{-1} \in H$ and so
 259 $\overline{g_p g_q^{-1}} \in L(\mathcal{S}(H))$ by Proposition 2.5. On the other hand, since $\mathcal{S}(H)$ is complete, we
 260 have a path

$$q_0 \xrightarrow{g_p g_q^{-1}} r$$

261 for some $r \in Q$. In view of Proposition 2.1, and by determinism, we get $r = q_0$. Hence
 262 we have paths

$$p \xrightarrow{g_q^{-1}} q_0, \quad q \xrightarrow{g_q^{-1}} q_0.$$

263 Since $\mathcal{S}(H)$ is inverse, we get $p = q$ as required. \square

264 **Example 2.4.** Since the Stallings automaton constructed in Example 2.1 is not complete,
265 it follows that $\langle a^{-1}ba, ba^2 \rangle$ is not a finite index subgroup of F_2 .

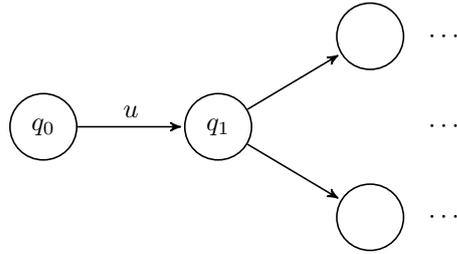
266 **Corollary 2.9.** *If $H \leq F_A$ has index n , then $\text{rk } H = 1 + n(\#A - 1)$.*

267 *Proof.* By Proposition 2.8, the automaton $\mathcal{S}(H)$ has n vertices and $n\#A$ positive edges.
268 A spanning tree has $n - 1$ positive edges, so $\text{rk } H = n\#A - (n - 1) = 1 + n(\#A - 1)$
269 by Proposition 2.6. \square

270 2.4 Conjugacy

271 We start now a brief discussion of conjugacy. Recall that the *outdegree* of a vertex q is the
272 number of edges starting at q and the *geodesic distance* in a connected graph is the length
273 of the shortest undirected path connecting two vertices.

Since the original generating set is always taken in reduced form, it follows easily that there is at most one vertex in a Stallings automaton having outdegree < 2 : the basepoint q_0 . Assuming that H is nontrivial, $\mathcal{S}(H)$ must always be of the form



274 where q_1 is the closest vertex to q_0 (in terms of geodesic distance) having outdegree > 2
275 (since there is at least one vertex having such outdegree). Note that $q_1 = q_0$ if q_0 has
276 outdegree > 2 itself. We call $q_0 \xrightarrow{u}$ the *tail* (which is empty if $q_1 = q_0$) and the remaining
277 subgraph the *core* of $\mathcal{S}(H)$.

278 Note that $\mathcal{S}(H)$, and its core, may be understood as follows. Consider the graph with
279 vertex set $F_A/H = \{gH \mid g \in F_A\}$, with an edge from gH to agH for each generator
280 $a \in A$. Then this graph, called the *Schreier graph* (see §24.1) of $H \setminus F_A$, consists of
281 finitely many trees attached to the core of $\mathcal{S}(H)$.

282 **Theorem 2.10.** *There is an algorithm that decides whether or not two finitely generated
283 subgroups of F_A are conjugate.*

284 *Proof.* Finitely generated subgroups G, H are conjugate if and only if the cores of $\mathcal{S}(G)$
285 and $\mathcal{S}(H)$ are equal (up to their basepoints). \square

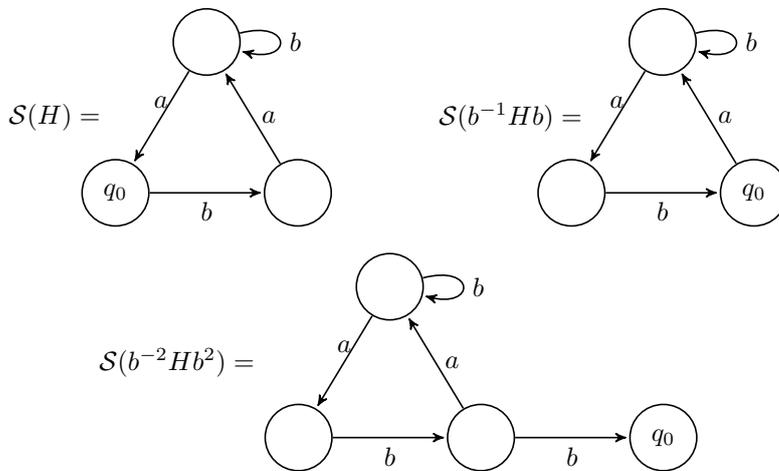
286 The Stallings automata of the conjugates of H can be obtained in the following ways:
287 (1) declaring a vertex in the core C to be the basepoint; (2) gluing a tail to some vertex in
288 the core C and taking its other endpoint to be the basepoint.

289 Note that the tail must be glued in some way that keeps the automaton inverse, so
 290 in particular this second type of operation can only be performed if the automaton is
 291 not complete, or equivalently, if H has infinite index. An immediate consequence is the
 292 following classical

293 **Proposition 2.11.** *A finite rank normal subgroup of a free group is either trivial or has*
 294 *finite index.*

295 Moreover, a finite index subgroup H is normal if and only if its Stallings automaton
 296 is *vertex-transitive*, that is, if all choices of basepoint yield the same automaton.

297 **Example 2.5.** Stallings automata of some conjugates of $H = \langle a^{-1}ba, ba^2 \rangle$:



298 We can also use the previous discussion on the structure of (finite) Stallings automata to
 299 provide them with an abstract characterization.

300 **Proposition 2.12.** *A finite inverse automata with a basepoint is a Stallings automaton if*
 301 *and only if it has at most one vertex of outdegree 1: the basepoint.*

302 *Proof.* Indeed, for any such automaton we can take a spanning tree and use it to construct
 303 a basis for the subgroup as in the proof of Proposition 2.6. □

304 2.5 Further algebraic properties

305 The study of intersections of finitely generated subgroups of F_A provides further applica-
 306 tions of Stallings automata. Howson's classical theorem admits a simple proof using the
 307 *direct product* of two Stallings automata; it is also an immediate consequence of Theorem
 308 3.1 and Corollary 3.4(ii).

309 **Theorem 2.13** (Howson). *If $H, K \leq_{f.g.} F_A$, then also $H \cap K \leq_{f.g.} F_A$.*

310 Stallings automata are also naturally related to the famous Hanna Neumann con-
 311 jecture: given $H, K \leq_{f.g.} F_A$, then $\text{rk}(H \cap K) - 1 \leq (\text{rk } H - 1)(\text{rk } K - 1)$. The conjecture
 312 had arisen in a paper of Hanna Neumann [33], where the inequality $\text{rk}(H \cap K) - 1 \leq$
 313 $2(\text{rk } H - 1)(\text{rk } K - 1)$ was also proved. In one of the early applications of Stallings’
 314 approach, Gersten provided an alternative geometric proof of Hanna Neumann’s inequal-
 315 ity [14].

316 A *free factor* of a free group F_A can be defined as a subgroup H generated by a subset
 317 of a basis of F_A . This is equivalent to say that there exists a *free product decomposition*
 318 $F_A = H * K$ for some $K \leq F_A$.

319 Since the rank of a free factor never exceeds the rank of the ambient free group, it is
 320 easy to construct examples of subgroups which are not free factors: it follows easily from
 321 Proposition 2.6 that any free group of rank ≥ 2 can have subgroups of arbitrary finite rank
 322 (and even infinite countable).

323 The problem of identifying free factors has been given a simple solution based on
 324 Stallings automata in [50]: one must check whether or not a prescribed number of vertex
 325 identifications in the Stallings automaton can lead to a bouquet. However, the most ef-
 326 ficient solution, due to Roig, Ventura and Weil [39], involves Whitehead automorphisms
 327 and will therefore be postponed to §2.7.

328 Given a morphism $\varphi : \mathcal{A} \rightarrow \mathcal{B}$ of inverse automata, let the *morphic image* $\varphi(\mathcal{A})$ be
 329 the subautomaton of \mathcal{B} induced by the image by φ of all the successful paths of \mathcal{A} .

330 The following classical result characterizes the extensions of $H \leq_{f.g.} F_A$ contained
 331 in F_A . We present the proof from [31]:

332 **Theorem 2.14** (Takahasi [54]). *Given $H \leq_{f.g.} F_A$, one can effectively compute finitely*
 333 *many extensions $K_1, \dots, K_m \leq_{f.g.} F_A$ of H such that the following conditions are equiv-*
 334 *alent for every $K \leq_{f.g.} F_A$:*

- 335 (i) $H \leq K$;
 336 (ii) K_i is a free factor of K for some $i \in \{1, \dots, m\}$.

