Groups defined by automata

Laurent Bartholdi[1]
Pedro V. Silva[2*]

[1] Mathematisches Institut
Georg-August Universität zu Göttingen
Bunsenstraße 3–5
D-37073 Göttingen, Germany
email: laurent.bartholdi@gmail.com

[2] Centro de Matemática, Faculdade de Ciências
Universidade do Porto
R. Campo Alegre 687
4169-007 Porto, Portugal
email: pvsilva@fc.up.pt

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Contents

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Finite automata have been used effectively in recent years to define infinite groups. The two main lines of research have as their most representative objects the class of automatic groups (including "word-hyperbolic groups" as a particular case) and automata groups (singled out among the more general "self-similar groups").

The first approach is studied in §1 and implements in the language of automata some tight constraints on the geometry of the group's Cayley graph. Automata are used to define a normal form for group elements, and to execute the fundamental group operations.

The second approach is developed in §2 and focuses on groups acting in a finitely constrained manner on a regular rooted tree. The automata define sequential permutations of the tree, and can even represent the group elements themselves.

1 The geometry of the Cayley graph

Since its inception at the beginning of the 19th century, group theory has been recognized as a powerful language to capture *symmetries* of mathematical objects: crystals in the early 19th century, for Hessel and Frankenheim [53, page 120]; roots of a polynomial, for Galois and Abel; solutions of a differential equation, for Lie, Painlevé, etc. It is only later, mainly through the work of Klein and Poincaré, that the tight connections between group theory and geometry were brought to light.

Topology and group theory are related as follows. Consider a space X, on which a group G acts *freely*: for every $g \neq \mathbb{1} \in G$ and $x \in X$, we have $x \cdot g \neq x$. If the quotient space $Z = X/G$ is compact, then G "looks very much like" X, in the following sense: choose any $x \in X$, and consider the orbit $x \cdot G$. This identifies G with a roughly evenly distributed subset of X.

Conversely, consider a "nice" compact space Z with *fundamental group* G: then $X = \widetilde{Z}$, the *universal cover* of Z, admits a free G-action. In conclusion, properties of the fundamental group of a compact space Z reflect geometric properties of the space's universal cover.

We recall that finitely generated groups were defined in §23.1: they are groups G admitting a surjective map $\pi : F_A \twoheadrightarrow G$, where F_A is the free group on a finite set A.

Definition 1.1. A group G is *finitely presented* if it is finitely generated, say by $\pi : F_A \twoheadrightarrow G$, and if there exists a finite subset $\mathscr{R} \subset F_A$ such the kernel $\ker(\pi)$ is generated by the F_A-conjugates of $\mathscr{R}$, that is, $\ker(\pi) = \langle\langle \mathscr{R} \rangle\rangle$; one then has $G = F_A/\langle\langle \mathscr{R} \rangle\rangle$. These $r \in \mathscr{R}$ are called *relators* of the presentation; and one writes

$$G = \langle A \mid \mathscr{R} \rangle.$$

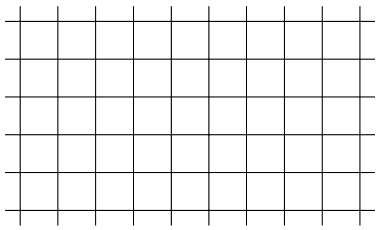
Sometimes it is convenient to write a relator in the form '$a = b$' rather than the more exact form 'ab^{-1}'.

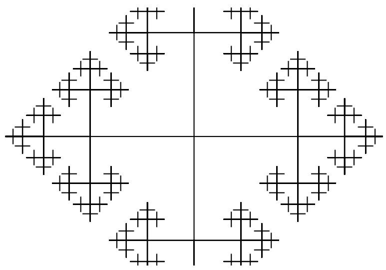
Let G be a finitely generated group, with generating set A. Its *Cayley graph* $\mathscr{C}(G, A)$, introduced by Cayley in [44], is the graph with vertex set G and edge set $G \times A$; the edge (g, s) starts at vertex g and ends at vertex gs.

In particular, the group G acts freely on $\mathscr{C}(G, A)$ by left translation; the quotient $\mathscr{C}(G, A)/G$ is a graph with one vertex and $\#A$ loops.

Assume moreover that G is finitely presented, with relator set $\mathscr{R}$. For each $r = r_1 \dots r_n \in \mathscr{R}$ and each $g \in G$, the word r traces a closed path in $\mathscr{C}(G, A)$, starting at g and passing successively through $gr_1, gr_1r_2, \dots, gr_1r_2 \dots r_n = g$. If one "glues" for each such r, g a closed disk to $\mathscr{C}(G, A)$ by identifying the disks' boundary with that path, one obtains a 2-dimensional cell complex in which each loop is contractible — this is a direct translation of the fact that the normal closure of $\mathscr{R}$ is the kernel of the presentation homomorphism $F_A \to G$.

For example, consider $G = \mathbb{Z}^2$, with generating set $A = \{(0, 1), (1, 0)\}$. Its Cayley graph is the standard square grid. The Cayley graph of a free group F_A, generated by A, is a tree.





More generally, consider a right G-set X, for instance the coset space $H\backslash G$. The *Schreier graph* $\mathscr{C}(G, X, A)$ of X is then the graph with vertex set X and edge set $X \times A$; the edge (x, s) starts in x and ends in xs.

1.1 History of geometric group theory

In a remarkable series of papers, Dehn [48–50], see also [51] initiated the geometric study of infinite groups, by trying to relate algorithmic questions on a group G and geometric questions on its Cayley graph. These problems were described in Definition 23.1.1, to which we refer. For instance, the word problem asks if one can determine whether a path in the Cayley graph of G is closed, knowing only the path's labels.

It is striking that Dehn used, for Cayley graph, the German *Gruppenbild*, literally "group picture". We must solve the word problem in a group G to be able to draw bounded portions of its Cayley graph; and some algebraic properties of G are tightly bound to the algorithmic complexity of the word problem, see §23.3.4. For example, Muller&Schupp prove (see Theorem 23.3.9) that a push-down automaton recognizes precisely the trivial elements of G if and only if G admits a free subgroup of finite index.

We consider now a more complicated example. Let $\mathcal{S}_g$ be an oriented surface of genus $g \geqslant 2$, and let J_g denote its fundamental group. Recall that $[x, y]$ denotes in a group the commutator $x^{-1}y^{-1}xy$. We have a presentation

$$J_g = \langle a_1, b_1, \dots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \rangle. \tag{1.1}$$

Let $r = [a_1, b_1] \cdots [a_g, b_g]$ denote the relator, and let $\mathscr{R}^*$ denote the set of cyclic permutations of $r^{\pm 1}$. The word problem in J_g is solvable in polynomial time by the following algorithm: let u be a given word. Freely reduce u by removing all aa^{-1} subwords. Then, if u contains a subword v_1 such that $v_1v_2 \in \mathscr{R}^*$ and v_1 is longer than v_2, replace v_1 by v_2^{-1} in u and repeat. Eventually, u represents $\mathbb{1} \in G$ if and only if it is the empty word.

The validity of this algorithm relies on a lemma by Dehn, that every nontrivial word representing the identity contains more than half of the relator as a subword.

Incidentally, the Cayley graph of J_g is a tiling of the hyperbolic plane by $4g$-gons, with $4g$ meeting at each vertex.

Tartakovsky [122], Greendlinger [67, 68] and Lyndon [98, 99] then devised "small cancellation" conditions on a group presentation that guarantee that Dehn's algorithm will succeed. Briefly said, they require the relators to have small enough overlaps. These conditions are purely combinatorial, and are described in §1.3.

Cannon and Thurston, on the other hand, sought a formalism that would encode the "periodicity of pictures" of a group's Cayley graph. Treating the graph as a metric space with geodesic distance d, already seen in §23.2.4, they make the following definition: the *cone type* of $g \in G$ is

$$C_g = \{h \in G \mid d(\mathbb{1}, gh) = d(\mathbb{1}, g) + d(g, gh)\}; \tag{1.2}$$

the translate gC_g is the set of vertices that may be connected to $\mathbb{1}$ by a geodesic passing through g. Their intuition is that the cone type of a vertex v remembers, for points near v, whether they are closer or further to the origin than v; for example, $\mathbb{Z}^2$ with its standard generators has 9 cone types: the cone type of the origin (the whole plane), those of vertices on the axes (half-planes), and those of other vertices (quadrants).

Thurston's motivation was to get a good, algorithmic understanding of fundamental groups of threefolds. They should be made of nilpotent (or, more generally, solvable) groups on the one hand, and "automatic" groups on the other hand.

Definition 1.2. Let $G = \langle A \rangle$ be a finitely generated group, and recall that $\tilde{A}$ denotes $A \sqcup A^{-1}$. The *word metric* on G is the geodesic distance in G's Cayley graph $\mathscr{C}(G, A)$. It may be defined directly as

$$d(g, h) = \min\{n \mid g = hs_1 \dots s_n \text{ with all } s_i \in \tilde{A}\},$$

and is left-invariant: $d(xg, xh) = d(g, h)$. The *ball of radius* n is the set

$$B_{G,A}(n) = \{g \in G \mid d(\mathbb{1}, g) \leqslant n\}.$$

The *growth function* of G is the function

$$\gamma_{G,A}(n) = \#B_{G,A}(n).$$

The *growth series* of G is the power series

$$\Gamma_{G,A}(z) = \sum_{g \in G} z^{d(\mathbb{1}, g)} = \sum_{n \geqslant 0} \gamma_{G,A}(n) z^n (1 - z).$$

Growth functions are usually compared as follows: $\gamma \precsim \delta$ if there is a constant $C \in \mathbb{N}$ such that $\gamma(n) \leqslant \delta(Cn)$ for all $n \in \mathbb{N}$; and $\gamma \sim \delta$ if $\gamma \precsim \delta \precsim \gamma$. The equivalence class of $\gamma_{G,A}$ is independent of A.

Cannon observed (in an unpublished 1981 manuscript; see also [40]) that, if a group has finitely many cone types, then its growth series satisfies a finite linear system and is therefore a rational function of z. For J_g, for instance, he computes

$$\Gamma_{J_g,A} = \frac{1 + 2z + \cdots + 2z^{2g-1} + z^{2g}}{1 + (2-4g)z + \cdots + (2-4g)z^{2g-1} + z^{2g}}.$$

This notion was formalized by Thurston in 1984 using automata, and is largely the topic of the next section. We will return to growth of groups in §2.5; see however [27] for a good example of growth series of groups computed thanks to a description of the Cayley graph by automata.

Gromov emphasized the relevance to group theory of the following definition, attributed to Margulis:

Definition 1.3 ([83]). A map $f : X \to Y$ between two metric spaces is a *quasi-isometry* if for a constant $C > 0$ one has

$$C^{-1}d(x,y) - C \leqslant d(f(x), f(y)) \leqslant Cd(x,y) + C, \qquad \forall y \in Y : d(f(X), y) \leqslant C.$$

Two spaces are quasi-isometric if there exists a quasi-isometry between them; this is an equivalence relation.

A property of finitely generated groups is *geometric* if it only depends on the quasi-isometry class of its Cayley graph.

Thus for instance the inclusion $\mathbb{Z} \to \mathbb{R}$, and the map $\mathbb{R} \to \mathbb{Z}, x \mapsto \lfloor x \rfloor$ are quasi-isometries.

Being finite, having a finite-index subgroup isomorphic to $\mathbb{Z}$, and being finitely presented are geometric properties. The asymptotics of the growth function is also a geometric invariant; thus for instance having growth function $\precsim n^2$ is a geometric property.

