Turing machines and bimachines

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February 27, 2008

2000 Mathematics Subject Classification: 18B40, 20E08, 20E18, 20E22, 20F69, 20M20, 20M35, 68Q15, 68Q17, 68Q70

ABSTRACT

We associate the iterated block product of a bimachine with a deterministic Turing machine. This allows us to introduce new algebraic notions to study the behavior of the Turing machine. Namely, we introduce double semidirect products through matrix multiplication of upper triangular matrices with coefficients in certain semigroups, which leads in turn to the study of the iterations of bimachines. By passing to the profinite (or projective) limit, we obtain an algebraic profinite description of the limit behavior of the Turing machine. Finally, we analyze the proof that all languages in NP can be reduced to CIRCUIT SAT from this viewpoint.

Contents

1 Introduction

1.1 Finite-state automata, bimachines and length-preserving maps

Let A, B be nonempty finite alphabets. We consider A^+, the set of all nonempty finite strings on A (i. e., the free semigroup with generators A), and consider maps $\alpha : A^+ \to B^+$ (often with $A = B$) which preserve length (*lp-mappings*). We are interested when α can be computed with a finite number of states.

We start with a finite-state automaton given by a right A-automaton $\mathcal{A}_R = (I_R, Q_R, S_R)$ (Q_R is a set, $I_R \in Q_R$ and S_R is an A-semigroup acting on Q_R on the right – see Section 2) together with an output function $f : Q_R \times A \to B$. Then $(\mathcal{A}_R, f)$ determines the lp-mapping $\alpha(\mathcal{A}_R, f) \equiv \alpha : A^+ \to B^+$ defined through its domain extension $\alpha : A^* \times A \times A^* \to B$ (see the beginning of Section 2 for notation) by $\alpha(u, a, v) = f(I_R u, a)$ for $u, v \in A^*$, $a \in A$ (so independent of v, i. e., (right) causal). So

$$a_1 \qquad a_2 \qquad a_3 \qquad \dots \qquad a_n$$

goes to

$$\begin{array}{ccccc} b_1 & b_2 & b_3 & \dots & b_n \\ \| & \| & \| & & \| \\ f(I_R, a_1) & f(I_R a_1, a_2) & f(I_R a_1 a_2, a_3) & \dots & f(I_R a_1, \dots, a_{n-1}, a_n) \end{array}$$

Notice this computation of $a_1, \dots, a_j$ to $b_1, \dots, b_j$ is linear-time in j for all $j \leq n$. An A, B-bimachine $\mathcal{B}$ (see Section 2) is given by a right A-automaton $\mathcal{A}_R = (I_R, Q_R, S_R)$, a left A-automaton $\mathcal{A}_L = (I_L, Q_L, S_L)$, and a function $f : Q_R \times A \times Q_L \to B$, and it determines $\alpha_{\mathcal{B}} : A^* \times A \times A^* \to B$ by

$$\alpha_{\mathcal{B}}(u, a, v) = f(I_R u, a, v I_L) \text{ for } u, v \in A^*,\ a \in A.$$

For notation, see the beginning of Section 2. Thus,

$$a_1 \qquad a_2 \qquad \dots \qquad a_n$$

goes to

$$\begin{array}{cccc} b_1 & b_2 & \dots & b_n \\ \| & \| & & \| \\ f(I_R, a_1, a_2 \cdots a_n I_L) & f(I_R a_1, a_2, a_3 \cdots a_n I_L) & \dots & f(I_R a_1 \cdots a_{n-1}, a_n, I_L) \end{array}$$

Given $\beta : A^+ \to B^+$, there is a unique minimal bimachine $\mathcal{B}(\beta)$ so $\alpha_{\mathcal{B}(\beta)} = \beta$. See Proposition 2.4.

Thus, a finite-state bimachine computes b_i from the input string

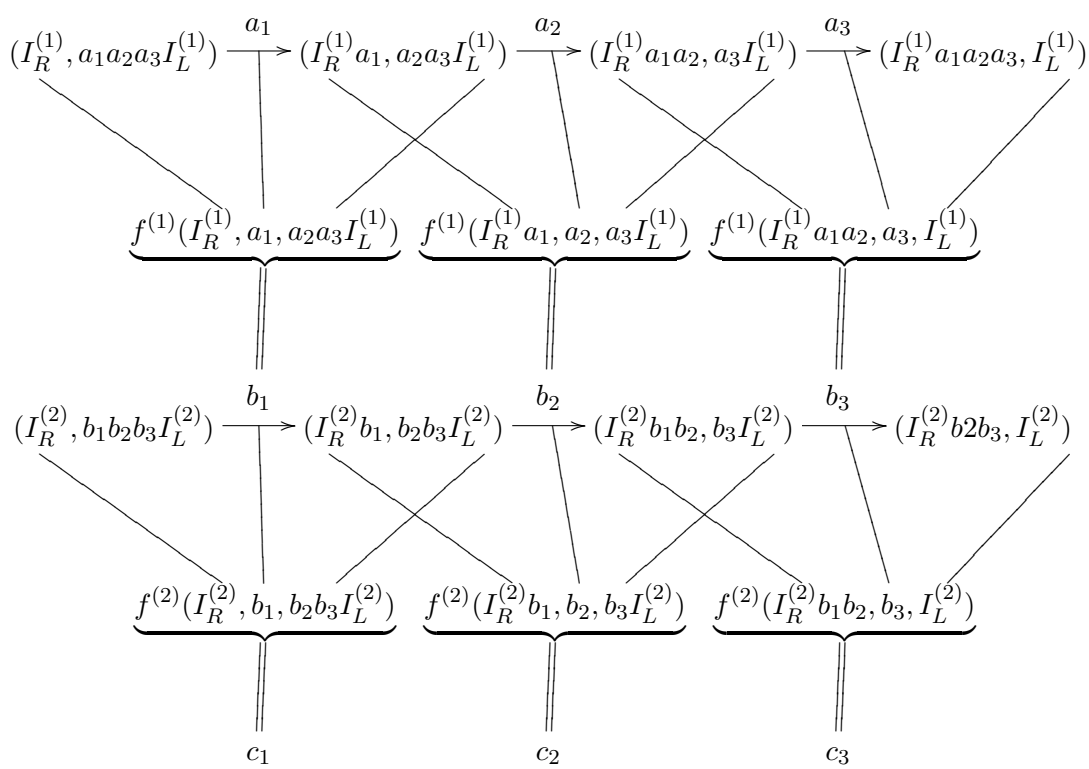
$$a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots, a_n$$

by running the right automata $\mathcal{A}_n$ starting at the left of $a_1, \ldots, a_{i-1}$ and running right, which is linear-time in $i-1$, and running the left automata $\mathcal{A}_L$ starting at the right of $a_{i+1}, \ldots, a_n$ (again, linear-time of length $n-i$) and running left and then determining b_i as $f(I_R a_1 \cdots a_{i-1}, a_i, a_{i+1} \cdots a_n I_L)$ (i. e., as (result of $\mathcal{A}_R, a_i,$ result of $\mathcal{A}_L$)).