337 *Proof.* Let $\mathcal{A}_1, \dots, \mathcal{A}_m$ denote all the morphic images of $\mathcal{S}(H)$, up to isomorphism.
 338 Since a morphic image cannot have more vertices than the original automaton, there are
 339 only finitely many isomorphism classes. Moreover, it follows from Proposition 2.12 that,
 340 for $i = 1, \dots, m$, $\mathcal{A}_i = \mathcal{S}(K_i)$ for some $K_i \leq_{f.g.} F_A$. Since $L(\mathcal{S}(H)) \subseteq L(\mathcal{A}_i) =$
 341 $L(\mathcal{S}(K_i))$, it follows from Proposition 2.5 that $H \leq K_i$. Clearly, we can construct all \mathcal{A}_i
 342 and therefore all K_i .

343 (i) \Rightarrow (ii). If $H \leq K$, it follows from Stallings’ construction that $L(\mathcal{S}(H)) \subseteq$
 344 $L(\mathcal{S}(K))$ and so there is a morphism $\varphi : \mathcal{S}(H) \rightarrow \mathcal{S}(K)$ by Proposition 2.2. Let \mathcal{A}_i
 345 be, up to isomorphism, the morphic image of $\mathcal{S}(H)$ through φ . Since $\mathcal{A}_i = \mathcal{S}(K_i)$ is a
 346 subautomaton of $\mathcal{S}(K)$, it follows easily from Proposition 2.6 that K_i is a free factor of
 347 K : it suffices to take a spanning tree for $\mathcal{S}(K_i)$, extend it to a spanning tree for $\mathcal{S}(K)$,
 348 and the induced basis of K_i will be contained in the induced basis of K .

349 (ii) \Rightarrow (i) is immediate. □

350 An interesting research line related to this result is built over the concept of algebraic
 351 extension, introduced in [19] by Kapovich and Miasnikov, and inspired by the homony-

352 mous field-theoretical classical notion. Given $H \leq K \leq F_A$, we say that K is an
 353 *algebraic* extension of H if no proper free factor of K contains H . Miasnikov, Ventura
 354 and Weil proved in [31] that the set of algebraic extensions of H is finite and effectively
 355 computable, and it constitutes the minimum set of extensions K_1, \dots, K_m satisfying the
 356 conditions of Theorem 2.14.

357 Consider a subgroup H of a group G . The *commensurator* of H in G , is

$$\text{Comm}_G(H) = \{g \in G \mid H \cap H^g \text{ has finite index in } H \text{ and } H^g\}. \quad (2.4)$$

358 For example, the commensurator of $\text{GL}_n(\mathbb{Z})$ in $\text{GL}_n(\mathbb{R})$ is $\text{GL}_n(\mathbb{Q})$.

359 The special case of finite-index extensions, $H \leq_{f.i.} K \leq F_A$ is of special interest,
 360 and can be interpreted in terms of commensurators. It can be proved (see [19, Lemma
 361 8.7] and [51]) that every $H \leq_{f.g.} F_A$ has a maximum finite-index extension inside F_A ,
 362 denoted by $H_{f.i.}$; and $H_{f.i.} = \text{Comm}_{F_A}(H)$. Silva and Weil proved in [51] that $\mathcal{S}(H_{f.i.})$
 363 can be constructed from $\mathcal{S}(H)$ using a simple automata-theoretic algorithm:

- 364 (1) The standard minimization algorithm is applied to the core of $\mathcal{S}(H)$, *taking all*
 365 *vertices as final*.
- 366 (2) The original tail of $\mathcal{S}(H)$ is subsequently reinstated in this new automaton, at the
 367 appropriate vertex.

368 We present now an application of different type, involving transition monoids. It
 369 follows easily from the definitions that the transition monoid of a finite inverse automaton
 370 is always a *finite inverse monoid*. Given a group G , we say that a subgroup $H \leq G$ is
 371 *pure* if the implication

$$g^n \in H \Rightarrow g \in H \quad (2.5)$$

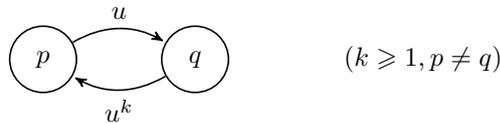
372 holds for all $g \in F_A$ and $n \geq 1$. If p is a prime, we say that H is *p-pure* if (2.5) holds
 373 when $(n, p) = 1$.

374 The next result is due to Birget, Margolis, Meakin and Weil, and is the only natural
 375 problem among applications of Stallings automata that is known so far to be PSPACE-
 376 complete [7].

377 **Proposition 2.15.** *For every $H \leq_{f.g.} F_A$, the following conditions hold:*

- 378 (i) *H is pure if and only if the transition monoid of $\mathcal{S}(H)$ is aperiodic.*
- 379 (ii) *H is p-pure if and only if the transition monoid of $\mathcal{S}(H)$ has no subgroups of order*
 380 *p .*

Proof. Both conditions in (i) are easily proved to be equivalent to the inexistence in $\mathcal{S}(H)$
 of a cycle of the form



381 where u can be assumed to be cyclically reduced. The proof of (ii) runs similarly. \square

2.6 Topological properties

We require for this subsection some basic topological concepts, which the reader can recover from Chapter 17.

For all $u, v \in F_A$, written in reduced form as elements of R_A , let $u \wedge v$ denote the longest common prefix of u and v . The *prefix metric* d on F_A is defined, for all $u, v \in F_A$, by

$$d(u, v) = \begin{cases} 2^{-|u \wedge v| - 1} & \text{if } u \neq v \\ 0 & \text{if } u = v \end{cases}$$

It follows easily from the definition that d is an ultrametric on F_A , satisfying in particular the axiom

$$d(u, v) \leq \max\{d(u, w), d(w, v)\}.$$

The *completion* of this metric space is compact; its extra elements are *infinite reduced words* $a_1 a_2 a_3 \dots$, with all $a_i \in \tilde{A}$, and constitute the *hyperbolic boundary* ∂F_A of F_A , see §24.1.5. Extending the operator \wedge to $F_A \cup \partial F_A$ in the obvious way, it follows easily from the definitions that, for every infinite reduced word α and every sequence $(u_n)_n$ in F_A ,

$$\alpha = \lim_{n \rightarrow +\infty} u_n \quad \text{if and only if} \quad \lim_{n \rightarrow +\infty} |\alpha \wedge u_n| = +\infty. \quad (2.6)$$

The next result shows that Stallings automata are given a new role in connection to the prefix metric. We denote by $\text{cl } H$ the closure of H in the completion of F_A .

Proposition 2.16. *If $H \leq_{f.g.} F_A$, then $\text{cl } H$ is the union of H with the set of all $\alpha \in \partial F_A$ that label paths in $\mathcal{S}(H)$ out of the basepoint.*

Proof. Since the topology of F_A is discrete, we have $\text{cl } H \cap F_A = H$.

(\subseteq): If $\alpha \in \partial F_A$ does not label a path in $\mathcal{S}(H)$ out of the basepoint, then $\{|\alpha \wedge h|; h \in H\}$ is finite and so no sequence of H can converge to α by (2.6).

(\supseteq): Let $\alpha = a_1 a_2 a_3 \dots \in \partial F_A$, with $a_i \in \tilde{A}$, label a path in $\mathcal{S}(H)$ out of the basepoint. Let m be the number of vertices of $\mathcal{S}(H)$. For every $n \geq 1$, there exists some word w_n of length $< m$ such that $a_1 \dots a_n w_n \in H$. Now $\alpha = \lim_{n \rightarrow +\infty} a_1 \dots a_n w_n$ by (2.6) and so $\alpha \in \text{cl } H$. \square

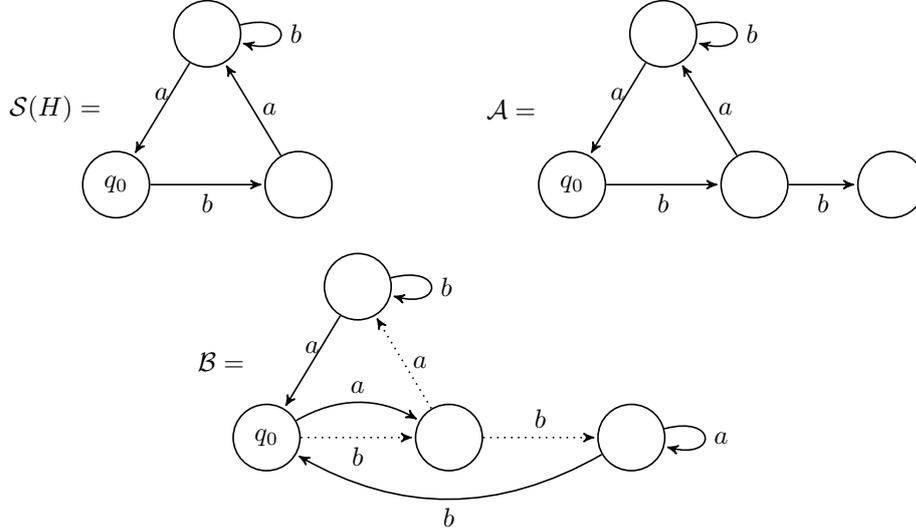
The *profinite topology* on F_A is defined in Chapter 17: for every $u \in F_A$, the collection $\{Ku \mid K \leq_{f.i.} F_A\}$ constitutes a basis of clopen neighbourhoods of u . In his seminal 1983 paper [53], Stallings gave an alternative proof of Marshall Hall's Theorem:

Theorem 2.17 (M. Hall). *Every finitely generated subgroup of F_A is closed for the profinite topology.*

Proof. Fix $H \leq_{f.g.} F_A$ and let $u \in F_A \setminus H$ be written in reduced form as an element of R_A . In view of Proposition 2.5, u does not label a loop at the basepoint q_0 of $\mathcal{S}(H)$. If there is no path $q_0 \xrightarrow{u} \dots$ in $\mathcal{S}(H)$, we add new edges to $\mathcal{S}(H)$ to get a finite inverse automaton \mathcal{A} having a path $q_0 \xrightarrow{u} q \neq q_0$. Otherwise just take $\mathcal{A} = \mathcal{S}(H)$. Next add new edges to \mathcal{A} to get a finite complete inverse automaton \mathcal{B} . In view of Propositions 2.8 and 2.12, we have $\mathcal{B} = \mathcal{S}(K)$ for some $K \leq_{f.i.} F_A$. Hence Ku is open and contains u .

417 Since $H \cap Ku \neq \emptyset$ yields $u \in K^{-1}H = K$, contradicting Proposition 2.5, it follows that
 418 $H \cap Ku = \emptyset$ and so H is closed as claimed. \square

Example 2.6. We consider the above construction for $H = \langle a^{-1}ba, ba^2 \rangle$ and $u = b^2$:



419 If we take the spanning tree defined by the dotted lines in \mathcal{B} , it follows from Proposi-
 420 tion 2.6 that

$$K = \langle ba^{-1}, b^3, b^2ab^{-2}, ba^2, baba^{-1}b^{-1} \rangle$$

421 is a finite index subgroup of F_2 such that $H \cap Kb^2 = \emptyset$.

422 We recall that a group G is *residually finite* if its finite index subgroups have trivial
 423 intersection. Considering the trivial subgroup in Theorem 2.17, we deduce the

424 **Corollary 2.18.** F_A is residually finite.

425 We remark that Ribes and Zalessky extended Theorem 2.17 to products of finitely
 426 many finitely generated subgroups of F_A , see [37]. This result is deeply connected to the
 427 solution of Rhodes' Type II conjecture, see [36, Chapter 4]

428 If \mathbf{V} denotes a pseudovariety of finite groups (see Chapter 16), the *pro- \mathbf{V} topology* on
 429 F_A is defined by considering that each $u \in F_A$ has

$$\{Ku \mid K \trianglelefteq_{f.i.} F_A, F_A/K \in \mathbf{V}\}$$

430 as a basis of clopen neighbourhoods. The closure for the pro- \mathbf{V} topology of $H \leq_{f.g.} F_A$
 431 can be related to an extension property of $\mathcal{S}(H)$, and Margolis, Sapir and Weil used
 432 automata to prove that efficient computation can be achieved for the pseudovarieties of
 433 finite p -groups and finite nilpotent groups [27]. The original computability proof for the
 434 p -group case is due to Ribes and Zalessky [38].

2.7 Dynamical properties

We shall mention briefly some examples of applications of Stallings automata to the study of endomorphism dynamics, starting with Gersten's solution of the subgroup orbit problem [15].

The subgroup orbit problem consists in finding an algorithm to decide, for given $H, K \leq_{f.g.} F_A$, whether or not $K = \varphi(H)$ for some automorphism φ of F_A . Equivalently, this can be described as deciding whether or not the automorphic orbit of a finitely generated subgroup is recursive.

Gersten's solution adapts to the context of Stallings automata Whitehead's idea to solve the orbit problem for words [58]. Whitehead's proof relies on a suitable decomposition of automorphisms as products of elementary factors (which became known as *Whitehead automorphisms*), and on using these as a tool to compute the elements of minimum length in the automorphic orbit of the word. In the subgroup case, word length is replaced by the number of vertices of the Stallings automaton.

The most efficient solution to the problem of identifying free factors [39], mentioned in §2.5, relies also on this approach: $H \leq_{f.g.} F_A$ is a free factor if and only if the Stallings automaton of some automorphic image of H has a single vertex (that is, a bouquet).

Another very nice application is given by the following theorem of Goldstein and Turner [16]:

Theorem 2.19. *The fixed point subgroup of an endomorphism of F_A is finitely generated.*

Proof. Let φ be an endomorphism of F_A . For every $u \in F_A$, define $Q(u) = \varphi(u)u^{-1}$. We define a potentially infinite automaton \mathcal{A} by taking

$$\{Q(u) \mid u \in F_A\} \subseteq F_A$$

as vertex set, all edges of the form $Q(u) \xrightarrow{a} Q(au)$ with $u \in F_A$, $a \in \tilde{A}$, and fixing $\mathbb{1}$ as basepoint. Then \mathcal{A} is a well-defined inverse automaton.

Next we take \mathcal{B} to be the subautomaton of \mathcal{A} obtained by retaining only those vertices and edges that lie in successful paths labelled by reduced words. Clearly, \mathcal{B} is still an inverse automaton, and it is easy to check that it must be the Stallings automaton of the fixed point subgroup of φ .

It remains to be proved that \mathcal{B} is finite. We define a subautomaton \mathcal{C} of \mathcal{B} by removing exactly one edge among each inverse pair

$$Q(u) \xrightarrow{a} Q(au), \quad Q(au) \xrightarrow{a^{-1}} Q(u)$$

with $a \in A$ as follows: if a^{-1} is the last letter of $Q(au)$, we remove $Q(u) \xrightarrow{a} Q(au)$; otherwise, we remove $Q(au) \xrightarrow{a^{-1}} Q(u)$.

Let M denote the maximum length of the image of a letter by φ . We claim that, whenever $|Q(v)| > 2M$, the vertex $Q(v)$ has outdegree at most 1.

Indeed, if $Q(v) \xrightarrow{a^{-1}} Q(a^{-1}v)$ is an edge in \mathcal{C} for $a \in A$, then a^{-1} is the last letter of $Q(v)$. On the other hand, if $Q(v) \xrightarrow{b} Q(bv)$ is an edge in \mathcal{C} for $b \in A$, then b^{-1} is not the last letter of $Q(bv)$. Since $Q(bv) = \varphi(b)Q(v)b^{-1}$ and $|Q(v)| > 2|\varphi(b)|$, then b must be the last letter of $Q(v)$ in this case. Since $Q(v)$ has at most one last letter, it follows that its outdegree is at most 1.

474 Let \mathcal{D} be a finite subautomaton of \mathcal{C} containing all vertices $Q(v)$ such that $|Q(v)| \leq$
 475 $2M$. Suppose that $p \rightarrow q$ is an edge in \mathcal{C} not belonging to \mathcal{D} . Since $p \rightarrow q$, being an edge
 476 of \mathcal{B} , must lie in some reduced path, and by the outdegree property of \mathcal{C} , it is easy to see
 477 that there exists some path in \mathcal{C} of the form

$$p' \rightarrow p \rightarrow q \rightarrow r \leftarrow r'$$

478 where p', r' are vertices in \mathcal{D} . Since there are only finitely many directed paths out of \mathcal{D} ,
 479 it follows that \mathcal{C} is finite and so is \mathcal{B} . Therefore the fixed point subgroup of φ is finitely
 480 generated. \square

481 Note that this proof is not by any means constructive. Indeed, the only known al-
 482 gorithm for computing the fixed point subgroup of a free group automorphism is due to
 483 Maslakova [30] and relies on the sophisticated *train track* theory of Bestvina and Han-
 484 del [6] and other algebraic geometry tools. The general endomorphism case remains
 485 open.

486 Stallings automata were also used by Ventura in the study of various properties of
 487 fixed subgroups, considering in particular arbitrary families of endomorphisms [56, 29]
 488 (see also [57]).

489 Automata play also a part on the study of *infinite fixed points*, taken over the continu-
 490 ous extension of a monomorphism to the hyperbolic boundary (see for example [48]).