1.2 Automatic groups

Let $G = \langle A \rangle$ be a finitely generated group. We will consider the formal alphabet $\hat{A} = A \sqcup A^{-1} \sqcup \{\mathbb{1}\}$, where $\mathbb{1}$ is treated as a "padding" symbol. Following the main reference [54] by Epstein *et al.*:

Definition 1.4 ([22, 54, 55]). The group G is *automatic* if there are finite-state automata $\mathcal{L}, \mathcal{M}$, the *language* and *multiplication* automata, with the following properties:

(i) $\mathcal{L}$ is an automaton with alphabet $\tilde{A}$;

(ii) $\mathcal{M}$ has alphabet $\hat{A} \times \hat{A}$, and has for each $s \in \hat{A}$ an accepting subset T_s of states; call $\mathcal{M}_s$ the automaton with accepting states T_s;

(iii) the language of $\mathcal{L}$ surjects onto G by the natural map $f : \tilde{A} \to F_A \to G$; words in $L(\mathcal{L})$ are called *normal forms*;

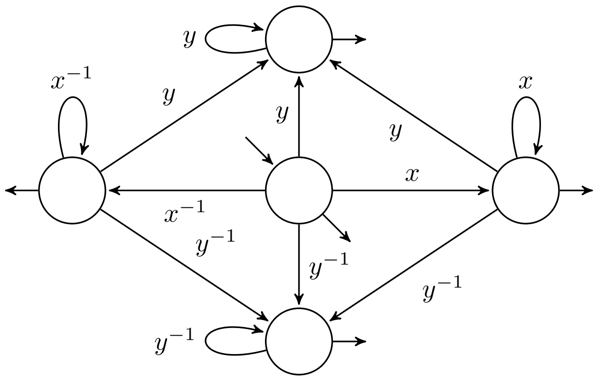
(iv) for any two normal forms $u, v \in L(\mathcal{L})$, consider the word

$$w = (u_1, v_1)(u_2, v_2) \dots (u_n, v_n) \in (\hat{A} \times \hat{A})^*,$$

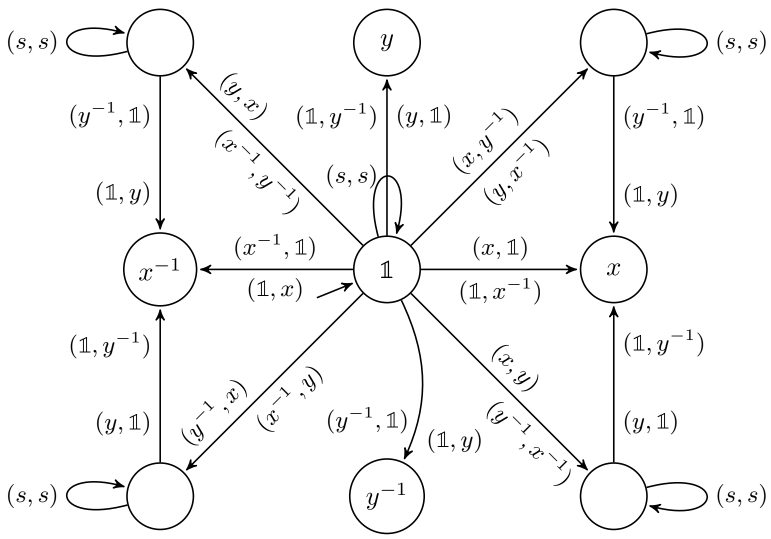
where $n = \max\{|u|, |v|\}$ and $u_i, v_j = \mathbb{1}$ if $i > |u|, j > |v|$. Then $\mathcal{M}_s$ accepts w if and only if $\pi(u) = \pi(vs)$.

In words, G is automatic if the automaton $\mathcal{L}$ singles out sufficiently many words which may be used to represent all group elements; and the automaton $\mathcal{M}_s$ recognizes when two such singled out words represent group elements differing by a generator. The pair $(\mathcal{L}, \mathcal{M})$ is an *automatic structure* for G.

We will give numerous examples of automatic groups in §1.3. Here is a simple one that contains the main features: the group $G = \mathbb{Z}^2$, with standard generators x, y. The language accepted by $\mathcal{L}$ is $(x^* \cup (x^{-1})^*)(y^* \cup (y^{-1})^*)$:



The multiplication automaton, in which states in T_s are labeled s, is



The definition we gave is purely automata-theoretic. It does, however, have a more geometric counterpart. A word $w \in \tilde{A}^*$ represents in a natural way a path in the Cayley graph $\mathscr{C}(G, A)$, starting at $\mathbb{1}$ and ending at $\pi(w)$. If $w = w_1 \dots w_n$, we write $w(j) = w_1 \dots w_j$ the vertex of $\mathscr{C}(G, A)$ reached after j steps; if $j > n$ then $w(j) = w$. For two paths $u, v \in \tilde{A}^*$, we say they *k-fellow-travel* if $d(u(j), v(j)) \leqslant k$ for all $j \in \{1, \dots, \max\{|u|, |v|\}\}$.

Proposition 1.1. *A group G is automatic if and only if there exists a rational language $L \subseteq \tilde{A}^*$, mapping onto G, and a constant k, such that for any $u, v \in L$ with $d(\pi u, \pi v) \leqslant 1$ the paths u, v k-fellow-travel.*

Sketch of proof. Assume first that G has automatic structure $(\mathcal{L}, \mathcal{M})$, and let c denote the number of states of $\mathcal{M}$. If $u, v \in L(\mathcal{L})$ satisfy $\pi(u) = \pi(vs)$, let s_j denote the state $\mathcal{M}$ is in, after having read $(u_1, v_1) \dots (u_j, v_j)$. There is a path of length $< c$, in $\mathcal{M}$, from s_j to an accepting state (labeled s); let its label be (p, q). Then $\pi(u(j)p) = \pi(v(j)qs)$, so $u(j)$ and $v(j)$ are at distance at most $2c - 1$ in $\mathscr{C}(G, A)$.

Conversely, assume that paths k-fellow-travel and that an automaton $\mathcal{L}$, with state set Q is given, with language surjecting onto G. Recall that $B(k)$ denotes the set of group elements at distance $\leqslant k$ from $\mathbb{1}$ in $\mathscr{C}(G, A)$. Consider the automaton with state set $Q \times Q \times B_k$. Its initial state is $(*, *, \mathbb{1})$, where $*$ is the initial state of $\mathcal{L}$; its alphabet is $\hat{A} \times \hat{A}$, and its transitions are given by $(p, q, g) \cdot (s, t) = (p \cdot s, q \cdot t, s^{-1}gt)$ whenever these are defined. Its accepting set of states, for $s \in \hat{A}$, is $T_s = Q \times Q \times \{s\}$. □

Corollary 1.2. *If the finitely generated group $G = \langle A \rangle$ is automatic, and if B is another finite generating set for G, then there also exists an automatic structure for G using the alphabet B.*

Sketch of proof. Note first that a trivial generator may be added or removed from A or B, using an appropriate finite transducer for the latter.

There exists then $M \in \mathbb{N}$ such that every $a \in \tilde{A}$ can be written as a word $w_a \in \tilde{B}^*$ of length precisely M. Accept as normal forms all $w_{a_1} \dots w_{a_n}$ such that $a_1 \dots a_n$ is a normal form in the original automatic structure $\mathcal{L}$. The new normal forms constitute a homomorphic image of $\mathcal{L}$ and therefore define a rational language. If paths in $L(\mathcal{L})$ k-fellow-travel, then their images in the new structure will kM-fellow-travel. □

Note that the language of normal forms is only required to contain "enough" expressions; namely that the evaluation map $L(\mathcal{L}) \to G$ is onto. We may assume that it is bijective, by the following lemma. The language $L(\mathcal{L})$ is then called a "rational cross-section" by Gilman [63]; and $(\mathcal{L}, \mathcal{M})$ is called an *automatic structure with uniqueness*.

Lemma 1.3. *Let G be an automatic group. Then G admits an automatic structure with uniqueness.*

Sketch of proof. Consider $(\mathcal{L}', \mathcal{M})$ an automatic structure. Recall the "short-lex" ordering on words: $u \leqslant v$ if $|u| < |v|$, or if $|u| = |v|$ and u comes lexicographically before v. The language $\{(u, v) \in \hat{A}^* \times \hat{A}^* \mid u \leqslant v\}$ is rational. The language

$$L = L(\mathcal{L}') \cap \{u \in \tilde{A}^* \mid \forall v \in \hat{A}^* : (u, v) \in L(\mathcal{M}_{\mathbb{1}}) \Rightarrow u \leqslant v\}$$

is then also rational, of the form $L(\mathcal{L})$. The automaton $\mathcal{M}$ need not be changed. □

Various notions related to automaticity have emerged, some stronger, some weaker:

- One may require the words accepted by $\mathcal{L}$ to be representatives of minimal length; the automatic structure is then called *geodesic*. It would then follow that the growth series $\Gamma_{G,A}(z)$ of G, which is the growth series of $\mathcal{L}$, is a rational function. Note that there is a constant K such that, for the language produced by Lemma 1.3, all words $u \in L(\mathcal{L})$ satisfy $|u| \leqslant Kd(\mathbb{1}, \pi(u))$.

- The definition is asymmetric; more precisely, we have defined a *right automatic* group, in that the automaton $\mathcal{M}$ recognizes multiplication on the right. One could similarly define *left automatic groups*; then a group is right automatic if and only if it is left automatic.

  Indeed, let $(\mathcal{L}, \mathcal{M})$ be an automatic structure where $\mathcal{L}$ recognizes a rational cross section. Then $L' = \{u^{-1} \mid u \in L(\mathcal{L})\}$ and $M' = \{(u^{-1}, v^{-1}) \mid (u, v) \in L(\mathcal{M})\}$ are again rational languages. Indeed, since rational languages are closed under reversion and morphisms, it follows easily that L' is rational. On the other hand, using the pumping lemma and the fact that group elements admit unique representatives in $L(\mathcal{L})$, the amount of padding at the end of word-pairs in $L(\mathcal{M})$ is bounded, and can be moved from the beginning to the end of the word-pairs in M' by a finite transducer. Therefore, L', M' are the languages of a right automatic structure.

  However, one could require both properties simultaneously, namely, on top of an automatic structure, a third automaton $\mathcal{N}$ accepting (in state $s \in \hat{A}$) all pairs of normal forms (u, v) with $\pi(u) = \pi(sv)$. Such groups are called *biautomatic*. No example is known of a group that is automatic but not biautomatic.
- One might also only keep the geometric notion of "combing": a *combing* on a group is a choice, for every $g \in G$, of a word $w_g \in \tilde{A}^*$ evaluating to g, such that the words w_g and w_{gs} fellow-travel for all $g \in G$, $s \in \tilde{A}$.

  In that sense, a group is automatic if and only if it admits a combing whose words form a rational language; see [30] for details.

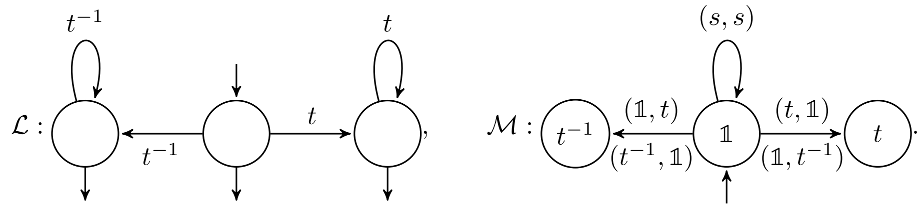
  One may again require the combing lines to be geodesics, i.e., words of minimal length; see Hermiller's work [87–89].

  One may also put weaker constraints on the words of the combing; for example, require it to be an indexed language. Bridson and Gilman proved in [31] that all geometries of threefolds, in particular the Nil (1.3) and Sol geometry that are not automatic, fall in this framework.
- Another relaxation is to allow the automaton $\mathcal{M}$ to read at will letters from the first or the second word; groups admitting such a structure are called *asynchronously automatic*. Among fundamental groups of threefolds, there is no difference between these definitions [31], but for more general groups there is.
- Finally, Definition 1.4 can be adapted to define automatic semigroups. Properties from automatic groups that can be proved within the automata-theoretic framework can often be generalized to automatic semigroups, or at least monoids [39]. However, establishing an alternative geometric approach has proved to be a tough task and success was reached only in restricted cases [90, 119].

1.3 Main examples of automatic groups

From the very definition, it is clear that finite groups are automatic: one chooses a word representing each group element, and these necessarily form a fellow-travelling rational language.

It is also clear that $\mathbb{Z}$ is automatic: write t for the canonical generator of $\mathbb{Z}$; the language $t^* \cup (t^{-1})^*$ maps bijectively to $\mathbb{Z}$; and the corresponding paths 1-fellow-travel. The automata are



Simple constructions show that the direct and free products of automatic groups are again automatic. Finite extensions and finite-index subgroups of automatic groups are automatic. It is however still an open problem whether a direct factor of an automatic group is automatic.

Recall that we glued disks, one for each $g \in G$ and each $r \in \mathscr{R}$, to the Cayley graph of a finitely presented group $G = \langle A \mid \mathscr{R} \rangle$, so as to obtain a 2-complex $\mathscr{K}$. The *small cancellation conditions* express a combinatorial form of non-positive curvature of $\mathscr{K}$: roughly, $C(p)$ means that every proper edge cycle in $\mathscr{K}$ has length $\geqslant p$, and $T(q)$ means that every proper edge cycle in the dual $\mathscr{K}^\vee$ has length $\geqslant q$; see [98, Chapter V] for details. If G satisfies $C(p)$ and $T(q)$ where $p^{-1} + q^{-1} \leqslant \frac{1}{2}$, then G is automatic.