This can be illustrated as follows:

$$\underbrace{\overrightarrow{\text{(linear time)}}^{\mathcal{A}_R}\, a_i\, \overleftarrow{\text{(linear time)}}^{\mathcal{A}_L}}_{b_i}.$$

The first course of business is to prove that the composition of two lp-maps given by finite-state bimachines is also given by a finite-state bimachine. It will turn out the semigroups of some (non-minimal) bimachine computing the composition can be taken as double semidirect products through matrix multiplication of upper triangular matrices with coefficients in some semiring (see Section 3). The pictures are as follows:



So

$$c_2 = f^{(2)}(I_R^{(2)} f^{(1)}(I_R^{(1)}, a_1, a_2a_3I_L^{(1)}), f^{(1)}(I_R^{(1)}a_1, a_2, a_3I_L^{(1)}), f^{(1)}(I_R^{(1)}a_1a_2, a_3, I_L^{(1)})I_L^{(2)}),$$

and this corresponds to the matrix product (see Section 3, Lemma 3.4)

$$\begin{pmatrix} (a_1)_L & 0 \\ f_{a_1} & (a_1)_R \end{pmatrix} \cdot \begin{pmatrix} (a_2)_L & 0 \\ f_{a_2} & (a_2)_R \end{pmatrix} = \begin{pmatrix} (a_1)_L(a_2)_L & 0 \\ f_{a_1}(a_2)_L + (a_1)_R f_{a_2} & (a_1)_R(a_2)_R \end{pmatrix},$$

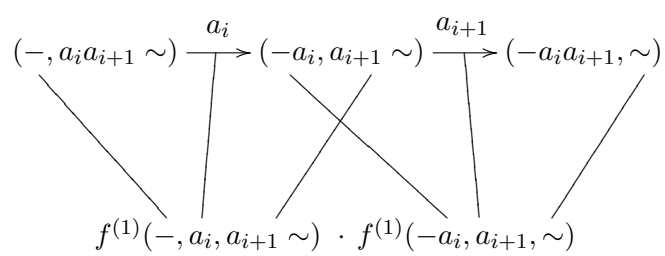
where

$$f_{a_i} : Q_R^{(1)} \times Q_L^{(1)} \to B,$$
$$f_{a_i} = f^{(1)}(-, a_i, \sim) \text{ written as } [-, a_i, \sim].$$

So

$$-(f_{a_1}(a_2)_L + (a_1)_R f_{a_2}) \sim = [-, a_1, a_2 \sim] \cdot [-a_1, a_2, \sim]$$

by considering this picture:



1.2 Associating a finite-state bimachine with a deterministic Turing machine

This is exposited in detail in Section 7. The reader should read this section after reading Sections 2 and 3. The following is a brief overview.

Say we are given the *instantaneous description* (ID) of the Turing machine (TM), say

$$\cdots \boxed{a_1} \cdots \boxed{a_{i-1}}\boxed{\begin{matrix} q \\ a_i \end{matrix}}\boxed{a_{i+1}} \cdots \boxed{a_n} \cdots,$$

where everything to the left of a_1 and to the right of a_n is a blank ($\boxed{B}$), q is the state at the position of the reading head (reading a_i), and $a_1, \ldots, a_n$ are arbitrary tape symbols (including blanks). We consider

$$\boxed{B}\boxed{a_1} \cdots \boxed{a_{i-1}}\boxed{\begin{matrix} q \\ a_i \end{matrix}}\boxed{a_{i+1}} \cdots \boxed{a_n}\boxed{B}.$$

Do one move of M, yielding

$$\begin{array}{c} \boxed{B}\boxed{a_1} \cdots \boxed{a_{i-1}}\boxed{\begin{matrix} q \\ a_i \end{matrix}}\boxed{a_{i+1}} \cdots \boxed{a_n}\boxed{B} \\ M_1 \downarrow \\ \boxed{a'_0}\boxed{a'_1} \cdots \boxed{a'_{i-1}}\boxed{a'_i}\boxed{a'_{i+1}} \cdots \boxed{a'_n}\boxed{a'_{n+1}} \end{array},$$

where $\boxed{a'_j}$ are tape symbols with one or fewer reading heads attached. Then

$$\beta_0 : \text{ID} \to \text{ID}$$

given by

$$\beta_0(a_1, \ldots, a_n) = a'_1, \ldots, a'_n$$

is the associated bimachine map (i. e., we chop off a'_0 and a'_{n+1}).

Example 1.1

$$\beta_0\left(\begin{array}{|c|c|c|}\hline & \overset{\leftarrow}{\boxed{\begin{array}{c} q \\ b \end{array}}} & \\ \hline a & b & c \\ \hline \end{array}\right) = \begin{array}{|c|c|c|}\hline \boxed{\begin{array}{c} q' \\ a \end{array}} & & \\ \hline & b' & c \\ \hline \end{array},$$

where $\overset{\leftarrow}{\boxed{\begin{array}{c} q \\ b \end{array}}}$ in this picture means the TM will move left, print b' over b, and go into new state q':

$$\beta_0\left(\begin{array}{|c|c|c|c|}\hline \overset{\leftarrow}{\boxed{\begin{array}{c} q \\ b \end{array}}} & & & \\ \hline & c & d & e \\ \hline \end{array}\right) = \begin{array}{|c|c|c|c|}\hline b' & c & d & e \\ \hline \end{array},$$

and

$$\beta_0\left(\begin{array}{|c|c|c|c|c|}\hline & \overset{\leftarrow}{\boxed{\begin{array}{c} q \\ b \end{array}}} & & & \\ \hline B & b & c & d & e \\ \hline \end{array}\right) = \begin{array}{|c|c|c|c|c|}\hline \boxed{\begin{array}{c} q' \\ B \end{array}} & & & & \\ \hline & b' & c & d & e \\ \hline \end{array},$$

and dually for $\overset{\rightarrow}{\boxed{\begin{array}{c} q \\ b \end{array}}}$.

If S is the set of all the symbols of the Turing machine, then $\beta_0 : S^+ \to S^+$ satisfies:

1. β_0 is the finite-state bimachine map corresponding to the following bimachine (see Definition 2.1): $S_R = S^r$, with S^r being the (right zero) semigroup with elements in S and $s_1 s_2 = s_2$; $S_L = S^l$, with S^l being the (left zero) semigroup with elements in S and $s_1 s_2 = s_1$; $Q_R = Q_L = S^I = S \cup \{I\}$; $f : S^I \times S \times S^I \to S$ is essentially defined by the transitions of M.

2. With suitable stopping conventions (see Section 7 for details),

$$\lim_{t \to \infty} \beta_0^t = P,$$

the problem. Here, $\lim \beta_0^t(w) = \beta_0^{t(w)}(w) = \beta_0(\beta_0^{t(w)}(w)) = \beta_0^{t(w)}(w)$. Time$(w)$ is the smallest $t(w)$ which works, and similarly for space. (See Section 7.)

Going from $M \to b_i \cong \beta_0$ (and we could go back: $\beta_0 \cong b_i \to M$) is essentially an (obvious!) equivalent formulation of Turing machines, so why do it? The taking of powers of β_0 under composition (i. e., running the Turing machine) leads to *algebra*, namely double semidirect products (of semigroups) as evidenced by multiplication in upper triangular matrices with coefficients in some semiring as was discussed in Subsection 1.5 before and continued in Sections 3 and 4. Two and three iterations of a bimachine (map) are considered in Sections 3 and 4. An arbitrary number of iterations is considered in Section 6. Infinite profinite limits of a bimachine (map) are considered in Section 8.

2 Bimachines

Let A, A' be finite nonempty alphabets. A function $\alpha : A^+ \to A'^+$ is said to be *length-preserving* if $|\alpha(w)| = |w|$ for every $w \in A^+$. It shall be usually referred as *lp-mapping*.