491 3 Rational and recognizable subsets

492 Rational subsets generalize the notion of finitely generated from subgroups to arbitrary
 493 subsets of a group, and can be quite useful in establishing inductive procedures that need
 494 to go beyond the territory of subgroups. Similarly, recognizable subsets extend the notion
 495 of finite index subgroups. Basic properties and results can be found in [5] or [42].

496 We consider a finitely generated group $G = \langle A \rangle$, with the canonical map $\pi : F_A \rightarrow G$.
 497 A subset of G is *rational* if it is the image by $\rho = \pi\theta$ of a rational subset of \tilde{A}^* , and is
 498 *recognizable* if its full preimage under ρ is rational in \tilde{A}^* .

499 For every group G , the classes $\text{Rat } G$ and $\text{Rec } G$ satisfy the following closure proper-
 500 ties:

- 501 • $\text{Rat } G$ is (effectively) closed under union, product, star, morphisms, inversion, sub-
 502 group generating.
- 503 • $\text{Rec } G$ is (effectively) closed under boolean operations, translation, product, star,
 504 inverse morphisms, inversion, subgroup generating.

505 Kleene's Theorem is not valid for groups: $\text{Rat } G = \text{Rec } G$ if and only if G is finite.
 506 However, if the class of rational subsets of G possesses some extra algorithmic properties,
 507 then many decidability/constructibility results can be deduced for G . Two properties are
 508 particularly coveted for $\text{Rat } G$:

- 509 • (effective) closure under complement (yielding closure under all the boolean oper-
 510 ations);
- 511 • decidable membership problem for arbitrary rational subsets.

512 In these cases, one may often solve problems (e.g. equations, or systems of equations)
 513 whose statement lies far out of the rational universe, by proving that the solution is a
 514 rational set.

515 3.1 Rational and recognizable subgroups

516 We start by some basic, general facts. The following result is essential to connect language
 517 theory to group theory.

518 **Theorem 3.1** (Anisimov & Seifert). *A subgroup H of a group G is rational if and only if*
 519 *H is finitely generated.*

520 *Proof.* (\Rightarrow): Let H be a rational subgroup of G and let $\pi : F_A \rightarrow G$ denote a morphism.
 521 Then there exists a finite \tilde{A} -automaton \mathcal{A} such that $H = \rho(L(\mathcal{A}))$. Assume that \mathcal{A} has m
 522 vertices and let X consist of all the words in $\rho^{-1}(H)$ of length $< 2m$. Since \mathcal{A} is finite,
 523 so is X . We claim that $H = \langle \rho(X) \rangle$. To prove it, it suffices to show that

$$u \in L(\mathcal{A}) \Rightarrow \rho(u) \in \langle \rho(X) \rangle \quad (3.1)$$

524 holds for every $u \in \tilde{A}^*$. We use induction on $|u|$. By definition of X , (3.1) holds for
 525 words of length $< 2m$. Assume now that $|u| \geq 2m$ and (3.1) holds for shorter words.
 526 Write $u = vw$ with $|w| = m$. Then there exists a path

$$\rightarrow q_0 \xrightarrow{v} q \xrightarrow{z} t \rightarrow$$

527 in \mathcal{A} with $|z| < m$. Thus $vz \in L(\mathcal{A})$ and by the induction hypothesis $\rho(vz) \in \langle \rho(X) \rangle$.
 528 On the other hand, $|z^{-1}w| < 2m$ and $\rho(z^{-1}w) = \rho(z^{-1}v^{-1})\rho(vw) \in H$, hence $z^{-1}w \in$
 529 X and so $\rho(u) = \rho(vz)\rho(z^{-1}w) \in \langle \rho(X) \rangle$, proving (3.1) as required.

530 (\Leftarrow) is trivial. □

531 It is an easier task to characterize recognizable subgroups:

532 **Proposition 3.2.** *A subgroup H of a group G is recognizable if and only if it has finite*
 533 *index.*

534 *Proof.* (\Rightarrow): In general, a recognizable subset of G is of the form NX , where $N \trianglelefteq_{f.i.} G$
 535 and $X \subseteq G$ is finite. If $H = NX$ is a subgroup of G , then $N \subseteq H$ and so H has finite
 536 index as well.

537 (\Leftarrow): This follows from the well-known fact that every finite index subgroup H of G
 538 contains a finite index normal subgroup N of G , namely $N = \bigcap_{g \in G} gHg^{-1}$. Since N
 539 has finite index, H must be of the form NX for some finite $X \subseteq G$. □

540 3.2 Benoiss' Theorem

541 The central result in this subsection is Benoiss' Theorem, the cornerstone of the whole
 542 theory of rational subsets of free groups:

543 **Theorem 3.3** (Benois).

- 544 (i) If $L \subseteq \tilde{A}^*$ is rational, then \bar{L} is also rational, and can be effectively constructed
 545 from L .
 546 (ii) A subset of R_A is a rational language as a subset of \tilde{A}^* if and only if it is rational
 547 as a subset of F_A .

548 We illustrate this in the case of finitely generated subgroups: temporarily calling
 549 “Benois automata” those automata recognizing rational subsets of R_A , we may convert
 550 them to Stallings automata by “folding” them, making at the same time sure they are in-
 551 verse automata. Given a Stallings automaton, one intersects it with R_A to obtain a Benois
 552 automaton.

553 *Proof.* (i) Let $\mathcal{A} = (Q, I, E, T)$ be a finite automaton recognizing L . We define a
 554 sequence $(\mathcal{A}_n)_n$ of finite automata with ε -transitions as follows. Let $\mathcal{A}_0 = \mathcal{A}$. As-
 555 suming that $\mathcal{A}_n = (Q, I, E_n, T)$ is defined, we consider all instances of ordered pairs
 556 $(p, q) \in Q \times Q$ such that

$$\text{there exists a path } p \xrightarrow{aa^{-1}} q \text{ in } \mathcal{A}_n \text{ for some } a \in \tilde{A}, \text{ but no path } p \xrightarrow{1} q. \quad (\text{P})$$

557 Clearly, there are only finitely many instances of (P) in \mathcal{A}_n . We define E_{n+1} to be the
 558 union of E_n with all the new edges $(p, 1, q)$, where $(p, q) \in Q \times Q$ is an instance of
 559 (P). Finally, we define $\mathcal{A}_{n+1} = (Q, I, E_{n+1}, T)$. In particular, note that $\mathcal{A}_n = \mathcal{A}_{n+k}$ for
 560 every $k \geq 1$ if there are no instances of (P) in \mathcal{A}_n .

561 Since Q is finite, the sequence $(\mathcal{A}_n)_n$ is ultimately constant, say after reaching \mathcal{A}_m .
 562 We claim that

$$\bar{L} = L(\mathcal{A}_m) \cap R_A. \quad (3.2)$$

563 Indeed, take $u \in L$. There exists a sequence of words $u = u_0, u_1, \dots, u_{k-1}, u_k = \bar{u}$
 564 where each term is obtained from the preceding one by erasing a factor of the form aa^{-1}
 565 for some $a \in \tilde{A}$. A straightforward induction shows that $u_i \in L(\mathcal{A}_i)$ for $i = 0, \dots, k$,
 566 since the existence of a path $p \xrightarrow{aa^{-1}} q$ in \mathcal{A}_i implies the existence of a path $p \xrightarrow{1} q$ in \mathcal{A}_{i+1} .
 567 Hence $\bar{u} = u_k \in L(\mathcal{A}_k) \subseteq L(\mathcal{A}_m)$ and it follows that $\bar{L} \subseteq L(\mathcal{A}_m) \cap R_A$.

568 For the opposite inclusion, we start by noting that any path $p \xrightarrow{u} q$ in \mathcal{A}_{i+1} can be
 569 lifted to a path $p \xrightarrow{v} q$ in \mathcal{A}_i , where v is obtained from u by inserting finitely many factors
 570 of the form aa^{-1} . It follows that

$$\overline{L(\mathcal{A}_m)} = \overline{L(\mathcal{A}_{m-1})} = \dots = \overline{L(\mathcal{A}_0)} = \bar{L}$$

571 and so $L(\mathcal{A}_m) \cap R_A \subseteq \overline{L(\mathcal{A}_m)} = \bar{L}$. Thus (3.2) holds.

572 Since

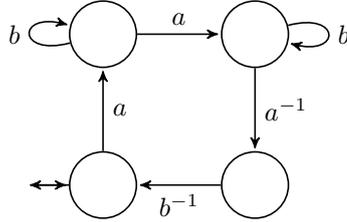
$$R_A = \tilde{A}^* \setminus \bigcup_{a \in \tilde{A}} \tilde{A}^* aa^{-1} \tilde{A}^*$$

573 is obviously rational, and rational languages are closed for intersection, it follows that \bar{L} is
 574 rational. Moreover, we can effectively compute the automaton \mathcal{A}_m and a finite automaton
 575 recognizing R_A , hence the direct product construction can be used to construct a finite
 576 automaton recognizing the intersection $\bar{L} = L(\mathcal{A}_m) \cap R_A$.