Consider the configurations defined by n strings in $\mathbb{R}^2 \times [0,1]$, with string $\#i$ starting at $(i,0,0)$ and ending at $(i,0,1)$; these configurations are viewed up to isotopy preserving the endpoints. They can be multiplied (by stacking them above each other) and inverted (by flipping them up-down), yielding a group, the *pure braid group*; if the strings are allowed to end in an arbitrary permutation, one obtains the *braid group*. This group B_n is generated by elementary half-twists of strings $\#i, i+1$ around each other, and admits the presentation

$$B_n = \langle \sigma_1, \dots, \sigma_{n-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, [\sigma_i, \sigma_j] \text{ whenever } |i-j| \geqslant 2 \rangle.$$

More generally, consider a surface $\mathcal{S}$ of genus g, with n punctures and b boundary components. The *mapping class group* $M_{g,n,b}$ is the group of maps $\mathcal{S} \to \mathcal{S}$ modulo isotopy, and B_n is the special case $M_{0,n,1}$ of mapping classes of the n-punctured disk. All mapping class groups $M_{g,n,b}$ are automatic groups [107].

As another generalization of braid groups, consider *Artin groups*. Let (m_{ij}) be a symmetric $n \times n$-matrix with entries in $\mathbb{N} \cup \{\infty\}$. The *Artin group* of type (m_{ij}) is the group with presentation

$$A(m) = \langle s_1, \dots, s_n \mid (s_is_j)^{\lfloor m_{ij}/2 \rfloor} = (s_js_i)^{\lfloor m_{ij}/2 \rfloor} \text{ whenever } m_{ij} < \infty \rangle.$$

The corresponding *Coxeter group* has presentation

$$C(m) = \langle s_1, \dots, s_n \mid s_i^2, (s_is_j)^{m_{ij}/2} = (s_js_i)^{m_{ij}/2} \text{ whenever } m_{ij} < \infty \rangle.$$

An Artin group $A(m)$ has *finite type* if $C(m)$ is finite. Artin groups of finite type are biautomatic [45]. Coxeter groups are automatic [33].

Fundamental groups of threefolds, except those with a piece modelled on Nil or Sol geometry [54, chapter 12], are automatic.

1.4 Properties of automatic groups

The definition of automatic groups, by automata, has a variety of interesting consequences. First, automatic groups are finitely presented; more generally, combable groups are finitely presented:

Proposition 1.4 ([2]). *Let G be a combable group. Then G has type F_∞, namely, there exists a contractible cellular complex with free G-action and finitely many G-orbits of cells in each dimension.*

(Finite presentation is equivalent to "finitely many G-orbits of cells in dimension $\leqslant 2$").

Sketch of proof. By assumption, G is finitely generated. Therefore, the Cayley graph contains one G-orbit of 0-cells (vertices), and $\#A$ orbits of 1-cells (edges). Consider all pairs of paths u, v in the combing that have neigbouring extremities. They k-fellow-travel by hypothesis; so there are for all j paths $w(j)$ of length $\leqslant k$ connecting $u(j)$ to $v(j)$. The closed paths $u(j) - v(j) - v(j+1) - u(j+1) - u(j)$ have length $\leqslant 2k+2$, so trace finitely many words in F_A. Taking them as relators defines a finite presentation for G. The process may be continued with higher-dimensional cells. □

Proposition 1.5. *Automatic groups satisfy a* quadratic isoperimetric inequality*; that is, for any finite presentation $G = \langle A \mid \mathscr{R}\rangle$ there is a constant k such that, if $w \in F_A$ is a word evaluating to $\mathbb{1}$ in G, then*

$$w = \prod_{i=1}^{\ell} r_i^{w_i} \text{ for some } r_i \in \mathscr{R}^{\pm 1}, w_i \in F_A \text{ and } \ell \leqslant k|w|^2.$$

Sketch of proof. Write $n = |w|$, and draw the combing lines between $\mathbb{1}$ and $w(j)$. There are n combing lines, which have length $\mathcal{O}(n)$; so the gap between neighbouring combing lines can be filled by $\mathcal{O}(n)$ relators. This gives $\mathcal{O}(n^2)$ relators in total. □

Note that being finitely presented is usually of little value as far as algorithmic questions are concerned: there are finitely presented groups whose word problem cannot be solved by a Turing machine [25, 110]. In contrast:

Proposition 1.6. *The word problem in a group given by an automatic structure is solvable in quadratic time. A word may even be put into canonical form in quadratic time.*

Sketch of proof. We may assume, by Lemma 1.3, that every $g \in G$ admits a unique normal form. Now, given a word $u = a_1 \dots a_n \in \hat{A}^*$, construct the following words: $w_0 \in L(\mathcal{L})$ is the representative of $\mathbb{1}$. Treating M_a as a non-deterministic automaton in its second variable, find for $i = 1, \dots, n$ a word $w_i \in \hat{A}^*$ such that the padding of (w_{i-1}, w_i) is accepted by $\mathcal{M}_{a_i}$. Then $\pi(u) = \mathbb{1} \in G$ if and only if $w_n = w_0$.

Clearly the w_i have linear length in i, so the total running time is quadratic in n. □

In general, finitely generated subgroups and quotients of automatic groups need not be automatic — they need not even be finitely presented. A subgroup H of a finitely generated group $G = \langle A \rangle$ is *quasi-convex* if there exists a constant δ such that every $h \in H$ is connected to $\mathbb{1} \in G$ by a geodesic in $\mathscr{C}(G, A)$ that remains at distance $\leqslant \delta$ from H. Typical examples are finite-index subgroups, free factors, and direct factors.

On the other hand, a subgroup H of an automatic group G with language $L(\mathcal{L})$ is *$\mathcal{L}$-rational* if the full preimage of H in $L(\mathcal{L})$ is rational. The following is easy but fundamental:

Lemma 1.7 ([60]). *A subgroup H of an automatic group is quasi-convex if and only if it is $\mathcal{L}$-rational.*

It is still unknown whether automatic groups have solvable conjugacy problem; however, there are asynchronously automatic groups with unsolvable conjugacy problem, for instance appropriate amalgamated products of two free groups over finitely generated subgroups. These groups are asynchronously automatic by [22, Theorem E], and have unsolvable conjugacy problem by [102].

Theorem 1.8 (Gersten-Short). *Biautomatic groups have solvable conjugacy problem.*

Sketch of proof; see [59]. Consider two words $x, y \in \tilde{A}^*$. Using the biautomatic structure, the language

$$C(x, y) = \{(u, v) \in \hat{A}^* \times \hat{A}^* \mid u, v \in \mathcal{L} \text{ and } \pi(u) = \pi(xvy)\}$$

is rational. Now x, y are conjugate if and only if $C(x^{-1}, y) \cap \{(w, w) \mid w \in \mathcal{L}\}$ is nonempty. The problem of deciding whether a rational language is empty is algorithmically solvable. □

In fact, the centralizer of an element of a biautomatic group is a quasi-convex subgroup, and is thus biautomatic [60] (but we remark that it is still unknown whether a quasi-convex subgroup of an automatic group is necessarily automatic). There is therefore a good algorithmic description of *all* elements that conjugate x to y.

1.5 Word-hyperbolic groups

Gromov introduced in [80] the fundamental concept of "negative curvature" to group theory. This goes further in the direction of viewing groups as metric spaces, through the geodesic distance on their Cayley graph. The definition is given for *geodesic* metric spaces, i.e., metric spaces in which any two points can be joined by a geodesic segment:

Definition 1.5 ([3,46,61]). Let X be a geodesic metric space, and let $\delta > 0$ be given. The space X is *δ-hyperbolic* if, for any three points $A, B, C \in X$ and geodesics arcs a, b, c joining them, every $P \in a$ is at distance at most δ from $b \cup c$.

The space X is *hyperbolic* if it is δ-hyperbolic for some δ. The finitely generated group $G = \langle A \rangle$ is *word-hyperbolic* if it acts by isometries on a hyperbolic metric space X with discrete orbits, finite point stabilizers, and compact quotient X/G.

Equivalently, G is word-hyperbolic if and only if $\mathscr{C}(G, A)$ is hyperbolic.

Gilman gives in [62] a purely automata-theoretic definition of word-hyperbolic groups: G is word-hyperbolic if and only if, for some regular combing $\mathcal{M} \subset \tilde{A}^*$, the language $\{u\mathbb{1}v\mathbb{1}w \mid u, v, w \in \mathcal{M}, \pi(uvw) = \mathbb{1}\} \subset \hat{A}^*$ is context-free. Using the geometric definition, we note immediately the following examples: first, the hyperbolic plane $\mathbb{H}^2$ is hyperbolic (with $\delta = \log 3$); so is $\mathbb{H}^n$. Any discrete, cocompact group of isometries of $\mathbb{H}^n$ is word-hyperbolic. This applies in particular to the surface group J_g from (1.1), if $g \geqslant 2$. Note however that some word-hyperbolic groups are not small cancellation groups, for instance because for small cancellation groups the complex in Proposition 1.4 has trivial homology in dimension $\geqslant 3$; yet the complex associated with a cocompact group acting on $\mathbb{H}^n$ has infinite cyclic homology in degree n (see [57] for applications of topology to group theory).

It is also possible to define δ-hyperbolicity for spaces X that are not geodesic (as, e.g., a group):

Definition 1.6. Let X be a metric space, and let $\delta' > 0$ be given. The space X is δ'-*hyperbolic* if, for any four points $A, B, C, D \in X$, the numbers

$$\{d(A,B) + d(C,D), d(A,C) + d(B,D), d(A,D) + d(B,C)\}$$

are such that the largest two differ by at most δ'.

Word-hyperbolic groups arise naturally in geometry, in the following way: let $\mathcal{M}$ be a compact Riemannian manifold with negative (not necessarily constant) sectional curvature. Then $\pi_1(\mathcal{M})$ is a word-hyperbolic group.

Word-hyperbolic groups are also "generic" among finitely-presented groups, in the following sense: fix a number k of generators, and a constant $\epsilon \in [0, 1]$. For large N, there are $\approx (2k-1)^N$ words of length $\leqslant N$ in F_k; choose a subset $\mathscr{R}$ of size $\approx (2k-1)^{\epsilon N}$ of them uniformly at random, and consider the group G with presentation $\langle A \mid \mathscr{R}\rangle$.

Then, with probability $\to 1$ as $N \to \infty$, the group G is word-hyperbolic. Furthermore, if $\epsilon < \frac{1}{2}$, then with probability $\to 1$ the group G is infinite, while if $\epsilon > \frac{1}{2}$, then with probability $\to 1$ the group G is trivial [111].

Word-hyperbolic groups provide us with a large number of examples of automatic groups. Better:

Theorem 1.9 (Gersten-Short, Gromov). *Let G be a word-hyperbolic group. Then G is biautomatic. Moreover, the normal form $\mathcal{L}$ may be chosen to consist of geodesics.*

Even better, the automatic structure is, in some precise sense, unique [28].

Sketch of proof. In a δ-word-hyperbolic group G, geodesics $(2\delta + 1)$-fellow-travel. On the other hand, G has a finite number of cone types (1.2), so the language of geodesics is rational, recognized by an automaton with as many states as there are cone types. □

Hyperbolic spaces X have a natural *hyperbolic boundary* ∂X: fix a point $x_0 \in X$, and consider *quasi-geodesics at* x_0, namely quasi-isometric embeddings $\gamma : \mathbb{N} \to X$ starting at x_0. Declare two such quasi-geodesics γ, δ to be equivalent if $d(\gamma(n), \delta(n))$ is bounded. The set of equivalence classes, with its natural topology, is the boundary ∂X of X. The fundamental tool in studying hyperbolic spaces is the following

Lemma 1.10 (Morse). *Let X be a hyperbolic space. Then all quasi-geodesics (for given C) between two given points $x, y \in X$ are at bounded distance from one another.*

The hyperbolic boundary ∂X is compact, under appropriate conditions satisfied e.g. by $X = \mathscr{C}(G, A)$, and $X \cup \partial X$ is a compactification of X. Now, in that case, the automaton $\mathcal{L}$ provides a symbolic coding of ∂X as a finitely presented shift space (where the shift action is the "geodesic flow", following one step along infinite paths $\in \hat{A}^\infty$ representing quasi-geodesics).

We note that, for word-hyperbolic groups, the word and conjugacy problem admit extremely efficient solutions: they are both solvable in linear time by a Turing machine. The word problem is actually solvable in real time, namely with a bounded amount of calculation allowed between inputs [92]. The isomorphism problem is decidable for word-hyperbolic groups, say given by a finite presentation [47] . Word-hyperbolic groups also satisfy a linear isoperimetric inequality, in the sense that every $w \in F_A$ that evaluates to $\mathbb{1}$ in G is a product of $\mathcal{O}(|w|)$ conjugates of relators. Better:

Proposition 1.11. *A finitely presented group is word-hyperbolic if and only if it satisfies a linear isoperimetric inequality, if and only if it satisfies a subquadratic isoperimetric inequality.*

Note that the generalized word problem is known to be unsolvable [113], but the order problem is on the other hand solvable in word-hyperbolic groups [26]. It follows that the generalized word problem is unsolvable for automatic groups as well.