Let $w \in A^+$ and $i \in \{1, \ldots, |w|\}$. We must define a factorization of w to isolate the letter in the ith position. More precisely, we define $\lambda_i(w) \in A^{i-1}$, $\sigma_i(w) \in A$ and $\mu_i(w) \in A^{|w|-i}$ by the equality

$$w = \lambda_i(w)\, \sigma_i(w)\, \mu_i(w).$$

Let $\alpha : A^+ \to A'^+$ be an lp-mapping. We extend the domain of α to $A^* \times A \times A^*$ as follows. Given $u, v \in A^*$ and $a \in A$, we write

$$\alpha(u, a, v) = \sigma_{|u|+1}\alpha(uav),$$

i.e. the symbol of the output string in the $|u|+1$ position. Note that this domain extension brings no inconsistency. Since

$$\alpha(w) = \prod_{i=1}^{|w|} \sigma_i\alpha(w) = \prod_{i=1}^{|w|} \alpha(\lambda_i(w), \sigma_i(w), \mu_i(w)) \tag{1}$$

for every $w \in A^+$, it follows that an lp-mapping $A^+ \to A'^+$ is uniquely determined by the mapping $\alpha(\_,\_,\_) : A^* \times A \times A^* \to A'$ and vice-versa. More generally, given $u, w \in A^*$ and $v \in A^+$, we write

$$\alpha(u, v, w) = \prod_{i=1}^{|v|} \alpha(u\lambda_i(v), \sigma_i(v), \mu_i(v)w).$$

A semigroup S is said to be A-generated (or an A-semigroup) if there exists a surjective homomorphism $\pi_S : A^+ \to S$. Given $w \in A^+$, we may write $w_S = \pi_S(w)$. As usual, we assume that π_S is implicitly determined by the mention of S and we drop the subscript $_S$ whenever possible.

Given A-semigroups S and S', we say that a semigroup morphism $\varphi : S \to S'$ is an *A-semigroup morphism* if $\varphi(a_S) = a_{S'}$ for every $a \in A$. Clearly, there is at most one A-semigroup morphism from an A-semigroup into another, and it must be necessarily surjective. Thus we can define a partial order on the set of all A-semigroups (up to isomorphism) by

$$S \geq S' \quad \Leftrightarrow \quad \exists \varphi : S \to S'.$$

This is equivalent to

$$\forall u, v \in A^+ \ (u_S = v_S \Rightarrow u_{S'} = v_{S'}).$$

A *right A-automaton* is a triple $\mathcal{A}_R = (I_R, Q_R, S_R)$ where Q_R is a set, $I_R \in Q_R$ and S_R is an A-semigroup acting on Q_R on the right, so

$$(q_R s_R) s'_R = q_R (s_R s'_R)$$

for all $q_R \in Q_R$ and $s_R, s'_R \in S_R$. We recall that this action is *faithful* if

$$(\forall q_R \in Q_R \ q_R s_R = q_R s'_R) \Rightarrow s_R = s'_R$$

holds for all $s_R, s'_R \in S_R$, i.e. different elements act differently on the set of states. The action in the right A-automaton $\mathcal{A}_R$ is NOT assumed to be faithful. We say that $\mathcal{A}_R$ is finite if Q_R and S_R are both finite. Clearly, the action of S_R on Q_R induces an action of A^+ on Q_R defined by $q_R u = q_R u_{S_R}$.

Let $\mathcal{A}_R = (I_R, Q_R, S_R)$ and $\mathcal{A}'_R = (I'_R, Q'_R, S'_R)$ be right A-automata. A morphism $\varphi : \mathcal{A}_R \to \mathcal{A}'_R$ is defined, whenever $S'_R \leq S_R$, via a mapping $\varphi : Q_R \to Q'_R$ such that

- $\varphi(I_R) = I'_R$;
- $\varphi(q_R u) = \varphi(q_R) u$ for all $q_R \in Q_R$ and $u \in A^+$.

This corresponds exactly to the statement that there exists a mapping on the states and an A-semigroup morphism preserving initial state and the action. If φ is onto, we say that $\mathcal{A}'_R$ is a *quotient* of $\mathcal{A}_R$. We say that the morphism φ is an *embedding* (respectively *isomorphism*) of right A-automata if $S' \cong S$ and φ is an injective (respectively bijective) mapping.

Given a semigroup S, we denote by S^I the semigroup obtained by adjoining an identity to S (even if S is a monoid). If S acts on some set Q, we assume that the new identity acts on Q as the identity.

The right automaton $\mathcal{A}_R = (I_R, Q_R, S_R)$ is said to be *trim* if $Q_R = I_R S_R^I$. The trim part of $\mathcal{A}_R$ is defined by

$$\mathrm{tr}(\mathcal{A}_R) = (I_R, I_R S_R^I, S_R).$$

Clearly, the inclusion map constitutes an embedding of $\mathrm{tr}(\mathcal{A}_R)$ into $\mathcal{A}_R$.

Dually, a *left A-automaton* is a triple (S_L, Q_L, I_L) where Q_L is a set, $I_L \in Q_L$ and S_L is an A-semigroup acting on Q_L on the left. The action induces canonically an action of A^+ on Q_L. Morphisms are defined dually.

Definition 2.1 *Let A, A' be finite alphabets. An A, A'-*bimachine *is a structure of the form*

$$\mathcal{B} = ((I_R, Q_R, S_R), f, (S_L, Q_L, I_L)),$$

where

- *(I_R, Q_R, S_R) is a right A-automaton;*
- *(S_L, Q_L, I_L) is a left A-automaton;*
- *$f : Q_R \times A \times Q_L \to A'$ a total map.*

We refer to the function f as the *output function*. We say that $\mathcal{B}$ is finite if (I_R, Q_R, S_R) and (S_L, Q_L, I_L) are both finite. We say that $\mathcal{B}$ is faithful if both actions in (I_R, Q_R, S_R) and (S_L, Q_L, I_L) are faithful.

Let $\mathcal{B} = ((I_R, Q_R, S_R), f, (S_L, Q_L, I_L))$ and $\mathcal{B}' = ((I'_R, Q'_R, S'_R), f', (S'_L, Q'_L, I'_L))$ be A, A'-bimachines. We say that $\varphi : \mathcal{B} \to \mathcal{B}'$ is a morphism of A, A'-bimachines if $\varphi = (\varphi_R, \varphi_L)$, where

- $\varphi_R : (I_R, Q_R, S_R) \to (I'_R, Q'_R, S'_R)$ is a morphism of right A-automata;
- $\varphi_L : (S_L, Q_L, I_L) \to (S'_L, Q'_L, I'_L)$ is a morphism of left A-automata;
- $\forall u, v \in A^* \; \forall a \in A \; f'(I'_R u, a, v I'_L) = f(I_R u, a, v I_L)$.

If φ_R and φ_L are both onto, we say that φ is onto and $\mathcal{B}'$ is a quotient of $\mathcal{B}$. If φ_R and φ_L are both embeddings, we say that φ is an embedding. We shall say that φ is an isomorphism if and only if φ_R and φ_L are both isomorphisms.

It is immediate that the class of all A, A'-bimachines and their morphisms constitutes a category.