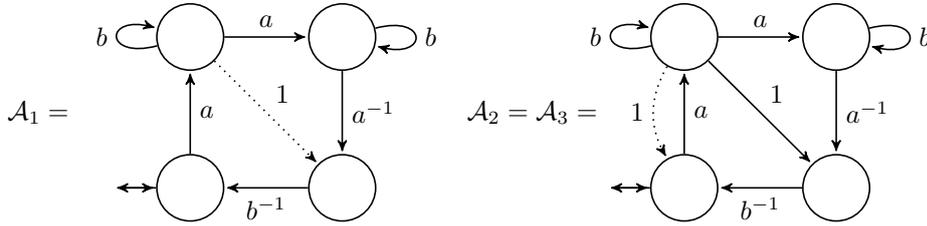
577 (ii) Consider $X \subseteq R_A$. If $X \in \text{Rat } \tilde{A}^*$, then $\theta(X) \in \text{Rat } F_A$ and so X is rational as
 578 a subset of F_A .

579 Conversely, if X is rational as a subset of F_A , then $X = \theta(L)$ for some $L \in \text{Rat } \tilde{A}^*$.
 580 Since $X \subseteq R_A$, we get $X = \bar{L}$. Now part (i) yields $\bar{L} \in \text{Rat } \tilde{A}^*$ and so $X \in \text{Rat } \tilde{A}^*$ as
 581 required. \square

Example 3.1. Let $\mathcal{A} = \mathcal{A}_0$ be depicted by



We get



582 and we can then proceed to compute $\bar{L} = L(\mathcal{A}_2) \cap R_2$.

583 The following result summarizes some of the most direct consequences of Benois'
 584 Theorem:

Corollary 3.4.

- 585 (i) F_A has decidable rational subset membership problem.
- 586 (ii) $\text{Rat } F_A$ is closed under the boolean operations.

587 *Proof.* (i) Given $X \in \text{Rat } F_A$ and $u \in F_A$, write $X = \theta(L)$ for some $L \in \text{Rat } \tilde{A}^*$.
 588 Then $u \in X$ if and only if $\bar{u} \in \bar{X} = \bar{L}$. By Theorem 3.3(i), we may construct a finite
 589 automaton recognizing \bar{L} and therefore decide whether or not $\bar{u} \in \bar{L}$.

590 (ii) Given $X \in \text{Rat } F_A$, we have $\overline{F_A \setminus X} = R_A \setminus \bar{X}$ and so $F_A \setminus X \in \text{Rat } F_A$ by
 591 Theorem 3.3. Therefore $\text{Rat } F_A$ is closed under complement.

592 Since $\text{Rat } F_A$ is trivially closed under union, it follows from De Morgan's laws that it
 593 is closed under intersection as well. \square

594 Note that we can associate algorithms to these boolean closure properties of $\text{Rat } F_A$ in
 595 a constructive way. We remark also that the proof of Theorem 3.3 can be clearly adapted
 596 to more general classes of rewriting systems. Theorem 3.3 and Corollary 3.4 have been
 597 generalized several times by Benois herself [4] and by Sénizergues, who obtained the most
 598 general versions. Sénizergues' results [43] hold for *rational length-reducing left basic*
 599 *confluent* rewriting systems and remain valid for the more general notion of *controlled*
 600 *rewriting system*.???rewriting systems in another chapter? or in "Nachum Dershowitz
 601



602 and Jean-Pierre Jouannaud. Chapter 6 of Handbook of Theoretical Computer Science
 603 B: Formal Methods and Semantics, J. van Leeuwen, ed., pages 243-320, North-Holland
 604 (1990).”

605 3.3 Rational versus recognizable

606 Since F_A is a finitely generated monoid, it follows that every recognizable subset of
 607 F_A is rational [5, Proposition III.2.4]. We turn to the problem of deciding which rat-
 608 ional subsets of F_A are recognizable. The first proof, using rewriting systems, is due
 609 to Sénizergues [44] but we follow the shorter alternative proof from [47], where a third
 610 alternative proof, of a more combinatorial nature, was also given.

611 Given a subset X of a group G , we define the *right stabilizer* of X to be the submonoid
 612 of G defined by

$$R(X) = \{g \in G \mid Xg \subseteq X\}.$$

Next let

$$K(X) = R(X) \cap (R(X))^{-1} = \{g \in G \mid Xg = X\}$$

613 be the largest subgroup of G contained in $R(X)$ and let

$$N(X) = \bigcap_{g \in G} gK(X)g^{-1}$$

614 be the largest normal subgroup of G contained in $K(X)$, and therefore in $R(X)$.

615 **Lemma 3.5.** *A subset X of a group G is recognizable if and only if $K(X)$ is a finite index*
 616 *subgroup of G .*

617 In fact, the Schreier graph (see §24.1) of $K(X) \backslash G$ is the underlying graph of an
 618 automaton recognizing X , and $G/N(X)$ is the syntactic monoid of X .

619 *Proof.* (\Rightarrow): If $X \subseteq G$ is recognizable, then $X = NF$ for some $N \trianglelefteq_{f.i.} G$ and $F \subseteq G$
 620 finite. Hence $N \subseteq R(X)$ and so $N \subseteq K(X)$ since $N \leq G$. Since N has finite index in
 621 G , so has $K(X)$.

622 (\Leftarrow): If $K(X)$ is a finite index subgroup of G , so is $N = N(X)$. Indeed, a finite
 623 index subgroup has only finitely many conjugates (having also finite index) and a finite
 624 intersection of finite index subgroups is easily checked to have finite index itself.

625 Therefore it suffices to show that $X = FN$ for some finite subset F of G . Since N
 626 has finite index, the claim follows from $XN = X$, in turn an immediate consequence of
 627 $N \subseteq R(X)$. \square

628 **Proposition 3.6.** *It is decidable whether or not a rational subset of F_A is recognizable.*

629 *Proof.* Take $X \in \text{Rat } F_A$. In view of Lemma 3.5 and Proposition 2.8, it suffices to show
 630 that $K(X)$ is finitely generated and effectively computable.

631 Given $u \in F_A$, we have

$$u \notin R(X) \Leftrightarrow Xu \not\subseteq X \Leftrightarrow Xu \cap (F_A \setminus X) \neq \emptyset \Leftrightarrow u \in X^{-1}(F_A \setminus X),$$

632 hence

$$R(X) = F_A \setminus (X^{-1}(F_A \setminus X)).$$

633 It follows easily from rational languages being closed under reversion and morphisms,
 634 as well as Theorem 3.3(ii), that $X^{-1} \in \text{Rat } F_A$. Since $\text{Rat } F_A$ is trivially closed for
 635 product, it follows from Corollary 3.4 that $R(X)$ is rational and effectively computable,
 636 and so is $K(X) = R(X) \cap (R(X))^{-1}$. By Theorem 3.1, the subgroup $K(X)$ is finitely
 637 generated and the proof is complete. \square

638 These results are related to the Sakarovitch conjecture [41], stating that every rational
 639 subset of F_A must be either recognizable or *disjunctive*: a subset X of a monoid M is
 640 disjunctive if it has trivial syntactic congruence, or equivalently, if any morphism $\varphi : M \rightarrow M'$
 641 recognizing X is necessarily injective.

642 In the group case, it follows easily from the proof of the direct implication of Lemma 3.5
 643 that the projection $G \rightarrow G/N$ recognizes $X \subseteq G$ if and only if $N \subseteq N(X)$. Thus X is
 644 disjunctive if and only if $N(X)$ is the trivial subgroup.

645 The Sakarovitch conjecture was first proved in [44], but once again we follow the
 646 shorter alternative proof from [47]:

647 **Theorem 3.7** (Sénizergues). *A rational subset of F_A is either recognizable or disjunctive.*

648 *Proof.* Since the only subgroups of \mathbb{Z} are the trivial subgroup and finite index subgroups,
 649 we may assume that $\#A > 1$.

650 Take $X \in \text{Rat } F_A$. By the proof of Proposition 3.6, the subgroup $K(X)$ is finitely
 651 generated. In view of Lemma 3.5, we may assume that $K(X)$ is not a finite index sub-
 652 group. Thus $\mathcal{S}(K(X))$ is not complete by Proposition 2.8. Let q_0 denote the basepoint of
 653 $\mathcal{S}(K(X))$. Since $\mathcal{S}(K(X))$ is not complete, $q_0 \cdot u$ is undefined for some reduced word u .