There are important weakenings of the definition of word-hyperbolic groups; we mention two. A *bicombing* is a choice, for every pair of vertices $g, h \in \mathscr{C}(G, A)$, of a path $\ell_{g,h}$ from g to h. Since G acts by left-translation on $\mathscr{C}(G, A)$, it also acts on bicombings. A bicombing satisfies the *k-fellow-traveller property* if for any neighbours x' of x and y' of y, the paths $\ell_{x,y}$ and $\ell_{x',y'}$ k-fellow-travel.

A *semi-hyperbolic group* is a group admitting an invariant bicombing by fellow-travelling words. See [32], or the older paper [4]. In particular, biautomatic, and therefore word-hyperbolic, groups are semi-hyperbolic.

Semi-hyperbolic groups are finitely presented and have solvable word and conjugacy problems. In fact, they even have the "monotone conjugation property", namely, if g and h are conjugate, then there exists a word w with $g^{\pi(w)} = h$ and $|g^{\pi(w(i))}| \leqslant \max\{|g|, |h|\}$ for all $i \in \{0, \dots, |w|\}$.

A group G is *relatively hyperbolic* [56] if it acts properly discontinuously on a hyperbolic space X, preserving a family $\mathcal{H}$ of separated horoballs, such that $(X \setminus \mathcal{H})/G$ is compact. All fundamental groups of finite-volume negatively curved manifolds are relatively hyperbolic.

A non-closed manifold has "cusps", going off to infinity, whose interpretation in the fundamental group are conjugacy classes of loops that may be homotoped arbitrarily far into the cusp. Farb [56] captures combinatorially the notion of relative hyperbolicity as follows: let $\mathscr{H}$ be a family of subgroups of a finitely generated group $G = \langle A \rangle$. Modify the Cayley graph of G as follows: for each coset gH of a subgroup $H \in \mathscr{H}$, add a vertex gH, and connect it by an edge to every $gh \in \mathscr{C}(G, A)$, for all $h \in H$. In addition, require that every edge in $\widehat{\mathscr{C}(G, A)}$ belong to only finitely many simple loops of any given length.

The group G is *weakly relatively hyperbolic*, relative to the family $\mathscr{H}$, if that modified Cayley graph $\widehat{\mathscr{C}(G,A)}$ is a hyperbolic metric space.

By virtue of its geometric characterization, being word-hyperbolic is a geometric property in the sense of Definition 1.3 (though beware that being hyperbolic is preserved under quasi-isometry only if the metric spaces are geodesic). Being combable and being bicombable are also geometric.

We finally remark that a notion of word-hyperbolicity has been defined for semigroups [52,91]; the definition uses context-free languages. As for automatic (semi)groups, the theory does not seem uniform enough to warrant a simultaneous treatment of groups and semigroups; again, there is no clear geometric counterpart to the definition of word-hyperbolic semigroups — except in particular cases, such as monoids defined through special confluent rewriting systems [43].

1.6 Non-automatic groups

All known examples of non-automatic groups arise as groups violating some interesting consequence of automaticity.

First, infinitely presented groups cannot be automatic. There are uncountably many finitely generated groups, and only countably many finitely presented groups; therefore automatic groups should be thought of as the rationals among the real numbers.

Groups with unsolvable word problem cannot be automatic.

If a nilpotent group is automatic, then it contains an abelian subgroup of finite index [64]); therefore, for instance, the discrete Heisenberg group

$$G=\begin{pmatrix}1&\mathbb{Z}&\mathbb{Z}\\0&1&\mathbb{Z}\\0&0&1\end{pmatrix}=\langle x,y\mid [x,[x,y]],[y,[x,y]]\rangle \tag{1.3}$$

is not automatic. Note also that G satisfies a cubic, but no quadratic, isoperimetric inequality.

Many solvable groups have larger-than-quadratic isoperimetric functions; they therefore cannot be automatic [84]. This applies in particular to the Baumslag-Solitar groups

$$BS_{1,n}=\langle a,t\mid a^n=a^t\rangle. \tag{1.4}$$

Similarly, $\mathrm{SL}_n(\mathbb{Z})$, for $n\geqslant 3$, or $\mathrm{SL}_n(\mathscr{O})$ for $n\geqslant 2$, where $\mathscr{O}$ are the integers in an imaginary number field, are not automatic.

Infinite, finitely generated torsion groups cannot be automatic: they cannot admit a rational normal form, because of the pumping lemma. We will see examples, due to Grigorchuk and Gupta-Sidki, in §2.1.

There are combable groups that are not automatic [29], for instance

$$G=\langle a_i,b_i,t_i,s\mid t_1a_1=t_2a_2,[a_i,s]=[a_i,t_i]=[b_i,s]=[b_i,t_i]=\mathbb{1}\quad(i=1,2)\rangle,$$

which satisfies only a cubic isoperimetric inequality. Finitely presented subgroups of automatic groups need not be automatic [23].

The following group is asynchronously automatic, but is not automatic: it does not satisfy a quadratic isoperimetric inequality [22, §11]:

$$G = \langle a, b, t, u \mid a^t = ab, b^t = a, a^u = ab, b^u = a \rangle.$$

2 Groups generated by automata

We now turn to another important class of groups related to finite-state automata. These groups act by permutations on a set A^* of words, and these permutations are represented by *Mealy automata*. These are deterministic, initial finite-state transducers $\mathcal{M}$ with input and output alphabet A, that are complete with respect to input; in other words,

$$\text{At every state and for each } a \in A, \text{ there is a unique outgoing edge with input } a. \tag{2.1}$$

The automaton $\mathcal{M}$ defines a transformation of A^*, which extends to a transformation of A^ω, as follows. Given $w = a_1 a_2 \cdots \in A^* \cup A^\omega$, there is by (2.1) a unique path in $\mathcal{M}$ starting at the initial state and with input labels w. The image of w under the transformation is the output label along that same path.

Definition 2.1. A map $f : A^* \to A^*$ is *automatic* if f is produced by a finite-state automaton as above.

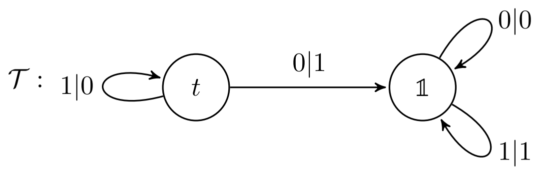
One may forget the initial state of $\mathcal{M}$, and consider the set of all transformations corresponding to all choices of initial state of $\mathcal{M}$; the *semigroup of the automaton* $S(\mathcal{M})$ is the semigroup generated by all these transformations. It is closely connected to Krohn-Rhodes Theory [96]. Its relevance to group theory was seen during Gluškov's seminar on automata [65].

The automaton $\mathcal{M}$ is *invertible* if furthermore it is complete with respect to output; namely,

$$\text{At every state and for each } a \in A, \text{ there is a unique outgoing edge with output } a; \tag{2.2}$$

the corresponding transformation of $A^* \cup A^\omega$ is then invertible; and the set of such permutations, for all choices of initial state, generate the *group of the automaton* $G(\mathcal{M})$. Note that $S(\mathcal{M})$ may be a proper subsemigroup of $G(\mathcal{M})$, even if $\mathcal{M}$ is *invertible*. General references on groups generated by automata are [14, 76, 108].

As our first, fundamental example, consider the automaton with alphabet $A = \{0, 1\}$



(2.3)

in which the input i and output o of an edge are represented as '$i|o$'. The transformation associated with state $\mathbb{1}$ is clearly the identity transformation, since any path starting from $\mathbb{1}$ is a loop with same input and output. Consider now the transformation t. One has, for instance, $t \cdot 111001 = 000101$, with the path consisting of three loops at t, the edge to $\mathbb{1}$, and two loops at $\mathbb{1}$. In particular, $G(\mathcal{T}) = \langle t \rangle$. We will see in §2.7 that it is infinite cyclic.

Lemma 2.1. *The product of two automatic transformations is automatic. The inverse of an invertible automatic transformation is automatic.*

The proof becomes transparent once we introduce a good notation. If in an automaton $\mathcal{M}$ there is a transition from state q to state r, with input i and output o, we write

$$q \cdot i = o \cdot r. \tag{2.4}$$

In effect, if the state set of $\mathcal{M}$ is Q, we are encoding $\mathcal{M}$ by a function $\tau : Q \times A \to A \times Q$. It then follows from (2.1) that, given $q \in Q$ and $v = a_1 \dots a_n \in A^*$, there are unique $w = b_1 \dots b_n \in A^*, r \in Q$ such that $q \cdot a_1 \dots a_n = b_1 \dots b_n \cdot r$. The image of v under the transformation q is w. We have in fact extended naturally the function τ to a function $\tau : Q \times A^* \to A^* \times Q$.

Proof of Lemma 2.1. Given $\mathcal{M}, \mathcal{N}$ initial automata with state sets Q, R respectively, consider the automaton $\mathcal{M}\mathcal{N}$ with state set $Q \times R$ and transitions defined by $(q, r) \cdot i = q \cdot (r \cdot i) = o \cdot (q', r')$. If q_0, r_0 be the initial states of $\mathcal{M}, \mathcal{N}$, then the transformation $q_0 \circ r_0$ is the transformation corresponding to state (q_0, r_0) in $\mathcal{M}\mathcal{N}$.

Similarly, if q_0 induces an invertible transformation, consider the automaton $\mathcal{M}^{-1}$ with state set $\{q^{-1} \mid q \in Q\}$ and transitions defined by $q^{-1} \cdot o = i \cdot r^{-1}$ whenever (2.4) holds. The transformation induced by q_0^{-1} is the inverse of q_0. □

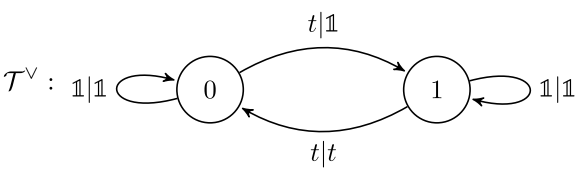
This construction applies naturally to any composition of finitely many automatic transformations. In case they all arise from the same machine $\mathcal{M}$, we *de facto* extend the function τ describing $\mathcal{M}$ to a function $\tau : Q^* \times A^* \to A^* \times Q^*$, and (if $\mathcal{M}$ is invertible) to a function $\tau : F_Q \times A^* \to A^* \times F_Q$. It projects to function $\tau : S(\mathcal{M}) \times A^* \to A^* \times S(\mathcal{M})$, and, if $\mathcal{M}$ is invertible, to a function $\tau : G(\mathcal{M}) \times A^* \to A^* \times G(\mathcal{M})$.

Note that a function $G(\mathcal{M}) \times A \to A \times G(\mathcal{M})$ naturally gives a function, still written $\tau : G(\mathcal{M}) \to G(\mathcal{M})^A \rtimes \mathrm{Sym}(A)$; this is the semidirect product of functions $A \to G(\mathcal{M})$ by the symmetric group of A (acting by permutation of coördinates), and is commonly called the *wreath product* $G(\mathcal{M}) \wr \mathrm{Sym}(A)$, see also Chapter 16.

This wreath product decomposition inspires also a convenient description of the function τ by a *matrix embedding*; the size and shape of the matrix is determined by the permutation of A, and the nonzero entries by the elements in $G(\mathcal{M})^A$; more precisely, assume $A = \{1, \dots, d\}$, and, for $\tau(q) = ((s_1, \dots, s_d), \pi) \in G(\mathcal{M})^A \rtimes \mathrm{Sym}(A)$, write $\tau'(q) =$ the permutation matrix with s_i at position $(i, i\pi)$. Then these matrices multiply as wreath product elements. More algebraically, we have defined a homomomorphism $\tau' : \Bbbk G \to M_d(\Bbbk G)$, where $\Bbbk G$ is the group ring of G over the field $\Bbbk$. Such an embedding defines an algebra acting on the linear span of A^*; this algebra has important properties, studied in [118] for Gupta-Sidki's example and in [12] for Grigorchuk's example.

The action of $g \in G(\mathcal{M})$ may be described as follows: given a sequence $u = a_1 \dots a_n$, compute $\tau(g, u) = (w, h)$. Then $g \cdot u = w$; and the image of $g \cdot (uv) = w(h \cdot v)$; that is, the action of g on sequences starting by u is defined by an element $h \in G(\mathcal{M})$ acting on the tail of the sequence. More geometrically, we can picture A^* as an infinite tree. The action of g carries the subtree uA^* to wA^*, and, within uA^* naturally identified with A^*, acts by the element h. For that reason, $G(\mathcal{M})$ is called a *self-similar group*.