We associate an lp-mapping $\alpha_{\mathcal{B}} : A^+ \to A'^+$ with the A, A'-bimachine $\mathcal{B} = ((I_R, Q_R, S_R), f, (S_L, Q_L, I_L))$ by

$$\alpha_{\mathcal{B}}(u, a, v) = f(I_R u, a, v I_L) \quad (u, v \in A^*, a \in A).$$

Proposition 2.2 *Let $\varphi : \mathcal{B} \to \mathcal{B}'$ be a morphism of A, A'-bimachines. Then $\alpha_{\mathcal{B}} = \alpha_{\mathcal{B}'}$.*

Proof. Write $\mathcal{B} = ((I_R, Q_R, S_R), f, (S_L, Q_L, I_L))$ and $\mathcal{B}' = ((I'_R, Q'_R, S'_R), f', (S'_L, Q'_L, I'_L))$. For all $u, v \in A^*$ and $a \in A$, we have

$$\begin{aligned} \alpha_{\mathcal{B}'}(u, a, v) &= f'(I'_R u, a, v I'_L) = f(I_R u, a, v I_L) \\ &= \alpha_{\mathcal{B}}(u, a, v) \end{aligned}$$

and so $\alpha_{\mathcal{B}} = \alpha_{\mathcal{B}'}$. $\square$

A partial converse is given by:

Proposition 2.3 *Let $\mathcal{B} = ((I_R, Q_R, S_R), f, (S_L, Q_L, I_L))$ and $\mathcal{B}' = ((I'_R, Q'_R, S'_R), f', (S'_L, Q'_L, I'_L))$ be A, A'-bimachines such that $\alpha_{\mathcal{B}} = \alpha_{\mathcal{B}'}$. If $\varphi_R : (I_R, Q_R, S_R) \to (I'_R, Q'_R, S'_R)$ and $\varphi_L : (S_L, Q_L, I_L) \to (S'_L, Q'_L, I'_L)$ are morphisms of respectively right and left A-automata, then $\varphi = (\varphi_R, \varphi_L)$ is a morphism from $\mathcal{B}$ to $\mathcal{B}'$.*

Proof. For all $u, v \in A^*$ and $a \in A$, we have

$$\begin{aligned} f'(I'_R u, a, v I'_L) &= \alpha_{\mathcal{B}'}(u, a, v) = \alpha_{\mathcal{B}}(u, a, v) \\ &= f(I_R u, a, v I_L) \end{aligned}$$

and so $\varphi = (\varphi_R, \varphi_L)$ is a morphism. $\square$

An A, A'-bimachine $\mathcal{B} = ((I_R, Q_R, S_R), f, (S_L, Q_L, I_L))$ is said to be *trim* if both (I_R, Q_R, S_R) and (S_L, Q_L, I_L) are trim. The trim part of $\mathcal{B}$ is defined by

$$\mathrm{tr}(\mathcal{B}) = ((I_R, I_R S_R^I, S_R), f', (S_L, S_L^I I_L, I_L),$$

where f' is the restriction of f to $I_R S_R^I \times A \times S_L^I I_L$. Clearly, the ordered pair of inclusion maps $I_R S_R^I \to Q_R$, $S_L^I I_L \to Q_L$ constitutes an embedding of $\mathrm{tr}(\mathcal{B})$ into $\mathcal{B}$.

We show now we can associate in a canonical way a bimachine with an lp-mapping. Let $\beta : A^+ \to A'^+$ be an lp-mapping. Given $u, v \in A^+$, we write

$$\begin{array}{lll} u\rho_R v & \text{if} & \forall x, y, z \in A^* \, \forall a \in A \; \beta(xuy, a, z) = \beta(xvy, a, z); \\ u\rho_L v & \text{if} & \forall x, y, z \in A^* \, \forall a \in A \; \beta(x, a, yuz) = \beta(x, a, yvz); \\ u\tau_R v & \text{if} & \forall y, z \in A^* \, \forall a \in A \; \beta(uy, a, z) = \beta(vy, a, z); \\ u\tau_L v & \text{if} & \forall x, y \in A^* \, \forall a \in A \; \beta(x, a, yu) = \beta(x, a, yv). \end{array}$$

Clearly, ρ_R and ρ_L are congruences on A^+, and so $S_R = A^+/\rho_R$ and $S_L = A^+/\rho_L$ are A-semigroups. On the other hand, τ_R is a right congruence and τ_L a left congruence on A^+ satisfying

$$\rho_R \subseteq \tau_R, \quad \rho_L \subseteq \tau_L. \tag{2}$$

We can extend τ_R to a right congruence on A^* by defining $1\tau_R = \{1\}$. Let $Q_R = A^*/\tau_R$ and $I_R = 1\tau_R$. We can define a right action of S_R on Q_R by

$$(u\tau_R)(v\rho_R) = (uv)\tau_R \quad (u \in A^*, v \in A^+) :$$

indeed, if $u\tau_R u'$ and $v\rho_R v'$, then $(uv)\tau_R(u'v)\rho_R(u'v')$ and so $(uv)\tau_R(u'v')$ by (2).

Similarly, we extend τ_L to A^* and let $Q_L = A^*/\tau_L$ and $I_L = 1\tau_L$. We define a left action of S_L on Q_L by

$$(u\rho_L)(v\tau_L) = (uv)\tau_L \quad (u \in A^+, v \in A^*).$$

Let $f : Q_R \times A \times Q_L \to A'$ be defined by

$$f(u\tau_R, a, v\tau_L) = \beta(u, a, v).$$

It follows easily from the definition of τ_R and τ_L that f is well defined. Therefore

$$\mathcal{B}_\beta = ((I_R, Q_R, S_R), f, (S_L, Q_L, I_L))$$

is a well-defined trim A, A'-bimachine.

The following result shows that we can view $\mathcal{B}_\beta$ as the *minimum* bimachine of β.

Proposition 2.4 *Let $\beta : A^+ \to A'^+$ be an lp-mapping. Then:*

(i) $\alpha_{\mathcal{B}_\beta} = \beta$.

(ii) If $\mathcal{B}'$ is a trim A, A'-bimachine such that $\alpha_{\mathcal{B}'} = \beta$, then there exists a (surjective) morphism $\varphi : \mathcal{B}' \to \mathcal{B}_\beta$.

(iii) Up to isomorphism, $\mathcal{B}_\beta$ is the unique trim A, A'-bimachine satisfying (ii).

Proof. (i) Given $u, v \in A^*$ and $a \in A$, we have

$$\begin{aligned}\alpha_{\mathcal{B}_\beta}(u,a,v) &= f(I_R u, a, v I_L) = f(u\tau_R, a, v\tau_L)\\ &= \beta(u,a,v)\end{aligned}$$

and so $\alpha_{\mathcal{B}_\beta} = \beta$.

(ii) Assume that $\mathcal{B}' = ((I'_R, Q'_R, S'_R), f', (S'_L, Q'_L, I'_L))$ is a trim A, A'-bimachine such that $\alpha_{\mathcal{B}'} = \beta$. We define mappings $\varphi_R : Q'_R \to Q_R$ and $\psi_R : S'_R \to S_R$ by

$$\varphi_R(I'_R u) = u\tau_R, \quad \psi_R(v_{S'_R}) = v\rho_R \quad (u \in A^*, v \in A^+).$$

Suppose that $v_{S'_R} = w_{S'_R}$. Let $x, y, z \in A^*$ and $a \in A$. We have $(xvy)_{S'_R} = (xwy)_{S'_R}$ and so

$$\begin{aligned}\beta(xvy,a,z) &= \beta_{\mathcal{B}'}(xvy,a,z) = f'(I'_R xvy, a, zI'_L) = f'(I'_R xwy, a, zI'_L)\\ &= \alpha_{\mathcal{B}'}(xwy,a,z) = \beta(xwy,a,z)\end{aligned}$$

and so $v\rho_R = w\rho_R$ and so ψ_R is well defined. Similarly, we can show that φ_R is well defined. It is immediate that ψ_R is an A-semigroup morphism and φ_R an onto morphism of right A-automata.