654 Let w be an arbitrary nonempty reduced word. We must show that $w \notin N(X)$.
 655 Suppose otherwise. Since u, w are reduced and $\#A > 1$, there exist enough letters to
 656 make sure that there is some word $v \in R_A$ such that $uvwv^{-1}u^{-1}$ is reduced. Now
 657 $w \in N(X)$, hence $uvwv^{-1}u^{-1} \in N(X) \subseteq K(X)$ by normality. Since $uvwv^{-1}u^{-1}$ is
 658 reduced, it follows from Proposition 2.5 that $uvwv^{-1}u^{-1}$ labels a loop at q_0 in $\mathcal{S}(K(X))$,
 659 contradicting $q_0 \cdot u$ being undefined. Thus $w \notin N(X)$ and so $N(X) = 1$. Therefore X
 660 is disjunctive as required. \square

661 3.4 Beyond free groups

662 Let $\pi : F_A \twoheadrightarrow G$ be a morphism onto a group G . We consider the *word problem sub-*
 663 *monoid* of a group G , defined as

$$W_\pi(G) = (\pi\theta)^{-1}(\mathbb{1}). \quad (3.3)$$

664 **Proposition 3.8.** *The language $W_\pi(G)$ is rational if and only if G is finite.*

665 *Proof.* If G is finite, it is easy to check that $W_\pi(G)$ is rational by viewing the Cayley
 666 graph of G (see §24.1) as an automaton. Conversely, if $W_\pi(G)$ is rational, then $\pi^{-1}(\mathbb{1})$
 667 is a finitely generated normal subgroup of F_A , either finite index or trivial by the proof

668 of Proposition 3.7. It is well known that the *Dyck language* $D_A = \theta^{-1}(\mathbb{1})$ is not rational
 669 if $\#A > 0$, thus it follows easily that $\pi^{-1}(\mathbb{1})$ has finite index and therefore G must be
 670 finite. \square

671 How about groups with context-free $W_\pi(G)$? A celebrated result by Muller and
 672 Schupp [32], with contribution by Dunwoody [12], relates them to *virtually free groups*:
 673 these are groups with a free subgroup of finite index.

674 As usual, we focus on the case of G being finitely generated. We claim that G has a
 675 *normal* free subgroup F_A of finite index, with A finite. Indeed, letting F be a finite-index
 676 free subgroup of G , it suffices to take $F' = \bigcap_{g \in G} gFg^{-1}$. Since F has finite index, so
 677 has F' , see the proof of Lemma 3.5. Taking a morphism $\pi : F_B \rightarrow G$ with B finite, we
 678 get from Corollary 2.9 that $\pi^{-1}(F') \leq_{f.i.} F_B$ is finitely generated; so F' is itself finitely
 679 generated. Finally, F' is a subgroup of F , so F' is still free by Theorem 2.7, and we can
 680 write $F' = F_A$.

681 We may therefore decompose G as a finite disjoint union of the form

$$G = F_A b_0 \cup F_A b_1 \cup \cdots \cup F_A b_m, \quad \text{with } b_0 = 1. \quad (3.4)$$

682 **Theorem 3.9** (Muller & Schupp). *The language $W_\pi(G)$ is context-free if and only if G*
 683 *is virtually free.*

684 *Sketch of proof.* If G is virtually free, the rewriting system implicit in (3.4) provides a
 685 rational transduction between $W_\pi(G)$ and D_A .

686 The converse implication can be proved by arguing geometrical properties of the Cay-
 687 ley graph of G such as in Chapter 24; briefly said, one deduces from the context-freeness
 688 of $W_\pi(G)$ that the Cayley graph of G is close (more precisely, quasi-isometric) to a
 689 tree. \square

690 It follows that virtually free groups have decidable word problem. In Chapter 24, we
 691 shall discuss the word problem for more general classes of groups using other techniques.

692 Grunschlag proved that every rational (respectively recognizable) subset of the virtu-
 693 ally free group G admits a decomposition as a finite union $X_0 b_0 \cup \cdots \cup X_m b_m$, where
 694 the X_i are rational (respectively recognizable) subsets of F_A , see [17]. Thus basic re-
 695 sults such as Corollary 3.4 or Proposition 3.6 can be extended to virtually free groups
 696 (see [17, 46]). Similar generalizations can be obtained for free abelian groups of finite
 697 rank [46].

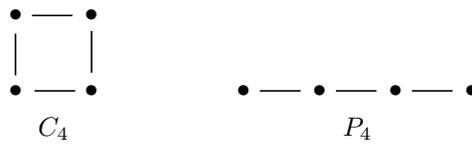
698 The fact that the strong properties of Corollary 3.4 do hold for both free groups and
 699 free abelian groups suggested considering the case of graph groups (also known as free
 700 partially abelian groups or right angled Artin groups), where we admit partial commuta-
 701 tion between letters.

702 An *independence graph* is a finite undirected graph (A, I) with no loops, that is, I is
 703 a symmetric anti-reflexive relation on A . The *graph group* $G(A, I)$ is the quotient F_A/τ ,
 704 where τ denotes the congruence generated by the relation

$$\{(ab, ba) \mid (a, b) \in I\}.$$

On both extremes, we have $F_A = G(A, \emptyset)$ and the free abelian group on A , which cor-
 responds to the complete graph on A . These turn out to be particular cases of *transitive*

forests. We can say that (A, I) is a transitive forest if it has no induced subgraph of either of the following forms:



705 We recall that an induced subgraph of (A, I) is formed by a subset of vertices $A' \subseteq A$
 706 and all the edges in I connecting vertices from A' .

707 The following difficult theorem, a group-theoretic version of a result on trace monoids
 708 by Aalbersberg and Hooeboom [1], was proved in [22]:

709 **Theorem 3.10** (Lohrey & Steinberg). *Let (A, I) be an independence graph. Then $G(A, I)$ ■
 710 has decidable rational subset membership problem if and only if (A, I) is a transitive for-
 711 est.*

712 They also proved that these conditions are equivalent to decidability of the member-
 713 ship problem for finitely generated submonoids. Such a ‘bad’ $G(A, I)$ gives an example
 714 of a finitely presented group with a decidable generalized word problem that does not
 715 have a decidable membership problem for finitely generated submonoids.

716 It follows from Theorem 3.10 that any group containing a direct product of two free
 717 monoids has undecidable rational subset membership problem, a fact that can be directly
 718 deduced from the undecidability of the Post correspondence problem.

719 Other positive results on rational subsets have been obtained for graphs of groups,
 720 HNN extensions and amalgamated free products by Kambites, Silva and Steinberg [18],
 721 or Lohrey and S nizergues [21]. Lohrey and Steinberg proved recently that the rational
 722 subset membership problem is recursively equivalent to the finitely generated submonoid
 723 membership problem for groups with two or more ends [23].

724 With respect to closure under complement, Lohrey and S nizergues [21] proved that
 725 the class of groups for which the rational subsets forms a boolean algebra is closed under
 726 HNN extension and amalgamated products over finite groups.

727 On the negative side, Bazhenova proved that rational subsets of finitely generated
 728 nilpotent groups do not form a boolean algebra, unless the group is virtually abelian [3].
 729 Moreover, Roman’kov proved in [40], via a reduction from Hilbert’s 10th problem, that
 730 the rational subset membership problem is undecidable for free nilpotent groups of any
 731 class ≥ 2 of sufficiently large rank

732 Last but not least, we should mention that Stallings’ construction was successfully
 733 generalized to prove results on both graph groups (by Kapovich, Miasnikov and Weid-
 734 mann [20]) and amalgamated free products of finite groups (by Markus-Epstein [28]).

735 3.5 Rational solution sets and rational constraints

736 In this final subsection we make a brief incursion in the brave new world of rational
 737 constraints. Rational subsets provide group theorists with two main assets:

- 738 • A concept which generalizes finite generation for subgroups and is much more fit
739 to stand most induction procedures.
- 740 • A systematic way of looking for solutions of the *right type* in the context of equa-
741 tions of many sorts.

742 This second feature leads us to the notion of *rational constraint*, when we restrict the set
743 of potential solutions to some rational subset. And there is a particular combination of
744 circumstances that can ensure the success of this strategy: if $\text{Rat } G$ is closed under inter-
745 section and we can prove that the solution set of problem P is an effectively computable
746 rational subset of G , then we can solve problem P with any rational constraint.

747 An early example is the adaptation by Margolis and Meakin of Rabin’s language and
748 Rabin’s tree theorem to free groups, where first-order formulae provide rational solution
749 sets [26]. The logic language considered here is meant to be applied to words, seen as
750 models, and consists basically of unary predicates that associate letters to positions in
751 each word, as well as a binary predicate for position ordering. Margolis and Meakin used
752 this construction to solve problems on combinatorial inverse semigroup theory [26].