The formalism expressing a Mealy machine as a map $\tau : Q\times A \to A\times Q$ is completely symmetric with respect to A and Q; the *dual* of the automaton $\mathcal{M}$ is the automaton $\mathcal{M}^\vee$ with state set A, alphabet Q, and transitions given by $i \cdot q = r \cdot o$ whenever (2.4) holds. For example, the dual of (2.3) is



(2.5)

In case the dual $\mathcal{M}^\vee$ of the automaton $\mathcal{M}$ is itself an invertible automaton, $\mathcal{M}$ is called *reversible*. If $\mathcal{M}$, $\mathcal{M}^\vee$ and $(\mathcal{M}^{-1})^\vee$ are all invertible, then $\mathcal{M}$ is *bireversible*; it then has eight associated automata, obtained through all combinations of $()^{-1}$ and $()^\vee$.

Note that $\mathcal{M}^\vee$ naturally encodes the action of $S(\mathcal{M})$ on A: it is a graph with vertex set A, and an edge, with (input) label q, from a to $q \cdot a$. More generally, $(\mathcal{M}^n)^\vee$ encodes the action of $S(\mathcal{M})$ on the set A^n of words of length n.

More generally, we will consider subgroups of $G(\mathcal{M})$, namely subgroups generated by a subset of the states of an automaton; we call these groups *automata groups*. This is a large class of groups, which contains in particular finitely generated linear groups, see Theorem 2.2 below or [35]. The elements of automata groups are, strictly speaking, automatic permutations of A^*. It is often convenient to identify them with a corresponding automaton, for instance constructed as a power of the original Mealy automaton (keeping in mind the construction for the composition of automatic transformations), with appropriate initial state.

Theorem 2.2 (Brunner-Sidki). *The affine group $\mathbb{Z}^n \rtimes \mathrm{GL}_n(\mathbb{Z})$ is an automata group for each n.*

This will be proven in more generality in §2.7.

We mention some closure properties of automata groups. Clearly a direct product of automata groups is an automata group (take the direct product of the alphabets). A more subtle operation, called *tree-wreathing* in [34, 115], gives wreath products with $\mathbb{Z}$.

A more general class of groups has also been considered, and is relevant to §2.6: *functionally recursive groups*. Let A denote a finite alphabet, Q a finite set, and $F = F_Q$ the free group on Q. The "automaton" now is given by a set of rules of the form

$$q \cdot a = b \cdot r$$

for all $q \in Q, a \in A$, where $b \in A$ and $r \in F$. In effect, in the dual $\mathcal{M}^\vee$ we are allowing arbitrary words over Q as output symbols.

2.1 Main examples

Automata groups gained significance when simple examples of finitely generated, infinite torsion groups, and groups of intermediate word-growth, were discovered. Alëshin studied in [6] the automaton (2.7), and showed that $\langle A, B\rangle$ is an infinite torsion group. Another of his examples is described in §2.8.

Grigorchuk [70–74] simplified Alëshin's example as follows: let $\mathcal{A}$ be obtained from the Alëshin automaton by removing the gray states; the stateset of $\mathcal{A}$ is $\{\mathbb{1}, a, b, c, d\}$. He showed that $G(\mathcal{A})$, which is known as the *Grigorchuk group*, is an infinite torsion group; see Theorem 2.9. In fact, $G(\mathcal{A})$ and $\langle A, B\rangle$ have isomorphic finite-index subgroups.

Gupta and Sidki [85, 86] constructed for all prime p an infinite, p-torsion group; their construction, for $p = 3$, is the automata group $G(\mathcal{G})$ generated by the automaton (2.8).

All invertible automata with at most three states and two alphabet letters have been listed in [24]; here are some important examples.

The affine group $BS_{1,3} = \{z \mapsto 3^p z + q/3^r \mid p, q, r \in \mathbb{Z}\}$, see (1.4) is a linear group, and an automata group by Theorem 2.15; see also [19]. It is generated by the automaton (2.9).

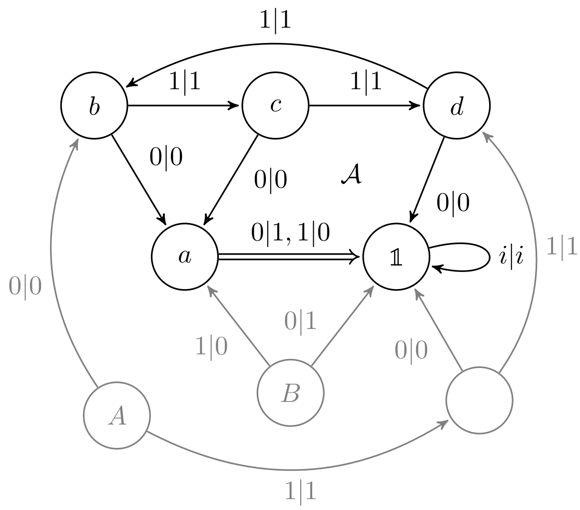
As another important example, consider the lamplighter group

$$G = (\mathbb{Z}/2)^{(\mathbb{Z})} \rtimes \mathbb{Z} = \langle a, t \mid a^2, [a, a^{t^n}] \text{ for all } n \in \mathbb{Z}\rangle. \tag{2.6}$$

It is an automata group [79], embedded as the set of maps

$$\{z \mapsto (t+1)^p z + q \mid p \in \mathbb{Z}, q \in \mathbb{F}_2[t+1, (t+1)^{-1}]\}$$

in the affine group of $\mathbb{F}_2[[t]]$. It is generated by the automaton $\mathcal{L}$ in (2.10).



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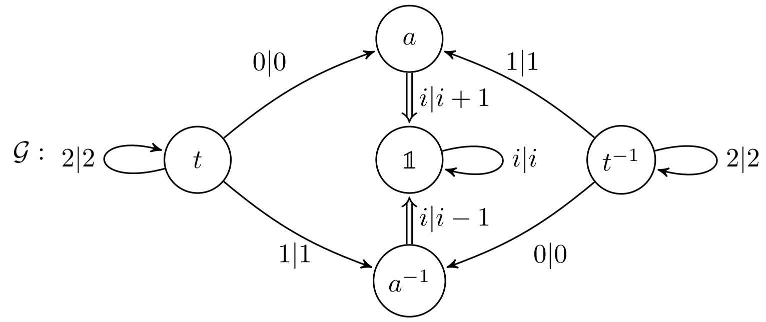

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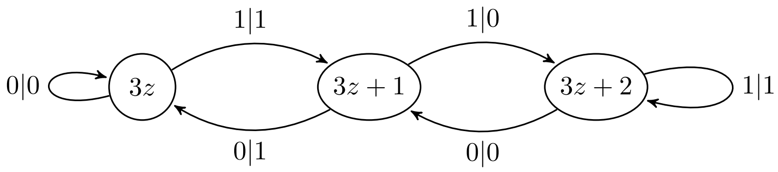

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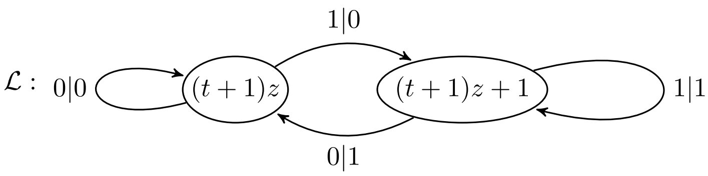
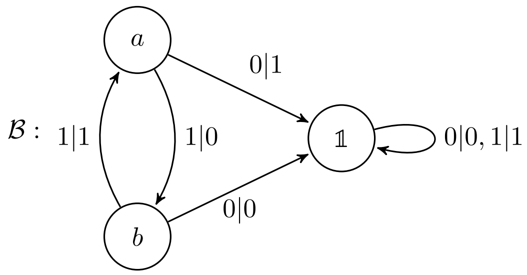


(2.10)

The Basilica group, see [21, 75], will appear again in §2.6. It is generated by the automaton (2.11).



(2.11)

There are (unpublished) lists by Sushchansky *et al.* of all (not necessarily invertible) automata with $\leqslant 3$ states, on a binary alphabet; there are more than 2000 such automata; the invertible ones are listed in [24].

How about groups that are *not* automata groups? Groups with unsolvable word problem (or more generally whose word problem cannot be solved in exponential time, see §2.2), and groups that are not residually finite (or more generally that are not residually (finite with composition factors of bounded order)) among the simplest examples. In fact, it is difficult to come up with any other ones.

2.2 Decision problems

One virtue of automata groups is that elements may easily be compared, since (Mealy) automata admit a unique minimized form, which furthermore may efficiently be computed in time $\mathcal{O}(\#A\#Q\log\#Q)$, see [93, 95].

Proposition 2.3. *Let G be an automata group. Then the word problem is solvable in G, in at worst exponential time.*

Proof. Let Q be a generating set for G, and for each $q \in Q$ compute the minimal automaton $\mathcal{M}_q$ representing q. Let C be an upper bound for the number of states of the $\mathcal{M}_q$.

Now given a word $w = q_1 \dots q_n \in (Q \sqcup Q^{-1})^*$, multiply the automata $\mathcal{M}_{q_1}, \dots, \mathcal{M}_{q_n}$. The result is an automaton with $\leqslant C^n$ states. Then w is trivial if and only if all states the initial state leads to have identical input and output symbols. □

It is unknown if the conjugacy or generalized word problem are solvable in general; though this is known in particular cases, such as the Grigorchuk group $G(\mathcal{A})$, see [78, 97, 114]. The conjugacy problem is solvable as soon as $G(\mathcal{A})$ is *conjugacy separable*, namely, for g, h non-conjugate in $G(\mathcal{A})$ there exists a finite quotient of $G(\mathcal{A})$ in which their images are non-conjugate. Indeed automata groups are recursively presented and residually finite.

It is also unknown whether the order problem is solvable in arbitrary automata groups; but this is known for particular cases, such as bounded automata groups, see §2.3.

Nekrashevych's limit space (see Theorem 2.14) may sometimes be used to prove that two contracting, self-similar groups are non-isomorphic: By [77], some groups admit essentially only one weakly branch self-similar action; if the group is also contracting, then its limit space is an isomorphism invariant.

On the other hand, in the more general class of functionally recursive groups, the very solvability of the word problem remains so far an open problem.

2.3 Bounded and contracting automata

As we saw in §2.2, it may be useful to note, and use, additional properties of automata groups.

Definition 2.2. An automaton $\mathcal{M}$ is *bounded* if the function which to $n \in \mathbb{N}$ associates the number of paths of length n in $\mathcal{M}$ that do not end at the identity state is a bounded function. A group is *bounded* if its elements are bounded automata; or, equivalently, if it is generated by bounded automata.

More generally, Sidki considered automata for which that function is bounded by a polynomial; see [116]. He showed in [117] that such groups cannot contain non-abelian free subgroups.

Definition 2.3. An automaton $\mathcal{M}$ is *nuclear* if the set of recurrent states of $\mathcal{M}\mathcal{M}$ spans an automaton isomorphic to $\mathcal{M}$; and, for invertible $\mathcal{M}$, if additionally $\mathcal{M} = \mathcal{M}^{-1}$. Recall that a state is *recurrent* if it is the endpoint of arbitrarily long paths.

An invertible automaton $\mathcal{M}$ is *contracting* if $G(\mathcal{M}) = G(\mathcal{N})$ for a (necessarily unique) nuclear automaton $\mathcal{N}$. The *nucleus* of $G(\mathcal{M})$ is then $\mathcal{N}$.

For example, the automata (2.7,2.8) are nuclear; the automata (2.3,2.11) are contracting, with nucleus $\{1, t, t^{-1}\}$ and $\{1, a^{\pm 1}, b^{\pm 1}, b^{-1}a, a^{-1}b\}$; the automaton (2.10) is not contracting.

If $\mathcal{M}$ is contracting, then for every $g \in G(\mathcal{M})$ there is a constant K such that (in the automaton describing g) all paths of length $\geqslant K$ end at a state in $\mathcal{M}$. It also implies that there are constants L, m and $\lambda < 1$ such that, for the word metric $\|\cdot\|$ on $G(\mathcal{M})$, whenever one has $g \cdot a_1 \dots a_m = b_1 \dots b_m \cdot h$ with $h, g \in G(\mathcal{M})$, one has $\|h\| \leqslant \lambda \|g\| + L$.

Proposition 2.4 ([108, Theorem 3.9.12]). *Finitely generated bounded groups are contracting.*

Consider the following graph $\mathscr{X}(\mathcal{M})$: its vertex set is A^*. It has two kinds of edges, *vertical* and *horizontal*. There is a vertical edge (u, ua) for all $u \in A^*, a \in A$, and a horizontal edge $(u, q \cdot u)$ for every $u \in A^*, q \in Q$. Note that the horizontal and vertical edges form squares labeled as in (2.4), and that the horizontal edges form the Schreier graphs of the action of $G(\mathcal{M})$ on A^n.