Similarly, we define an A-semigroup morphism $\psi_L : S'_L \to S_L$ and an onto morphism $\varphi_L : Q'_L \to Q_L$ of left A-automata by

$$\varphi_L(uI'_L) = u\tau_L, \quad \psi_L(v_{S'_L}) = v\rho_L \quad (u \in A^*, v \in A^+).$$

Since $\alpha_{\mathcal{B}'} = \beta$, it follows from Proposition 2.3 that $\varphi = (\varphi_R, \varphi_L)$ is an onto morphism of $\mathcal{B}'$ onto $\mathcal{B}_\beta$.

(iii) Suppose that $\mathcal{B}'$ is another trim A, A'-bimachine satisfying (ii). Then we have onto morphisms $\varphi : \mathcal{B}' \to \mathcal{B}_\beta$ and $\varphi' : \mathcal{B}_\beta \to \mathcal{B}'$. Since there is at most one morphism from one trim right A-automaton into another, it follows that $\varphi_R\varphi'_R$ and $\varphi'_R\varphi_R$ are both identity mappings, and so φ_R is an isomorphism. Similarly, φ_L is an isomorphism and so is φ. □

We end this section by remarking that changing the initial states in a bimachine may give a new perspective on the computation of the associated lp-mapping.

Proposition 2.5 *Let $\mathcal{B} = ((I_R, Q_R, S_R), f, (S_L, Q_L, I_L))$ be an A_1, A_2-bimachine and let $u, w \in A_1^*$, $v \in A_1^+$. If $\mathcal{B}' = ((I_R u, Q_R, S_R), f, (S_L, Q_L, wI_L))$, then*

$$\alpha_{\mathcal{B}}(u,v,w) = \alpha_{\mathcal{B}'}(1,v,1) = \alpha_{\mathcal{B}'}(v).$$

Proof. It follows from the definitions that

$$\begin{aligned}\alpha_{\mathcal{B}}(u,v,w) &= \textstyle\prod_{i=1}^{|v|} \alpha_{\mathcal{B}}(u\lambda_i(v), \sigma_i(v), \mu_i(v)w)\\ &= \textstyle\prod_{i=1}^{|v|} f(I_R u\lambda_i(v), \sigma_i(v), \mu_i(v)wI_L)\\ &= \textstyle\prod_{i=1}^{|v|} \alpha_{\mathcal{B}'}(\lambda_i(v), \sigma_i(v), \mu_i(v))\\ &= \alpha_{\mathcal{B}'}(1,v,1) = \alpha_{\mathcal{B}'}(v).\end{aligned}$$

□

An early reference on bimachines is [16]. Also see [9, vol. A] and [27]. A somewhat related approach is in [10].

3 The block product – composing two bimachines

We develop in this section a construction on bimachines appropriate do deal with composition.

Let

$$\mathcal{B}^{(i)} = ((I_R^{(i)}, Q_R^{(i)}, S_R^{(i)}), f^{(i)}, (S_L^{(i)}, Q_L^{(i)}, I_L^{(i)}))$$

be an A_i, A_{i+1}-bimachine for $i = 1, 2$. After some preparation, we shall define an A_1, A_3-bimachine

$$\mathcal{B}^{(2)} \square \mathcal{B}^{(1)} = \mathcal{B}^{(21)} = ((I_R^{(21)}, Q_R^{(21)}, S_R^{(21)}), f^{(21)}, (S_L^{(21)}, Q_L^{(21)}, I_L^{(21)}))$$

called the *block product* of $\mathcal{B}^{(2)}$ and $\mathcal{B}^{(1)}$.

The block product construction involves sets of mappings whose domain is often a direct product of the form $Q_R^{(1)} \times Q_L^{(1)}$. Following [27], we shall use the notation $q_R^{(1)} g q_L^{(1)} = g(q_R^{(1)}, q_L^{(1)})$ for $g \in U^{Q_R^{(1)} \times Q_L^{(1)}} = {}^{Q_R^{(1)}}U^{Q_L^{(1)}}$, $q_R^{(1)} \in Q_R^{(1)}$ and $q_L^{(1)} \in Q_L^{(1)}$. To be consistent, we shall write maps with domains of type $Q_R^{(1)}$ on the right and type $Q_L^{(1)}$ on the left.

We define

$$\overline{S_R^{(21)}} = \begin{pmatrix} S_L^{(1)} & 0 \\ {}^{Q_R^{(1)}}S_R^{(2)^{Q_L^{(1)}}} & S_R^{(1)} \end{pmatrix}.$$

A straightforward adaptation of [9, vol.B, p.142] shows that $\overline{S_R^{(21)}}$ is a semigroup for the product

$$\begin{pmatrix} s_L^{(1)} & 0 \\ g & s_R^{(1)} \end{pmatrix} \begin{pmatrix} s'^{(1)}_L & 0 \\ g' & s'^{(1)}_R \end{pmatrix} = \begin{pmatrix} s_L^{(1)} s'^{(1)}_L & 0 \\ g s'^{(1)}_L + s_R^{(1)} g' & s_R^{(1)} s'^{(1)}_R \end{pmatrix},$$

where

$$q_R^{(1)}(g s'^{(1)}_L + s_R^{(1)} g') q_L^{(1)} = (q_R^{(1)} g (s'^{(1)}_L q_L^{(1)})) + ((q_R^{(1)} s_R^{(1)}) g' q_L^{(1)}).$$

Following [9, vol. B], we use here + to denote the semigroup operation of $S_R^{(2)}$, whether it is commutative or not, to emphasize that we are doing the natural matrix multiplication. However, we shall revert to the more classical $\cdot$ notation in the sequel.

Let

$$Q_R^{(21)} = Q_R^{(2)^{Q_L^{(1)}}} \times Q_R^{(1)}.$$

It will often be convenient to represent the elements of $Q_R^{(21)}$, termed *R-generalized 2 step crossing sequences*, as 1×2 matrices (see Section 9 for more details). The semigroup $\overline{S_R^{(21)}}$ acts on $Q_R^{(21)}$ on the right by

$$\begin{pmatrix} \gamma & q_R^{(1)} \end{pmatrix} \begin{pmatrix} s_L^{(1)} & 0 \\ g & s_R^{(1)} \end{pmatrix} = \begin{pmatrix} \gamma s_L^{(1)} \cdot q_R^{(1)} g & q_R^{(1)} s_R^{(1)} \end{pmatrix},$$

where

$$(\gamma s_L^{(1)} \cdot q_R^{(1)} g)(q_L^{(1)}) = \gamma (s_L^{(1)} q_L^{(1)}) \cdot q_R^{(1)} g q_L^{(1)}.$$

Once again, we note that this is a form of matrix multiplication (but we refrain from using + for the action).

To show that this is indeed an action, we compute

$$\begin{aligned}
&\begin{pmatrix}\gamma & q_R^{(1)}\end{pmatrix}\left(\begin{pmatrix} s_L^{(1)} & 0 \\ g & s_R^{(1)}\end{pmatrix}\begin{pmatrix} {s'}_L^{(1)} & 0 \\ g' & {s'}_R^{(1)}\end{pmatrix}\right)\\
&=\begin{pmatrix}\gamma & q_R^{(1)}\end{pmatrix}\begin{pmatrix} s_L^{(1)}{s'}_L^{(1)} & 0 \\ g{s'}_L^{(1)}\cdot s_R^{(1)}g' & s_R^{(1)}{s'}_R^{(1)}\end{pmatrix}\\
&=\begin{pmatrix}\gamma(s_L^{(1)}{s'}_L^{(1)})\cdot q_R^{(1)}(g{s'}_L^{(1)}\cdot s_R^{(1)}g') & q_R^{(1)}(s_R^{(1)}{s'}_R^{(1)})\end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
&\left(\begin{pmatrix}\gamma & q_R^{(1)}\end{pmatrix}\begin{pmatrix} s_L^{(1)} & 0 \\ g & s_R^{(1)}\end{pmatrix}\right)\begin{pmatrix} {s'}_L^{(1)} & 0 \\ g' & {s'}_R^{(1)}\end{pmatrix}\\
&=\begin{pmatrix}\gamma s_L^{(1)}\cdot q_R^{(1)}g & q_R^{(1)}s_R^{(1)}\end{pmatrix}\begin{pmatrix} {s'}_L^{(1)} & 0 \\ g' & {s'}_R^{(1)}\end{pmatrix}\\
&=\begin{pmatrix}(\gamma s_L^{(1)}\cdot q_R^{(1)}g){s'}_L^{(1)}\cdot(q_R^{(1)}s_R^{(1)})g' & (q_R^{(1)}s_R^{(1)}){s'}_R^{(1)}\end{pmatrix}.
\end{aligned}$$