753 Diekert, Gutierrez and Hagenah proved that the existential theory of systems of equa-
754 tions with rational constraints is solvable over a free group [10]. Working basically on
755 a free monoid with involution, and adapting Plandowski’s approach [34] in the process,
756 they extended the classical result of Makanin [24] to include rational constraints, with
757 much lower complexity as well.

758 The proof of this deep result is well out of scope here, but its potential applications
759 are immense. Group theorists are only starting to discover its full strength.

760 The results in [21] can be used to extend the existential theory of equations with ra-
761 tional constraints to virtually free groups, a result that follows also from Dahmani and
762 Guirardel’s recent paper on equations over hyperbolic groups with quasi-convex rational
763 constraints [9]. Equations over graph groups with a restricted class of rational constraints
764 were also successfully considered by Diekert and Lohrey [11].

765 A somewhat exotic example of computation of a rational solution set arises in the
766 problem of determining which automorphisms of F_2 (if any) carry a given word into a
767 given finitely generated subgroup. The full solution set is recognized by a finite automa-
768 ton; its vertices are themselves structures named “finite truncated automata” [49].

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883 **Abstract.** This chapter is devoted to the study of rational subsets of groups, with particular em-
884 phasis on the automata-theoretic approach to finitely generated subgroups of free groups. Indeed,
885 Stallings' construction, associating a finite inverse automaton with every such subgroup, inaugu-
886 rated a complete rewriting of free group algorithmics, with connections to other fields such as
887 topology or dynamics.

888 Another important vector in the chapter is the fundamental Benois' Theorem, characterizing
889 rational subsets of free groups. The theorem and its consequences really explain why language
890 theory can be successfully applied to the study of free groups. Rational subsets of (free) groups can
891 play a major role in proving statements (*a priori* unrelated to the notion of rationality) by induction.
892 The chapter also includes related results for more general classes of groups, such as virtually free
893 groups or graph groups.

Index

- 894 *Aalbersberg, IJsbrand Jan*, 23
 895 *Abért, Miklós*, 123
 896 abelian group
 897 free, 23, 106
 898 action
 899 free, 102
 900 Alëshin group, 118, 122
 901 is free, 129
 902 *Aleshin, Stanislav V.* [Алëшин, Станислав
 903 Владимирович], 118, 122, 129
 904 algebraic extension, 12
 905 alphabet
 906 involutive, 2
 907 amalgamated free product, 24
 908 amenable group, 121
 909 *Amir, Gideon*, 122
 910 *Angel, Omer*, 122
 911 *Anisimov, Anatoly V.* [Анісімов, Анатолій
 912 Васильович], 17
 913 aperiodic monoid, 13
 914 Artin group, 109
 915 automata group, 3, 115–131
 916 [regular] [weakly] branch, 123
 917 contracting, 121
 918 nuclear, 120
 919 word problem, 120
 920 automatic group, 105–113
 921 biautomatic, 108
 922 normal forms, 105
 923 quadratic isoperimetric inequality, 110
 924 right/left, 108
 925 word problem in quadratic time, 110
 926 automatic structure, 106
 927 geodesic, 107
 928 with uniqueness, 107
 929 automaton
 930 bounded, 120
 931 finite truncated, 25
 932 flower, 5
 933 inverse, 4
 934 involutive, 4
 935 Stallings, 6
 936 automorphism
 937 orbits, 25
 938 Whitehead, 16
 939 *Bartholdi, Laurent*, 121
 940 Basilica group, 119, 121
 941 is amenable, 121
 942 is regular weakly branch, 123
 943 presentation, 122
 944 Baumslag-Solitar group, 114, 118
 945 *Bazhenova, Galina A.* [Баженова, Галина
 946 Александровна], 24
 947 *Benois, Michèle*, 18, 20
 948 *Bestvina, Mladen*, 17
 949 biautomatic group, 108
 950 conjugacy problem, 111
 951 bicombing, 113
 952 bireversible Mealy automaton, 117, 128–
 953 131
 954 *Birget, Jean-Camille*, 13
 955 boundary
 956 hyperbolic, 14
 957 bounded group, 120
 958 is amenable, 121
 959 is contracting, 121
 960 branched covering, 125
 961 combinatorially equivalent, 125
 962 branch group, 122–123
 963 *Bridson, Martin R.*, 108
 964 *Brunner, Andrew M.*, 117
 965 *Burger, Marc*, 131
 966 *Cannon, James W.*, 104, 105, 126
 967 *Cayley, Arthur*, 102
 968 Cayley automaton, 127
 969 Cayley graph, 23, 102
 970 combing of group, 108
 971 commensurator of group, 12
 972 commutator in groups, 2
 973 cone type, 104, 112

- 974 conjugacy problem, 2
 975 conjugate elements, 2
 976 context-free
 word problem submonoid, 23
 978 contracting automaton/group, 120, 121, 126
 979 Coxeter group, 109
- 980 *Dahmani, François*, 25, 113
 981 *Dehn, Max*, 103, 104
 982 *Diekert, Volker*, 24, 25
 983 disjunctive rational subset, 21
 984 distance
 geodesic, 10
 prefix metric, 13
 word metric, 104
 988 *Dunwoody, Martin J.*, 22
 989 Dyck language, 22
- 990 *Epstein, David B. A.*, 105
 991 equations
 with rational constraints, 24
 993 existential theory of equations, 24
 994 extension
 algebraic, 12
 finite-index, 12
 HNN, 24
- 998 fellow traveller property, 106
 999 F_∞ group, 110
 1000 finite order element, 2
 1001 *Floyd, William J.*, 126
 1002 free group, 3, 122, 130
 basis, 3
 free factor, 11
 generalized word problem, 7
 is residually finite, 15
 rank, 3
 1008 free product, 129
 amalgamated, 24
 1010 fundamental group, 102, 104
 of 3-fold is automatic, 109
 of negatively curved manifold, 113
 1013 generalized word problem, 2, 113
 1014 geodesic distance, 10
 1015 *Gersten, Stephen M.*, 11, 15, 111, 112
 1016 *Gilman, Robert H.*, 107, 108, 112
- 1017 *Glasner, Yair*, 131
 1018 *Gluškov, Victor M.* [Глушков, Виктор Михайлович],
 1019 115
 1020 *Goldstein, Richard Z.*, 16
 graph
 1022 Cayley, 102
 1023 Schreier, 10, 21, 103, 121
 1024 graphs of groups, 24
 1025 *Greendlinger, Martin D.*, 104
 1026 *Grigorchuk, Rostislav I.* [Григорчук, Рос-
 1027 тислав Иванович], 114, 118, 122,
 1028 124
 1029 Grigorchuk group, 114, 116, 118, 124
 conjugacy problem, 120
 generalized word problem, 120
 has intermediate growth, 124
 is regular branch, 123
 is torsion, 122
 presentation, 122
 1036 *Gromov, Mikhael L.* [Громов, Михаил Леонидович],
 1037 105, 111, 112, 124
 1038 group
 affine, 117, 127
 Alëshin, 118, *see* Alëshin group
 amenable, 121
 Artin, 23, 109
 asynchronously automatic, 108
 automata, 3, 117, *see* automata group
 automatic, 105, *see* automatic group
 of an automaton, 115
 ball of radius n in, 104
 Basilica, 119, *see* Basilica group
 Baumslag-Solitar, 114, 118, 127
 biautomatic, 108, *see* biautomatic group
 bounded, 120, *see* bounded group
 braid, 109
 branch, 122, 123
 conjugacy separable, 120
 Coxeter, 109
 F_∞ , 110
 finite, 22
 finitely presented, 3, 24, 102
 free, 3
 free abelian, 23, 106
 free partially abelian, 23
 functionally recursive, 117