Proposition 2.5 ([108, Theorem 3.8.6]). *If $G(\mathcal{M})$ is contracting then $\mathscr{X}(\mathcal{M})$ is a hyperbolic graph in the sense of Definition 1.5.*

Discrete groups may be broadly separated in two classes: *amenable* and *non-amenable* groups. A group G is *amenable* if it admits a normalized, invariant mean, that is, a map $\mu : \mathcal{P}(G) \to [0, 1]$ with $\mu(A \sqcup B) = \mu(A) + \mu(B)$, $\mu(G) = 1$ and $\mu(gA) = \mu(A)$ for all $g \in G$ and $A, B \subseteq G$. All finite and abelian groups are amenable; so are groups of subexponential word-growth (see §2.5). Extensions, quotients, subgroups, and directed unions of amenable groups are amenable. On the other hand, non-abelian free groups are non-amenable.

In understanding the frontier between amenable and non-amenable groups, the Basilica group $G(\mathcal{B})$ stands out as an important example: Bartholdi and Virág proved that it is amenable [21], but its amenability cannot be decided by the criteria of the previous paragraph. We now briefly indicate the core of the argument.

The matrix embedding $\tau' : \Bbbk G \to M_d(\Bbbk G)$ associated with a self-similar group (see page 116) extends to a map $\tau' : \ell^1(G) \to M_d(\ell^1(G))$ on measures on G. A measure μ gives rise to a random walk on G, with one-step transition probability $p_1(x, y) = \mu(xy^{-1})$. On the other hand, $\tau'(\mu)$ naturally defines a random walk on $G \times X$; treating the second variable as an "internal degree of freedom", one may sample the random walk on $G \times X$ each time it hits $G \times \{x_0\}$ for a fixed $x_0 \in X$. In favourable cases, the corresponding random walk on G is *self-similar*: it is a convex combination of $\mathbb{1}$ and μ. One may then deduce that its "asymptotic entropy" vanishes, and therefore that G is amenable. This strategy works in the following cases:

Theorem 2.6 (Bartholdi-Kaimanovich-Nekrashevych [15]). *Bounded groups are amenable.* ∎

Theorem 2.7 (Amir, Angel, Virág[7]). *Automata of linear growth generate amenable groups.*

Nekrashevych conjectures that contracting automata always generate amenable groups, and proves:

Proposition 2.8 (Nekrashevych, [109]). *A contracting self-similar group cannot contain a non-abelian free subgroup.*

We turn to the original claim to fame of automata groups:

Theorem 2.9 (Alëshin-Grigorchuk [6, 74], Gupta-Sidki [85]). *The Grigorchuk group $G(\mathcal{A})$ and the Gupta-Sidki group $G(\mathcal{G})$ are infinite, finitely generated torsion groups.*

Sketch of proof. To see that these groups G are infinite, consider their action on A^*, the stabilizer H of $0 \in A \subset A^*$, and the restriction θ of the action of H to $0A^*$. This defines a homomorphism $\theta : H \to \mathrm{Sym}(0A^*) \cong \mathrm{Sym}(A^*)$, which is in fact onto G. Therefore G possesses a proper subgroup mapping onto G, so is infinite.

To see that these groups are torsion, proceed by induction on the word-length of an element $g \in G$. The initial cases $a^2 = b^2 = c^2 = d^2 = \mathbb{1}$, respectively $a^3 = t^3 = \mathbb{1}$, are easily checked. Now consider again the action of g on $A \subset A^*$. If g fixes A, then its actions on the subsets iA^* are again defined by elements of G, which are shorter by the contraction property; so have finite order. It follows that g itself has finite order.

If, on the other hand, g does not fix A, then $g^{\#A}$ fixes A; the action of $g^{\#A}$ on iA^* is defined by an element of G, of length at most the length of g; and (by an argument that we skip) smaller in the induction order than g; so $g^{\#A}$ is torsion and so is g. □

Contracting groups have recursive presentations (meaning the relators $\mathscr{R}$ of the presentation form a recursive subset of F_Q); in favourable cases, such as branch groups [8], the set of relators is the set of iterates, under an endomorphism of F_Q, of a finite subset of F_Q. For example [100], Grigorchuk's group satisfies

$$G(\mathcal{A}) = \langle a, b, c, d \mid \sigma^n(bcd), \sigma^n(a^2), \sigma^n([d, d^a]), \sigma^n([d, d^{[a,c]a}]) \text{ for all } n \in \mathbb{N} \rangle,$$

where σ is the endomorphism of $F_{\{a,b,c,d\}}$

$$\sigma : a \mapsto aca, b \mapsto d \mapsto c \mapsto b. \tag{2.12}$$

A similar statement holds for the Basilica group (2.11):

$$G(\mathcal{B}) = \langle a, b \mid [a^p, (a^p)^{b^p}], [b^p, (b^p)^{a^{2p}}] \text{ for all } p = 2^n \rangle;$$

here the endomorphism is $\sigma : a \mapsto b \mapsto a^2$.

2.4 Branch groups

Some of the most-studied examples of automata groups are *branch groups*, see [69] or the survey [14]. We will define a strictly smaller class:

Definition 2.4. An automata group $G(\mathcal{M})$ is *regular weakly branch* if it acts transitively on A^n for all n, and if there exists a nontrivial subgroup K of $G(\mathcal{M})$ such that, for all $u \in A^*$ and all $k \in K$, the permutation

$$w \mapsto \begin{cases} u\,k(v) & \text{if } w = uv, \\ w & \text{else} \end{cases}$$

belongs to $G(\mathcal{M})$.

The group $G(\mathcal{M})$ is *regular branch* if furthermore K has finite index in $G(\mathcal{M})$.

If we view A^* as an infinite tree, a regular branch group G contains a rich supply of tree automorphisms in two manners: enough automorphisms to permute any two vertices of the same depth; and, for any disjoint subtrees of A^*, and for (up to finite index) any elements of G acting on these subtrees, an automorphism acting in that manner on A^*.

In particular, if G is a regular branch group, then G and $G \times \cdots \times G$, with $\#A$ factors, have isomorphic finite-index subgroups (they are *commensurable*, see (2.4)).

Proposition 2.10. *The Grigorchuk group $G(\mathcal{A})$ and the Gupta-Sidki group $G(\mathcal{G})$ are regular branch; the Basilica group $G(\mathcal{B})$ is regular weakly branch.*

Sketch of proof. For $G = G(\mathcal{A})$, note first that G acts transitively on A; since the stabilizer of 0 acts as G on $0A^*$, by induction G acts transively on A^n for all $n \in \mathbb{N}$.

Define then $x = [a, b]$ and $K = \langle\langle x \rangle\rangle$. Consider the endomorphism (2.12), and note that $\sigma(x) = [aca, d] = [x^{-1}, d] \in K$ using the relation $(ad)^4 = \mathbb{1}$, so σ restricts to an endomorphism $K \to K$, such that $\sigma(k)$ acts as k on $1A^*$ and fixes $0A^*$. Similarly, $\sigma^n(k)$ acts as k on 1^nA^*, so Definition 2.4 is fulfilled for $u = 1^n$. Since G acts transitively on A^n, the definition is also fulfilled for other $u \in A^n$.

Finally, a direct computation shows that K has index 16 in G.

The other groups $G(\mathcal{G})$ and $G(\mathcal{B})$ are handled similarly; for them, one takes $K = [G, G]$. $\square$

Various consequences may be derived from a group being a branch group; in particular,

Theorem 2.11 (Abért, [1]). *A weakly branch group satisfies no identity; that is, if G is a weakly branch group, then for every nontrivial word $w = w(x_1, \dots, x_k) \in F_k$, there are $a_1, \dots, a_k \in G$ such that $w(a_1, \dots, a_k) \neq \mathbb{1}$.*

2.5 Growth of groups

An important geometric invariant of a finitely generated group is the asymptotic behaviour of its growth function $\gamma_{G,A}(n)$. Finite groups, of course, have a bounded growth function. If G has a finite-index nilpotent subgroup, then $\gamma_{G,A}(n)$ is bounded by a polynomial, and one says G has *polynomial growth*; the converse is true [81].

On the other hand, if G contains a free subgroup, for example if G is word-hyperbolic and is not a finite extension of $\mathbb{Z}$, then $\gamma_{G,A}$ is bounded from above and below by exponential functions, and one says that G has *exponential growth*.

By a result of Milnor and Wolf [104, 128], if G has a solvable subgroup of finite index then G has either polynomial or exponential growth. The same conclusion holds, by Tits' alternative [123], if G is linear. Milnor asked in [103] whether there exist groups with growth strictly between polynomial and exponential.

Theorem 2.12 (Grigorchuk [73]). *The Grigorchuk group $G(\mathcal{A})$ has intermediate growth. More precisely, its growth function satisfies the following estimates:*

$$e^{n^\alpha} \precsim \gamma_{G,S}(n) \precsim e^{n^\beta},$$

with $\alpha = 0.515$ and $\beta = \log(2)/\log(2/\eta) \approx 0.767$, for $\eta \approx 0.811$ the real root of the polynomial $X^3 + X^2 + X - 2$.

Sketch of proof; see [10, 11]. Recall that G admits an endomorphism σ, see (2.12), such that $\sigma(g)$ acts as g on $1A^*$ and as an element of the finite dihedral group $D_8 = \langle a, d\rangle$ on $0A^*$.

Given $g_0, g_1 \in G$ of length $\leqslant N$, the element $g = a\sigma(g_0)a\sigma(g_1)$ has length $\leqslant 4N$, and acts (up to an element of D_8) as g_i on iA^* for $i = 0, 1$. It follows that g essentially (i.e., up to a 8 choices) determines g_0, g_1, and therefore that $\gamma_{G,S}(4N) \geqslant (\gamma_{G,S}(N)/8)^2$. The lower bound follows easily.

On the other hand, the Grigorchuk group G satisfies a stronger property than contraction; namely, for a well-chosen metric (which is equivalent to the word metric), one has that if $g \in G$ acts as $g_i \in G$ on iA^*, then

$$\|g_0\| + \|g_1\| \leqslant \eta(\|g\| + 1), \tag{2.13}$$

with η the constant above.

Then, to every $g \in G$ one associates a description by a finite, labeled binary tree $\iota(g)$. If $\|g\| \leqslant 1/(1-\eta)$, its description is a one-vertex tree with g at its unique leaf. Otherwise, let $i \in \{0, 1\}$ be such that ga^i fixes A, and write g_0, g_1 the elements of G defined by the actions of ga^i on $0A^*, 1A^*$ respectively. Construct recursively the descriptions $\iota(g_0), \iota(g_1)$. Then the description of g is a tree with i at its root, and two descendants $\iota(g_0), \iota(g_1)$.

By (2.13), the tree $\iota(g)$ has at most $\|g\|^\beta$ leaves; and $\iota(g)$ determines g. There are exponentially many trees with given number of leaves, and the upper bound follows. $\square$

Among groups of exponential growth, Gromov asked the following question [82]: is there a group G of exponential growth, namely such that $\lim \gamma_{G,Q}(n)^{1/n} > 1$ for all (finite) Q, but such that $\inf_{Q\subset G} \lim \gamma_{G,Q}(n)^{1/n} = 1$?

Such examples, called *groups of non-uniform exponential growth*, were first found by Wilson [126]; see [9] for a simplification. Both constructions are heavily based on groups generated by automata.

It is known that essentially any function growing faster than n^2 may be, asymptotically, the growth function of a semigroup. It is however notable that very small automata generate semigroups of growth $\sim e^{\sqrt{n}}$, and of polynomial growth of irrational degree [16, 18]. However, it is not known whether there exist groups whose growth function is strictly between polynomial and $e^{\sqrt{n}}$.

2.6 Dynamics and subdivision rules

We show, in this subsection, how automata naturally arise from geometric or topological situations. As a first step, we will obtain a functionally recursive action; in favourable cases it will be encoded by an automaton. We must first adopt a slightly more abstract point of view on functionally recursive groups:

Definition 2.5. A group G is *self-similar* if it is endowed with a *self-similarity biset*, that is, a set $\mathfrak{B}$ with commuting left and right actions, that is free qua right G-set.

The fundamental example is $G = G(\mathcal{M})$ and $\mathfrak{B} = A \times G$, with actions

$$g \cdot (a, h) = (b, kh) \text{ if } \tau(g, a) = (b, k), \qquad (a, g) \cdot h = (a, gh).$$

Conversely, given a self-similar group G, choose a *basis* A of its biset, i.e., express $\mathfrak{B} = A \times G$; then define $\tau(g, a) = (b, k)$ whenever $g \cdot (a, 1) = (b, k)$ in $\mathfrak{B}$. This vindicates the notation 2.4.