Since $S_R^{(1)}$ acts on $Q_R^{(1)}$, the second columns coincide. For the first columns, we compute

$$\begin{aligned}
[\gamma(s_L^{(1)}{s'}_L^{(1)})\cdot q_R^{(1)}(g{s'}_L^{(1)}\cdot s_R^{(1)}g')](q_L^{(1)}) &= \gamma((s_L^{(1)}{s'}_L^{(1)})q_L^{(1)})\cdot q_R^{(1)}(g{s'}_L^{(1)}\cdot s_R^{(1)}g')q_L^{(1)}\\
&=\gamma(s_L^{(1)}({s'}_L^{(1)}q_L^{(1)}))\cdot[q_R^{(1)}g({s'}_L^{(1)}q_L^{(1)})\cdot(q_R^{(1)}s_R^{(1)})g'q_L^{(1)}]\\
&=[\gamma(s_L^{(1)}({s'}_L^{(1)}q_L^{(1)}))\cdot q_R^{(1)}g({s'}_L^{(1)}q_L^{(1)})]\cdot(q_R^{(1)}s_R^{(1)})g'q_L^{(1)}\\
&=(\gamma s_L^{(1)}\cdot q_R^{(1)}g)({s'}_L^{(1)}q_L^{(1)})\cdot(q_R^{(1)}s_R^{(1)})g'q_L^{(1)}\\
&=[(\gamma s_L^{(1)}\cdot q_R^{(1)}g){s'}_L^{(1)}\cdot(q_R^{(1)}s_R^{(1)})g'](q_L^{(1)}),
\end{aligned}$$

hence we have indeed an action.

Let

$$I_R^{(21)} = (\gamma_0^{(21)}, I_R^{(1)}),$$

where $\gamma_0^{(21)} \in {Q_R^{(2)}}^{Q_L^{(1)}}$ is defined by $\gamma_0^{(21)}(q_L^{(1)}) = I_R^{(2)}$.

The semigroup $\overline{S_R^{(21)}}$ is not an A_1-semigroup, so let $\eta_R : A_1^+ \to \overline{S_R^{(21)}}$ be the homomorphism defined by

$$\eta_R(a) = \begin{pmatrix} a_{S_L^{(1)}} & 0 \\ g_a^{(1)} & a_{S_R^{(1)}}\end{pmatrix},$$

where

$$q_R^{(1)} g_a^{(1)} q_L^{(1)} = (f^{(1)}(q_R^{(1)}, a, q_L^{(1)}))_{S_R^{(2)}}$$

for all $q_R^{(1)} \in Q_R^{(1)}$ and $q_L^{(1)} \in Q_L^{(1)}$. We define

$$S_R^{(21)} = \eta_R(A_1^+).$$

It is clear that, given $w \in A_1^+$, we may write

$$\eta_R(w) = \begin{pmatrix} w_{S_L^{(1)}} & 0 \\ g_w^{(1)} & w_{S_R^{(1)}} \end{pmatrix}$$

for some $g_w^{(1)} \in {}^{Q_R^{(1)}} S_R^{(2)} {}^{Q_L^{(1)}}$. We have now completed the definition of the right A_1-automaton $(I_R^{(21)}, Q_R^{(21)}, S_R^{(21)})$.

The following straightforward lemmas will be useful throughout the paper.

Lemma 3.1 *For all $q_R^{(1)} \in Q_R^{(1)}$, $q_L^{(1)} \in Q_L^{(1)}$ and $u, v \in A_1^*$,*

$$q_R^{(1)} g_{uv}^{(1)} q_L^{(1)} = q_R^{(1)} g_u^{(1)} (v q_L^{(1)}) \cdot (q_R^{(1)} u) g_v^{(1)} q_L^{(1)}.$$

Proof. This follows immediately from

$$\begin{pmatrix} uv & 0 \\ g_{uv}^{(1)} & uv \end{pmatrix} = \begin{pmatrix} u & 0 \\ g_u^{(1)} & u \end{pmatrix} \begin{pmatrix} v & 0 \\ g_v^{(1)} & v \end{pmatrix} = \begin{pmatrix} uv & 0 \\ g_u^{(1)} v \cdot u g_v^{(1)} & uv \end{pmatrix}.$$

□

Lemma 3.2 *For all $u, v \in A_1^+$, $I_R^{(21)} u = I_R^{(21)} v$ if and only if the following conditions hold:*

(i) $I_R^{(1)} u = I_R^{(1)} v$;

(ii) $\forall q_L^{(1)} \in Q_L^{(1)} \quad I_R^{(2)} \cdot I_R^{(1)} g_u^{(1)} q_L^{(1)} = I_R^{(2)} \cdot I_R^{(1)} g_v^{(1)} q_L^{(1)}$.

Proof. We have $I_R^{(21)} u = I_R^{(21)} v$ if and only if $(\gamma_0, I_R^{(1)}) u = (\gamma_0, I_R^{(1)}) v$ if and only if

$$(\gamma_0 u \cdot I_R^{(1)} g_u^{(1)}, I_R^{(1)} u) = (\gamma_0 v \cdot I_R^{(1)} g_v^{(1)}, I_R^{(1)} v).$$

Clearly, this is equivalent to (i) and

$$\forall q_L^{(1)} \in Q_L^{(1)} \quad (\gamma_0 u \cdot I_R^{(1)} g_u^{(1)}) q_L^{(1)} = (\gamma_0 v \cdot I_R^{(1)} g_v^{(1)}) q_L^{(1)},$$

that is, (ii). □

Lemma 3.3 *For all $u, v \in A_1^+$, the equality $u = v$ holds in $S_R^{(21)}$ if and only if the following conditions hold:*

(i) $u = v$ *holds in* $S_R^{(1)}$*;*

(ii) $u = v$ *holds in* $S_L^{(1)}$*;*

(ii) $\forall q_R^{(1)} \in Q_R^{(1)} \quad \forall q_L^{(1)} \in Q_L^{(1)} \quad q_R^{(1)} g_u^{(1)} q_L^{(1)} = q_R^{(1)} g_v^{(1)} q_L^{(1)}$ *holds in* $S_R^{(2)}$.

Proof. Clearly, $u = v$ holds in $S_R^{(21)}$ if and only if

$$\begin{pmatrix} u_{S_L^{(1)}} & 0 \\ g_u^{(1)} & u_{S_R^{(1)}} \end{pmatrix} = \begin{pmatrix} v_{S_L^{(1)}} & 0 \\ g_v^{(1)} & v_{S_R^{(1)}} \end{pmatrix}.$$

□

Dually, we define

$$Q_L^{(21)} = Q_L^{(1)} \times^{Q_R^{(1)}} Q_L^{(2)}.$$

It will often be convenient to represent the elements of $Q_L^{(21)}$, termed *L-generalized 2 step crossing sequences*, as 2×1 matrices (see Section 9 for more details). Let

$$I_L^{(21)} = (I_L^{(1)}, \delta_0^{(21)}),$$

where $q_R^{(1)} \delta_0^{(21)} = I_L^{(2)}$.