- 1063 fundamental, *see* fundamental group 108
- 1064 graph, 23, 25 1109
- 1065 Grigorchuk, 118, *see* Grigorchuk group 1108
- 1066 growth, 123–125, *see* growth of groups 1108
- 1067 Gupta-Sidki, 118, *see* Gupta-Sidki group 1108
- 1068 Heisenberg, 114 1113
- 1069 iterated monodromy, 126 1114
- 1070 Kazhdan, 131 1115
- 1071 lamplighter, 118
- 1072 mapping class, 109 1116
- 1073 nilpotent, 15, 24, 114 1117
- 1074 p -, 15, 118, 122
- 1075 relatively hyperbolic, 113 1118
- 1076 relators of a , 102
- 1077 residually finite, 15 1119
- 1078 right angled Artin, 23 1120
- 1079 self-similar, 117, 125 1121
- 1080 semi-hyperbolic, 113 1122
- 1081 surface, 103 1123
- 1082 VH-, 131 1124
- 1083 virtually abelian, 24 1125
- 1084 virtually free, 22, 23, 25 1126
- 1085 word-hyperbolic, 25, 111 1127
- 1086 —, *see* word-hyperbolic group 1128
- 1087 growth function, 104 1129
- 1088 growth of groups, 123–125 1130
- 1089 exponential, 124 1131
- 1090 intermediate, 124 1132
- 1091 non-uniformly exponential, 124 1133
- 1092 polynomial, 123 1134
- 1093 growth series, 104
- 1094 Grunschlag, Zeph, 23 1135
- 1095 Guirardel, Vincent, 25, 113 1136
- 1096 Gupta, Narain D., 114, 118, 122 1137
- 1097 Gupta-Sidki group, 114, 116, 118 1138
- 1098 is regular branch, 123 1139
- 1099 is torsion, 122 1140
- 1100 Gutierrez, Claudio, 24 1141
- 1101 Hagenah, Christian, 24 1142
- 1102 Hall, Marshall, Jr., 14 1143
- 1103 Handel, Michael, 17 1144
- 1104 Heisenberg group, 114 1145
- 1105 Hermiller, Susan, 108 1146
- 1106 HNN extension, 24 1147
- 1107 Hoogeboom, Hendrik Jan, 23 1148
- Howson, Albert Geoffrey, 11 1149
- Hurwitz, Adolf, 131
- hyperbolic boundary, 14, 112
- group, *see* word-hyperbolic group
- space, 111
- hyperbolic graph, 121
- hyperbolic plane, 104
- isomorphism problem, 3, 113
- isoperimetric inequality, 110, 113
- Julia set, 126
- Kaimanovich, Vadim A. [Кайманович, Вадим Адольфович], 121
- Kambites, Mark, 24
- Kapovich, Ilya [Капович, Илья Эрикович], 4, 12, 24
- Kazhdan group, 131
- Kleene, Stephen C., 17
- Krohn, Kenneth B., 115, 127
- lamplighter group, 118
- language
- Dyck, 22
- rational, 17
- recognizable, 17
- limit space, 126
- Lohrey, Markus, 23–25
- Lyndon, Roger C., 104
- Makanin, Gennady S. [Маканин, Геннадий Семёнович], 25
- mapping
- automatic, 115
- Margolis, Stuart W., 13, 15, 24
- Margulis, Grigory A. [Маргулис, Григорий Александрович], 105
- Markus-Epstein, Luda, 24
- Maslakova, Olga S., 17
- matrix embedding, 116
- Meakin, John C., 13, 24
- Mealy automaton, 115
- bireversible, 117, 128–131
- Cayley automaton, 127
- contracting, 121

- 1150 dual, 117
- 1151 nuclear, 120
- 1152 reset machine, 127
- 1153 reversible, 117, 127
- 1154 membership problem, 2
- 1155 rational subset, 23
- 1156 *Miasnikov, Alexei G.* [Мясников, Алексей
1157 Георгиевич], 4, 12, 24
- 1158 *Milnor, John W.*, 124
- 1159 monoid
- 1160 aperiodic, 13
- 1161 automatic, 108
- 1162 inverse, 13
- 1163 syntactic, 21
- 1164 transition, 13
- 1165 monomorphism extension, 17
- 1166 *Morse, Marston*, 113
- 1167 *Mozes, Shahar*, 131
- 1168 *Muller, David E.*, 22, 103
- 1169 *Muntyan, Yevgen* [Мунтян, Евгений], 129
- 1170 *Nekrashevych, Volodymyr V.* [Некрасевич,
1171 Володимир Володимирович], 120, 121, 123,
1172 122, 126
- 1173 *Nerode, Anil*, 4
- 1174 *Neumann, Hanna*, 11
- 1175 *Nielsen, Jacob*, 8, 9
- 1176 nilpotent group, 15, 114
- 1177 nucleus of a Mealy automaton, 121
- 1178 order problem, 3, 113
- 1179 p -group, 15, 118, 122
- 1180 *Parry, Walter R.*, 126
- 1181 *Plandowski, Wojciech*, 25
- 1182 *Post, Emil*, 24
- 1183 prefix metric, 13
- 1184 problem
- 1185 Post correspondence, 24
- 1186 problem, decision, 2–3, 103
- 1187 conjugacy, 2
- 1188 isomorphism, 3
- 1189 membership, 2
- 1190 —, *see* membership problem
- 1191 order, 3, *see* order problem
- 1192 word, 2, *see* word problem
- 1193 generalized, 2
- 1194 —, *see* generalized word problem
- 1195 property
- 1196 fellow traveller, 106
- 1197 geometric, 105
- 1198 (T), 131
- 1199 [p]-pure subgroup, 13
- 1200 quasi-geodesic, 112
- 1201 quasi-isometry, 23, 105
- 1202 *Rabin, Michael O.*, 24
- 1203 random walk, 121
- 1204 self-similar, 121
- 1205 rational constraint, 24
- 1206 rational cross-section, 107
- 1207 rational language, 17
- 1208 recognizable language, 17
- 1209 residually finite group, 15
- 1210 reversible Mealy automaton, 117, 127
- 1211 rewriting system
- 1212 confluent, 3
- 1213 length-reducing etc., 20
- 1214 *Rhodes, John L.*, 15, 115, 127
- 1215 *Ribes, Luis*, 15
- 1216 *Roig, Abdó*, 12
- 1217 *Roman'kov, Vitalii A.* [Романьков, Виталий
1218 Анатольевич], 24
- 1219 *Sakarovitch, Jacques*, 21
- 1220 *Sapir, Mark* [Сапир, Марк Валентинович],
1221 15
- 1222 *Savchuk, Dmytro* [Савчук, Дмитро], 129
- 1223 *Schreier, Oscar*, 4, 9
- 1224 Schreier graph, 10, 21, 103, 121
- 1225 *Schupp, Paul E.*, 22, 103
- 1226 *Seifert, Franz D.*, 17
- 1227 self-similarity biset, 125
- 1228 semigroup
- 1229 automatic, 108
- 1230 of an automaton, 115
- 1231 inverse, 24
- 1232 *Sénizergues, Géraud*, 20, 22, 24
- 1233 *Serre, Jean-Pierre*, 4
- 1234 *Short, Hamish B.*, 111, 112
- 1235 *Sidki, Said N.*, 114, 117, 118, 120, 122, 129
- 1236 *Silva, Pedro V.*, 13, 24, 127

- 1237 small cancellation, 104, 109
 1238 spanning tree, 8
 1239 Stallings' construction, 9
 1240 amalgamated free products etc., 24
 1241 graph groups, 24
 1242 Stallings, John R., 4, 128
 1243 Stallings automaton, 6
 1244 Stallings construction, 5
 1245 Steinberg, Benjamin, 23, 24, 127
 1246 subgroup
 1247 finitely generated, 2
 1248 fixed point, 16
 1249 index of a , 2
 1250 intersection, 11
 1251 normal, 10
 1252 [p -]pure, 13
 1253 quasi-convex, 111
 1254 surface group, 103
 1255 growth series, 105
 1256 Sushchansky, Vitalij I. [Сушчанський, Віталій Іванович], 119
 1257 syntactic monoid, 21
 1259 Takahasi, Mutuo, 12
 1260 Tartakovskij, Vladimir A. [Тартаковский, Владимир Абрамович], 104
 1262 Thurston, William P., 104, 105, 109, 126
 1263 Tits, Jacques, 124
 1264 topology
 1265 pro- V , 15
 1266 Touikan, Nicholas W. M., 7
 1267 transformation
 1268 automatic, 115
 1269 transition monoid, 13
 1270 Turner, Edward C., 16
 1271 universal cover, 102
 1272 Ventura Capell, Enric, 12, 17
 1273 VH-group, 131
 1274 Virág, Bálint, 121, 122
 1275 virtually free group, 22, 23, 25
 1276 Vorobets, Mariya and Yaroslav [Воробець, Марія і Ярослав], 130
 1277
 1278 weakly branch group, 120, 122–123
 1279 satisfies no identity, 123
 1280 Weidmann, Richard, 24
 1281 Weil, Pascal, 12, 13, 15
 1282 Whitehead, John H. C., 12, 15
 1283 Wilson, John S., 124
 1284 Wise, Daniel T., 131
 1285 Wolf, Joseph A., 124
 1286 word
 1287 cyclically reduced, 3
 1288 reduced, 3
 1289 word-hyperbolic group, 25, 111–114
 1290 is biautomatic, 112
 1291 linear isoperimetric inequality, 113
 1292 word metric, 104
 1293 word problem, 2, 104, 110, 113
 1294 in automata groups, 120
 1295 submonoid, 22
 1296 context-free, 23
 1297 rational, 22
 1298 wreath product, 116, 118
 1299 Zalesskij, Pavel A. [Залесский, Павел Александрович], 15
 1300