Two bisets $\mathfrak{B}, \mathfrak{B}'$ are *isomorphic* if there is a map $\varphi : \mathfrak{B} \to \mathfrak{B}'$ with $g\varphi(b)h = \varphi(gbh)$ for all $g, h \in G, b \in \mathfrak{B}$. They are *equivalent* if there is a map $\varphi : \mathfrak{B} \to \mathfrak{B}'$ and an automorphism $\theta : G \to G$ with $\theta(g)\varphi(b)\theta(h) = \varphi(gbh)$.

Consider now X a topological space, and $f : X \to X$ a *branched covering*; this means that there is an open dense subspace $X_0 \subseteq X$ such that $f : f^{-1}(X_0) \to X_0$ is a covering. The subset $\mathscr{C} = X \setminus f^{-1}(X_0)$ is the *branch locus*, and $\mathscr{P} = \bigcup_{n \geqslant 1} f^n(\mathscr{C})$ is the *post-critical locus*. Write $\Omega = X \setminus \mathscr{P}$, and choose a basepoint $* \in \Omega$.

Two coverings $(f, \mathscr{P}_f)$ and $(g, \mathscr{P}_g)$ are *combinatorially equivalent* if there exists a path g_t through branched coverings, with $g_0 = f, g_1 = g$, such that the post-critical set of g_t varies continuously along the path.

We define a self-similarity biset for $G = \pi_1(\Omega, *)$: set

$$\mathfrak{B}_f = \{\text{homotopy classes of paths } \gamma : [0, 1] \to \Omega \mid \gamma(0) = f(\gamma(1)) = *\}.$$

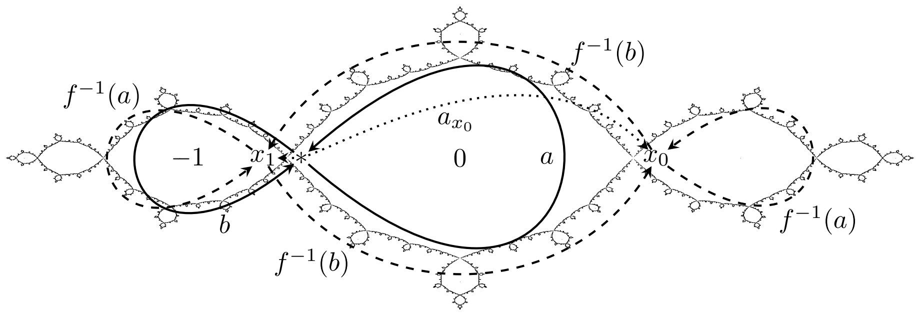
The right action of G prepends a loop at $*$ to γ; the left action appends the unique f-lift of the loop that starts at $\gamma(1)$ to γ.

A choice of basis for $\mathfrak{B}$ amounts to choosing, for each $x \in f^{-1}(*)$, a path $a_x \subset \Omega$ from $*$ to x. Set $A = \{a_x \mid x \in f^{-1}(*)\}$. Now, for $g \in G$, and $a_x \in A$, consider a path γ starting at x such that $f \circ \gamma = g$; such a path is unique up to homotopy, by the covering property of f. The path γ ends at some $y \in f^{-1}(*)$. Set then

$$\tau(g, a_x) = (a_y, a_y^{-1}\gamma a_x),$$

where we write concatenation of paths in reverse order, that is, $\gamma\delta$ is first δ, then γ.

For example, consider the sphere $X = \widehat{\mathbb{C}}$, with branched covering $f(z) = z^2 - 1$. Its post-critical locus is $\mathscr{P} = \{0, -1, \infty\}$. A direct calculation (see e.g. [13]) gives that its biset is the automaton (2.11); the relevant paths are shown here:



Branched self-coverings are encoded by self-similar groups in the following sense:

Theorem 2.13 (Nekrashevych). *Let f, g be branched coverings. Then f, g are combinatorially equivalent if and only if the bisets $\mathfrak{B}_f, \mathfrak{B}_g$ are equivalent.*

This result has been used to answer a long-standing open problem in complex dynamics [17].

If furthermore G is finitely generated and the map f expands a length metric, then the associated biset may be defined by a contracting automaton. This is, in particular, the case for all rational maps acting on the sphere $\widehat{\mathbb{C}}$.

Definition 2.6. Let $f : X \to X$ be a branched self-covering. The *iterated monodromy group* of f is the automata group $G(f) = G(\mathcal{M})$, where $\mathcal{M}$ is an automaton describing the biset $\mathfrak{B}_f$.

If $G = G(\mathcal{M})$ is a contracting self-similar group, consider the hyperbolic boundary $\mathscr{J} = \partial\mathscr{X}(\mathcal{M})$, called the *limit space* of G. It admits an expanding self-covering map $s : \mathscr{J} \to \mathscr{J}$, induced on vertices by the shift map $s(au) = u$.

Theorem 2.14 ([108, Theorems 5.2.6 and 5.4.3]). *The groups $G(s)$ and $G(\mathcal{M})$ are isomorphic.*

Conversely, suppose f is an analytic map, with Julia set *J, the points near which $\{f^{\circ n} \mid n \in \mathbb{N}\}$ does not form a normal family. Then (J, f) and $(\mathscr{J}, s)$ are homeomorphic and topologically conjugate.*

For instance, the Julia set of the Basilica map $f(z) = z^2 - 1$ is depicted above. Appropriately scaled and metrized, the Schreier graphs of the action of $G(\mathcal{M})$ on X^n converge to $\mathscr{J}$.

The first appearance of encodings of branched coverings by automata seems to be the “finite subdivision rules” by Cannon, Floyd and Parry [41]; they wished to know when a branched covering of the sphere may be realized as a conformal map. In their work, a finite subdivision rule is given by a finite subdivision of the sphere, a refinement of it, and a covering map from the refinement to the original subdivision; by iteration, one obtains finer and finer subdivisions of the sphere. The combinatorial information involved is essentially equivalent to a self-similarity biset. Contraction of $G(\mathcal{M})$ and combinatorial versions of expansion have been related in [42].

2.7 Reversible actions

Recall that an automaton $\mathcal{M}$ is *reversible* if its dual $\mathcal{M}^\vee$ is invertible. In other words, if $g \in G(\mathcal{M})$, the action of g is determined by the action on any subset uA^*, for $u \in A^*$.

We have already seen some examples of reversible automata, notably (2.9,2.10). That last example generalizes as follows: consider a finite group G, and set $A = Q = G$. Define an automaton $\mathcal{C}_G$, the "Cayley automaton" of G, by $\tau(q,a) = (qa, qa)$. This automaton seems to have first been considered in [96, page 358]. The automaton $\mathcal{L}$ in (2.10) is the special case $G = \mathbb{Z}/2\mathbb{Z}$. The inverse of the automaton $\mathcal{C}_G$ is a *reset machine*, in that the target of a transition depends only on the input, not on the source state. Silva and Steinberg prove in [120] that, if G is abelian, then $G(\mathcal{C}_G) = G \wr \mathbb{Z}$.

A large class of reversible automata are covered by the following construction. Let R be a ring, let M be an R-module, and let N be a submodule of M, with M/N finite. Let $\varphi : N \to M$ be an R-module homomorphism. Define a decreasing sequence of submodules M_i of M by $M_0 = M$ and $M_{n+1} = \varphi^{-1}(M_n)$, and denote by $\operatorname{End}_R(M,\varphi)$ the algebra of R-endomorphisms of M that map M_n into M_n for all n. Assume finally that there is an algebra homomorphism $\widehat{\varphi} : \operatorname{End}_R(M,\varphi) \to \operatorname{End}_R(M,\varphi)$ such that $\varphi(an) = \widehat{\varphi}(a)\varphi(n)$ for all $a \in \operatorname{End}_R(M,\varphi), n \in N$. Consider

$$T_M = \{z \mapsto az + m \mid a \in \operatorname{End}_R(M,\varphi), m \in M\}$$

the affine semigroup of M.

Theorem 2.15. *Let A be a transversal of N in M. Then the semigroup T_M acts self-similarly on A^*, by*

$$\tau(az+b, x) = (y, \widehat{\varphi}(a)z + \varphi(ax+b-y)) \text{ for the unique } y \in A \text{ with } ax+b-y \in N.$$

This action is

1. *(1) faithful if and only if $\bigcap_n M_n = 0$;*
2. *(2) reversible if and only if φ is injective;*
3. *(3) defined by a finite-state automaton if $\widehat{\varphi}$ is an automorphism of finite order, and there exists a norm $\|\cdot\| : M \to \mathbb{N}$ such that $\|a+b\| \leqslant \|a\| + \|b\|$, for all $K \in \mathbb{N}$ the ball $\{m \in M \mid K \geqslant \|m\|\}$ is finite, and a constant $\lambda < 1$ satisfies $\|\varphi(n)\| \leqslant \lambda\|n\|$ for all $n \in N$.*

We already saw some examples of this construction: the lamplighter automaton $\mathcal{L}$ is obtained by taking $R = M = \mathbb{F}_2[t], N = tM, \varphi(tm) = m, \widehat{\varphi} = 1$, and $\|f\| = 2^{\deg f}$ with $\lambda = \frac{1}{2}$. The semigroup $S(\mathcal{L})$ is contained in T_M, and the group $G(\mathcal{L})$ is contained in the affine group of $\mathbb{F}_2[[t]]$. More generally, the Cayley automaton of a finite group G is obtained by taking $R = G[[t]]$ with G viewed as a ring with product $xy = 0$ unless $x = 1$ or $y = 1$.

The adding machine (2.3) generates the subgroup of translations in the affine group of M with $R = M = \mathbb{Z}, N = 2M, \varphi(2m) = m$, and $\|m\| = |m|$. The same ring-theoretic data produce the Baumslag-Solitar group (2.9); as above, we use $R = \mathbb{Z}$ to obtain a semigroup, and $R = \mathbb{Z}_2$ (or any ring in which 3 is invertible) to obtain a group.

Consider, more generally, $R = \mathbb{Z}, M = \mathbb{Z}^n, N = 2M$, and $\varphi(2m) = m$. These data produce the affine group $\mathbb{Z}^n \rtimes \operatorname{GL}_n(\mathbb{Z})$, proving Theorem 2.2.

A finer construction, giving an action on the binary tree, is to take again $M = \mathbb{Z}^n$ and $N = \varphi^{-1}(M)$ with $\varphi^{-1}(x_1, \dots, x_n) = (2x_n, x_1, \dots, x_{n-1})$; here $\widehat{\varphi}(a) = \varphi \circ a \circ \varphi^{-1}$. This gives a faithful action, on the binary tree, of

$$\mathbb{Z}^n \rtimes \{a \in \mathrm{GL}_n(\mathbb{Z}) \mid a \bmod 2 \text{ is lower triangular}\}.$$

Sketch of proof. (1) The action is faithful if and only if the translation part $\{z \mapsto z + m\}$ acts faithfully; and $z \mapsto z + m$ acts trivially on A^* if and only if $m \in M_n$ for all $n \in \mathbb{N}$.

(2) For any $x \in A$, the map (not a homomorphism!) $T_M \to T_M$ which to $g \in T_M$ associates the permutation of A^* given by $A^* \to xA^* \xrightarrow{g} g(x)A^* \to A^*$ is injective precisely when φ is injective.

(3) Without loss of generality, suppose $\widehat{\varphi} = 1$. Consider $g = z \mapsto az + m \in T_M$. Let K be larger than the norms of $ax + y$ for all $x, y \in A$. Then the states of an automaton describing g are all of the form $z \mapsto az + m'$, with $\|m'\| \leqslant (\|m\| + K)/(1 - \lambda)$; there are finitely many possibilities for such m'. □

Note that the transversal A amounts to a choice of "digits": the analogy is clear in the case of the adding machine (2.3), which has digits $\{0, 1\}$ and "counts" in base 2. For more general radix representations and their association with automata, see e.g. [124].

2.8 Bireversible actions

Recall that an automaton $\mathcal{M}$ is bireversible if $\mathcal{M}, \mathcal{M}^\vee, (\mathcal{M}^{-1})^\vee, ((\mathcal{M}^\vee)^{-1})^\vee$ etc. are all invertible; equivalently, the map $\tau : Q \times A \to A \times Q$ is a bijection for Q the state set of $\mathcal{M} \sqcup \mathcal{M}^{-1}$.

Bireversible automata are interpreted in [101] in terms of *commensurators* of free groups, defined in (2.4) of Chapter 23. Consider a free group F_A on a set A. Its Cayley graph $\mathscr{C}$ is a tree, and F_A acts by isometries on $\mathscr{C}$, so we have $F_A \leqslant \mathrm{Isom}(\mathscr{C})$. Furthermore, $\mathscr{C}$ is oriented: its edges are labeled by $A \sqcup A^{-1}$, and we choose as orientation the edges labeled A. In this way, F_A is contained in the orientation-preserving subgroup of $\mathrm{Isom}(\mathscr{C})$, denoted $\overrightarrow{\mathrm{Isom}(\mathscr{C})}$.

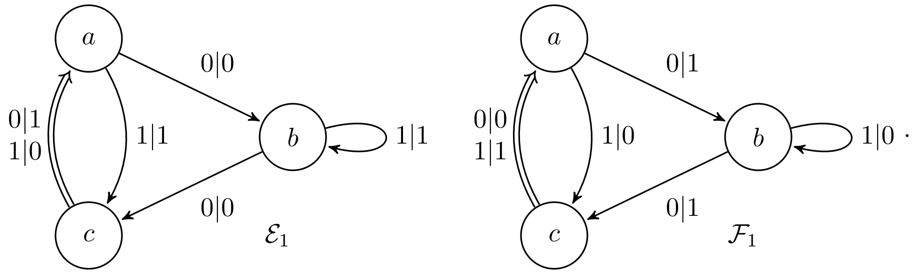
Proposition 2.16. *The stabilizer of* $\mathbb{1}$ *in* $\mathrm{Comm}_{\overrightarrow{\mathrm{Isom}(\mathscr{C})}}(F_A)$ *is the set of bireversible automata with alphabet* A.

Sketch of proof. The proof relies on an interpretation of finite-index subgroups of F_A as complete automata, see 23.2.2.

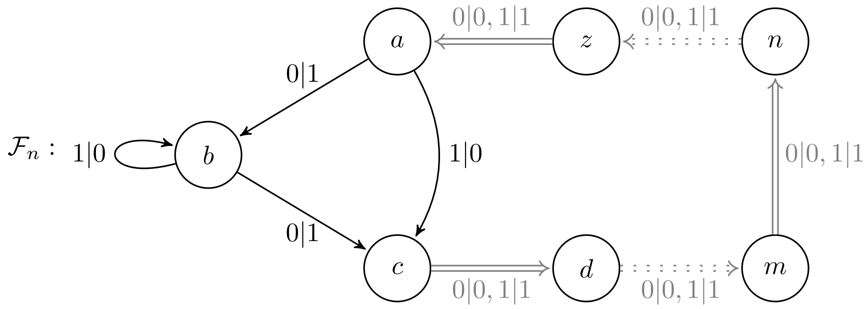
Let $\mathcal{M}$ be a bireversible automaton with alphabet A. Erase first the output labels from $\mathcal{M}$; this defines the Stallings automaton of a finite-index subgroup H_1 (of index $\#Q$) of F_A. Erase then the input labels from $\mathcal{M}$; this defines an isomorphic subgroup H_2 of F_A. The automaton $\mathcal{M}$ itself defines an isomorphism between these two subgroups, which preserves the Cayley graph.

Conversely, given an isometry g of the Cayley graph of F_A which restricts to an isomorphism $G \to H$ between finite-index subgroups of F_A, the Stallings graphs of G and H and put their labels together, as input and output, to construct a bireversible automaton. □

It is striking that all known bireversible automata generate finitely presented groups. There are, up to isomorphism, precisely two minimized bireversible automata with three states and two alphabet letters:



These automata are part of families, whose general term $\mathcal{E}_n, \mathcal{F}_n$ has $2n+1$ states. We describe only $\mathcal{F}_n$:



Alëshin proved in [5] that the group generated by the states b_1, b_2 in $\mathcal{F}_1, \mathcal{F}_2$ respectively is a free group on its two generators; but his argument (especially Lemma 8) has been considered incomplete, and a detailed proof appears in [121]. Alëshin's idea is to prove by induction that, for any reduced word $w \in \{b_1^{\pm 1}, b_2^{\pm 1}\}^*$, the syntactic monoid of the corresponding automaton acts transitively on its state set.

Sidki conjectured that in fact $G(\mathcal{F}_1)$ is a free group on its three generators; this has been proven in [125]. On the other hand, $G(\mathcal{E}_1)$ is a free product of three cyclic groups of order 2. Both proofs illustrate some techniques used to compute with bireversible automata. They rely on the following

Lemma 2.17. *Let $L \subset Q^*$ be a subset mapping to $G(\mathcal{M})$ through the evaluation map. If L is $G(\mathcal{M}^\vee)$-invariant, and every $G(\mathcal{M}^\vee)$-orbit contains a word mapping to a nontrivial element of $G(\mathcal{M})$, then L maps injectively onto $G(\mathcal{M})$.*

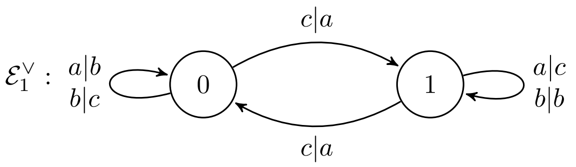
To derive the structure of a bireversible group, we therefore seek a $G(\mathcal{M}^\vee)$-invariant subset $L \subset Q^*$ that maps onto $G(\mathcal{M}) \setminus \{1\}$, and show that every $G(\mathcal{M})$-orbit contains a non-trivial element of $G(\mathcal{M})$.

Theorem 2.18 (Muntyan-Savchuk). $G(\mathcal{E}_1) = \langle a, b, c \mid a^2, b^2, c^2 \rangle$.

Note that this result generalizes: $G(\mathcal{E}_n)$ is a free product of $2n+1$ order-two groups.

Proof. Write $Q = \{a, b, c\}$. We first check the relations $a^2 = b^2 = c^2 = \mathbb{1}$ in $G = G(\mathcal{E}_1)$. Let $L \subset Q^*$ denote those sequences $s_1 \dots s_n$ with $s_i \neq s_{i+1}$ for all i.

Consider the group $G(\mathcal{E}_1^\vee)$, with generators $0, 1$. It acts on L, and acts transitively on $L \cap Q^n$ for all n; indeed already 0 acts transitively on $Q = L \cap Q^1$, and 1 acts on $\{a, c\}Q^{n-1} \cap L$ as a 2^n-cycle, conjugate to the action (2.3) in the sense that there is an identification of $\{a, c\}Q^{n-1} \cap L$ with $\{0, 1\}^n$ interleaving these actions. It follows that the $3 \cdot 2^{n-1}$ elements of $L \cap A^n$ are in the same orbit.



It remains to note that $L \cap A^n$ contains a word mapping to a nontrivial element of G; for example, $c(ab)^{(n-1)/2}$ or $c(ab)^{n/2-1}a$ depending on the parity of n; and to apply Lemma 2.17. ☐

Theorem 2.19 (Vorobets). $G(\mathcal{F}_2) = \langle a, b, c \mid \rangle \cong F_3$.

Note that this result generalizes: $G(\mathcal{F}_n)$ is a free group of rank $2n + 1$.

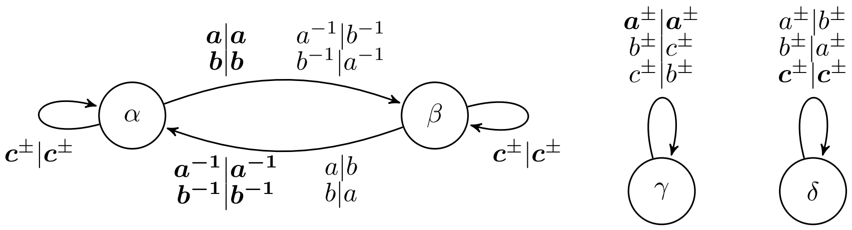
Sketch of proof. Again the key is to control the orbits of $G^\vee = G(\mathcal{F}_2^\vee) = \langle 0, 1 \rangle$ on the reduced words over $Q = \{a, b, c\}$ of any given length. Let $s \in (\pm 1)^n$ be a sequence of signs, and consider

$$L_s = \{w = w_1^{s_1} \dots w_n^{s_n} \in (Q \sqcup Q^{-1})^* \mid w_i^{s_i} \neq w_{i+1}^{-s_{i+1}} \text{ for all } i\}.$$

We show that $G^\vee$ acts transitively on L_s for all s, and that L_s contains a word mapping to a nontrivial element of G. Consider the elements

$$\alpha = 0^2 1^{-2} 0^2 1^{-1}, \quad \beta = 1^2 0^{-2} 1^2 0^{-1}, \quad \gamma = 1^{-1} 0, \quad \delta = 0 1^{-1}$$

of $G^\vee$, where the products are computed left-to-right; they are described by the automata



The elements γ, δ generate a copy of $\mathrm{Sym}(3)$, allowing arbitrary permutations of Q or Q^{-1}. In particular, $G^\vee$ acts transitively on L_s whenever $|s| \leqslant 1$, so we may proceed by induction on $|s|$. The elements α, β, on the other hand, fix a large set of sequences (following the bold edges in the automata).

Consider now $s = s_1 \dots s_n$, and $s' = s_1 \dots s_{n-1}$. If $s_{n-1} \neq s_n$, so that $\#L_s = 2\#L_{s'}$, then there exists $w = w_1^{s_1} \dots w_n^{s_n} \in L_s$, moved by α or β, and such that $w_1^{s_1} \dots w_{n-1}^{s_{n-1}} \in L_{s'}$ is fixed by α and β; so $G^\vee$ acts transitively on L_s.

If $s_1 \neq s_2$, apply the same argument to $L_{s_n^{-1}\dots s_1^{-1}}$ and $L_{s_n^{-1}\dots s_2^{-1}}$.

Finally, if $s_1 = s_2$ and $s_{n-1} = s_n$, consider a typical $w \in L_{s_2\dots s_{n-1}}$, and all $w_{qr} = q^{s_1} w r^{s_n}$, for $q, r \in Q$. Using the action of α and β, the words w_{qa} and w_{qb} are in the same $G^\vee$-orbit for all $q \in Q$, and similarly w_{ar} and w_{br} are in the same $G^\vee$-orbit for all $r \in Q$. For all $r \in Q$, finally, $w_{ar}, w_{br'}, w_{cr''}$ are in the same $G^\vee$-orbit for some $r', r'' \in Q$, and similarly $w_{qa,q'b,q''c}$ are in the same $G^\vee$-orbit. It follows that all w_{qr} are in the same $G^\vee$-orbit, so by induction L_s is a single orbit.

It remains to check that every L_s contains a word w mapping to a nontrivial group element. If n is odd, set $w_i = a$ if $s_i = 1$ and $w_i = b$ if $s_i = -1$; then $\overline{w}$ acts nontrivially on A. If n is even, change w_n to c^{s_n}; again $\overline{w}$ acts nontrivially on A. We are done by Lemma 2.17. □

Burger and Mozes have constructed in [36–38] some infinite, finitely presented simple groups, see also [112]. From this chapter's point of view, these groups are obtained as follows: one constructs an "appropriate" bireversible automaton $\mathcal{M}$ with state set Q and alphabet A, defines

$$G_0 = \langle A \cup Q \mid aq = rb \text{ whenever that relation holds in } \mathcal{M} \rangle,$$

and considers G a finite-index subgroup of G_0. We will not explicitly give here the conditions required on $\mathcal{M}$ for their construction to work; but note that automata groups can be understood as a byproduct of their work. Wise constructed finitely presented groups with non-residual finiteness properties that are also related to automata [127].

Burger and Mozes give the following algebraic construction: consider two primes $p, \ell \equiv 1 \pmod 4$. Let A (respectively Q) denote those integral quaternions, up to a unit $\pm 1, \pm i, \pm j, \pm k$, of norm p (respectively ℓ). By a result of Hurwitz, $\#A = p + 1$ and $\#Q = \ell + 1$. Furthermore [94], for every $q \in Q, a \in A$ there are unique (again up to units) $b \in A, r \in Q$ with $qa = br$. Use these relations to define an automaton $\mathcal{M}_{p,\ell}$. Clearly $\mathcal{M}_{p,\ell}$ is bireversible, with dual $\mathcal{M}_{p,\ell}^\vee = \mathcal{M}_{\ell,p}$. Again thanks to unique factorization of integral quaternions of odd norm,

Proposition 2.20. $G(\mathcal{M}_{p,\ell}) = F_{(\ell+1)/2}$.

Glasner and Mozes construct in [66] an example of a bireversible automata group with Kazhdan's property (T).

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Abstract.

Finite automata have been used effectively in recent years to define infinite groups. The two main lines of research have as their most representative objects the class of automatic groups (including word-hyperbolic groups as a particular case) and automata groups (singled out among the more general self-similar groups).

The first approach implements in the language of automata some tight constraints on the geometry of the group's Cayley graph, building strange, beautiful bridges between far-off domains. Automata are used to define a normal form for group elements, and to monitor the fundamental group operations.

The second approach features groups acting in a finitely constrained manner on a regular rooted tree. Automata define sequential permutations of the tree, and represent the group elements themselves. The choice of particular classes of automata has often provided groups with exotic behaviour which have revolutioned our perception of infinite finitely generated groups.

Index