We define

$$\overline{S_L^{(21)}} = \begin{pmatrix} S_L^{(1)} & 0 \\ {}^{Q_R^{(1)}}S_L^{(2)}{}^{Q_L^{(1)}} & S_R^{(1)} \end{pmatrix}.$$

Similarly, $\overline{S_L^{(21)}}$ is a semigroup for the product

$$\begin{pmatrix} s_L^{(1)} & 0 \\ h & s_R^{(1)} \end{pmatrix} \begin{pmatrix} s'^{(1)}_L & 0 \\ h' & s'^{(1)}_R \end{pmatrix} = \begin{pmatrix} s_L^{(1)} s'^{(1)}_L & 0 \\ h s'^{(1)}_L \cdot s_R^{(1)} h' & s_R^{(1)} s'^{(1)}_R \end{pmatrix},$$

where

$$q_R^{(1)}(h s'^{(1)}_L \cdot s_R^{(1)} h') q_L^{(1)} = (q_R^{(1)} h (s'^{(1)}_L q_L^{(1)}))((q_R^{(1)} s_R^{(1)}) h' q_L^{(1)}).$$

The semigroup $\overline{S_L^{(21)}}$ acts on $Q_L^{(21)}$ on the left by

$$\begin{pmatrix} s_L^{(1)} & 0 \\ h & s_R^{(1)} \end{pmatrix} \begin{pmatrix} q_L^{(1)} \\ \delta \end{pmatrix} = \begin{pmatrix} s_L^{(1)} q_L^{(1)} \\ h q_L^{(1)} \cdot s_R^{(1)} \delta \end{pmatrix},$$

where

$$q_R^{(1)}(h q_L^{(1)} \cdot s_R^{(1)} \delta) = q_R^{(1)} h q_L^{(1)} \cdot (q_R^{(1)} s_R^{(1)}) \delta.$$

We omit verifying that this is indeed an action.

Let $\eta_L : A_1^+ \to \overline{S_L^{(21)}}$ be the homomorphism defined by

$$\eta_L(a) = \begin{pmatrix} a_{S_L^{(1)}} & 0 \\ h_a^{(1)} & a_{S_R^{(1)}} \end{pmatrix},$$

where

$$q_R^{(1)} h_a^{(1)} q_L^{(1)} = (f^{(1)}(q_R^{(1)}, a, q_L^{(1)}))_{S_L^{(2)}}$$

for all $q_R^{(1)} \in Q_R^{(1)}$ and $q_L^{(1)} \in Q_L^{(1)}$. We define

$$S_L^{(21)} = \eta_L(A_1^+).$$

It is clear that, given $w \in A_1^+$, we may write

$$\eta_L(w) = \begin{pmatrix} w_{S_L^{(1)}} & 0 \\ h_w^{(1)} & w_{S_R^{(1)}} \end{pmatrix}$$

for some $h_w^{(1)} \in {}^{Q_R^{(1)}}S_L^{(2)}{}^{Q_L^{(1)}}$. We have now completed the definition of the left A_1-automaton $(S_L^{(21)}, Q_L^{(21)}, I_L^{(21)})$.

Clearly, Lemmas 3.1, 3.2 and 3.3 have appropriate duals.

Finally, the output function $f^{(21)} : Q_R^{(21)} \times A_1 \times Q_L^{(21)} \to A_3$ is defined by

$$f^{(21)}(\begin{pmatrix} \gamma & q_R^{(1)} \end{pmatrix}, a, \begin{pmatrix} q_L^{(1)} \\ \delta \end{pmatrix}) = f^{(2)}(\gamma(aq_L^{(1)}), f^{(1)}(q_R^{(1)}, a, q_L^{(1)}), (q_R^{(1)}a)\delta).$$

This completes the definition of the bimachine $\mathcal{B}^{(2)}\Box\mathcal{B}^{(1)}$. Note that if $\mathcal{B}^{(2)}$ and $\mathcal{B}^{(1)}$ are both finite, so is $\mathcal{B}^{(2)}\Box\mathcal{B}^{(1)}$.

Next we expose the nature of the functions $g_w^{(1)}$ that play an important part in the definition of η_R and $S_R^{(21)}$. In order to do so, we define for every $w \in A_1^*$ a mapping $G_w^{(1)} \in {}^{Q_R^{(1)}}A_2^{*}{}^{Q_L^{(1)}}$, depending only on $\mathcal{B}^{(1)}$ and w, by

$$q_R^{(1)} G_w^{(1)} q_L^{(1)} = \begin{cases} \prod_{i=1}^{|w|} f^{(1)}(q_R^{(1)}\lambda_i(w), \sigma_i(w), \mu_i(w)q_L^{(1)}) & \text{if } w \neq 1 \\ 1 & \text{if } w = 1 \end{cases}$$

We recall that $\pi_{S_R^{(2)}}$ denotes the canonical surjective homomorphism $A_2^+ \to S_R^{(2)}$.

Lemma 3.4 *For every $w \in A^+$, we have*

$$g_w^{(1)} = \pi_{S_R^{(2)}} G_w^{(1)}.$$

Proof. We use induction on $|w|$. The case $|w| = 1$ follows from the definition, hence we assume that $|w| > 1$ and the lemma holds for shorter words. We may write $w = va$ with $a \in A_1$. Thus

$$\begin{pmatrix} w_{S_L^{(1)}} & 0 \\ g_w^{(1)} & w_{S_R^{(1)}} \end{pmatrix} = w_{S_R^{(21)}} = v_{S_R^{(21)}} a_{S_R^{(21)}} = \begin{pmatrix} v_{S_L^{(1)}} & 0 \\ g_v^{(1)} & v_{S_R^{(1)}} \end{pmatrix} \begin{pmatrix} a_{S_L^{(1)}} & 0 \\ g_a^{(1)} & a_{S_R^{(1)}} \end{pmatrix}$$

$$= \begin{pmatrix} w_{S_L^{(1)}} & 0 \\ g_v^{(1)} a_{S_L^{(1)}} \cdot v_{S_R^{(1)}} g_a^{(1)} & w_{S_R^{(1)}} \end{pmatrix}.$$

By the induction hypothesis, we get for all $q_R^{(1)} \in Q_R^{(1)}$ and $q_L^{(1)} \in Q_L^{(1)}$

$$\begin{aligned} q_R^{(1)} g_w^{(1)} q_L^{(1)} &= q_R^{(1)} (g_v^{(1)} a_{S_L^{(1)}} \cdot v_{S_R^{(1)}} g_a^{(1)}) q_L^{(1)} \\ &= q_R^{(1)} g_v^{(1)} (a q_L^{(1)}) \cdot (q_R^{(1)} v) g_a^{(1)} q_L^{(1)} \\ &= [\textstyle\prod_{i=1}^{|v|} f^{(1)}(q_R^{(1)} \lambda_i(v), \sigma_i(v), \mu_i(v) a q_L^{(1)})]_{S_R^{(2)}} [f^{(1)}(q_R^{(1)} v, a, q_L^{(1)})]_{S_R^{(2)}} \\ &= [(\textstyle\prod_{i=1}^{|w|-1} f^{(1)}(q_R^{(1)} \lambda_i(w), \sigma_i(w), \mu_i(w) q_L^{(1)})) f^{(1)}(q_R^{(1)} \lambda_{|w|}(w), \sigma_{|w|}(w), \mu_{|w|}(w) q_L^{(1)})]_{S_R^{(2)}} \\ &= [\textstyle\prod_{i=1}^{|w|} f^{(1)}(q_R^{(1)} \lambda_i(w), \sigma_i(w), \mu_i(w) q_L^{(1)})]_{S_R^{(2)}} \\ &= \pi_{S_R^{(2)}} (q_R^{(1)} G_w^{(1)} q_L^{(1)}) \end{aligned}$$

and the lemma holds. □

Similarly, we get:

Lemma 3.5 *For every $w \in A^+$, we have*

$$h_w^{(1)} = \pi_{S_L^{(2)}} G_w^{(1)}.$$

Convention 3.6 *In view of Lemmas 3.4 and 3.5, we may from now on replace $g_w^{(1)}$ or $h_w^{(1)}$ by $G_w^{(1)}$ whenever convenient: we shall use the simplified notation*

$$w_{S_R^{(21)}} = \begin{pmatrix} w & 0 \\ G_w^{(1)} & w \end{pmatrix}, \quad w_{S_L^{(21)}} = \begin{pmatrix} w & 0 \\ G_w^{(1)} & w \end{pmatrix}$$

when no confusion will arise.

We may define $g_1^{(1)} = \pi_{S_R^{(2)}} G_w^{(1)}$ and $h_1^{(1)} = \pi_{S_L^{(2)}} G_w^{(1)}$. It is straightforward that the formulae

$$(\gamma, q_R^{(1)}) w = (\gamma w \cdot q_R^{(1)} g_w^{(1)}, q_R^{(1)} w) = (\gamma w \cdot q_R^{(1)} G_w^{(1)}, q_R^{(1)} w) \tag{3}$$

and

$$w(q_L^{(1)},\delta) = (wq_L^{(1)}, h_w^{(1)}q_L^{(1)}\cdot w\delta) = (wq_L^{(1)}, G_w^{(1)}q_L^{(1)}\cdot w\delta) \tag{4}$$

hold for every $w \in A^*$. These equalities will be systematically used throughout the paper, the corresponding reference being often omitted. Its importance can be summarized in asserting that $g_w^{(1)}$ or $h_w^{(1)}$ can be replaced by $G_w^{(1)}$ *whenever their result is supposed to act on some state of* $Q_R^{(2)}$ *or* $Q_L^{(2)}$ (thus enlightening Convention 3.6).

Our next result shows that the block product of bimachines is adequate to deal with the composition of lp-mappings:

Proposition 3.7 *Let* $\mathcal{B}^{(1)}$ *be an* A_1, A_2*-bimachine and let* $\mathcal{B}^{(2)}$ *be an* A_2, A_3*-bimachine. Then* $\alpha_{\mathcal{B}^{(2)}\Box\mathcal{B}^{(1)}} = \alpha_{\mathcal{B}^{(2)}}\alpha_{\mathcal{B}^{(1)}}$.

Proof. Keeping the same notation used so far, we fix $u, v \in A^*$ and $a \in A$. We have

$$\begin{aligned}
\alpha_{\mathcal{B}^{(21)}}(u,a,v) &= f^{(21)}(I_R^{(21)}u, a, vI_L^{(21)})\\
&= f^{(21)}(\begin{pmatrix}\gamma_0^{(21)} & I_R^{(1)}\end{pmatrix}\begin{pmatrix} u & 0\\ g_u^{(1)} & u\end{pmatrix}, a, \begin{pmatrix} v & 0\\ h_v^{(1)} & v\end{pmatrix}\begin{pmatrix} I_L^{(1)}\\ \delta_0^{(21)}\end{pmatrix})\\
&= f^{(21)}(\begin{pmatrix}\gamma_0^{(21)}u\cdot I_R^{(1)}g_u^{(1)} & I_R^{(1)}u\end{pmatrix}, a, \begin{pmatrix} vI_L^{(1)}\\ h_v^{(1)}I_L^{(1)}\cdot v\delta_0^{(21)}\end{pmatrix})\\
&= f^{(2)}((\gamma_0^{(21)}u\cdot I_R^{(1)}g_u^{(1)})(avI_L^{(1)}), f^{(1)}(I_R^{(1)}u, a, vI_L^{(1)}), (I_R^{(1)}ua)(h_v^{(1)}I_L^{(1)}\cdot v\delta_0^{(21)}))\\
&= f^{(2)}(I_R^{(2)}(I_R^{(1)}g_u^{(1)}(avI_L^{(1)})), f^{(1)}(I_R^{(1)}u, a, vI_L^{(1)}), ((I_R^{(1)}ua)h_v^{(1)}I_L^{(1)})I_L^{(2)}).
\end{aligned}$$

On the other hand, by (1) of Section 2, we have

$$\begin{aligned}
\alpha_{\mathcal{B}^{(1)}}(uav) &= \textstyle\prod_{i=1}^{|uav|}\alpha_{\mathcal{B}^{(1)}}(\lambda_i(uav), \sigma_i(uav), \mu_i(uav))\\
&= \textstyle\prod_{i=1}^{|uav|} f^{(1)}(I_R^{(1)}\lambda_i(uav), \sigma_i(uav), \mu_i(uav)\, I_L^{(1)})
\end{aligned}$$

and so

$$\begin{aligned}
\alpha_{\mathcal{B}^{(2)}}\alpha_{\mathcal{B}^{(1)}}(u,a,v) &= f^{(2)}(I_R^{(2)}\lambda_{|ua|}(\alpha_{\mathcal{B}^{(1)}}(uav)), \sigma_{|ua|}(\alpha_{\mathcal{B}^{(1)}}(uav)), \mu_{|ua|}(\alpha_{\mathcal{B}^{(1)}}(uav))\, I_L^{(2)}),\\
&= f^{(2)}(I_R^{(2)}\textstyle\prod_{i=1}^{|u|} f^{(1)}(I_R^{(1)}\lambda_i(uav), \sigma_i(uav), \mu_i(uav)\, I_L^{(1)}),\\
&\quad f^{(1)}(I_R^{(1)}u, a, vI_L^{(1)}), (\textstyle\prod_{i=|ua|+1}^{|uav|} f^{(1)}(I_R^{(1)}\lambda_i(uav), \sigma_i(uav), \mu_i(uav)\, I_L^{(1)}))\, I_L^{(2)})
\end{aligned}$$

Therefore, by Lemmas 3.4 and 3.5, we only need to show that

$$I_R^{(1)}G_u^{(1)}(avI_L^{(1)}) = \prod_{i=1}^{|u|} f^{(1)}(I_R^{(1)}\lambda_i(uav), \sigma_i(uav), \mu_i(uav)\, I_L^{(1)}) \tag{5}$$

and

$$(I_R^{(1)}ua)G_v^{(1)}I_L^{(1)} = \prod_{i=|ua|+1}^{|uav|} f^{(1)}(I_R^{(1)}\lambda_i(uav), \sigma_i(uav), \mu_i(uav)\, I_L^{(1)}). \tag{6}$$

Clearly,

$$\begin{aligned} I_R^{(1)}G_u^{(1)}(avI_L^{(1)}) &= \textstyle\prod_{i=1}^{|u|} f^{(1)}(I_R^{(1)}\lambda_i(u), \sigma_i(u), \mu_i(u)avI_L^{(1)}) \\ &= \textstyle\prod_{i=1}^{|u|} f^{(1)}(I_R^{(1)}\lambda_i(uav), \sigma_i(uav), \mu_i(uav)\, I_L^{(1)}) \end{aligned}$$

and so (5) holds.

Similarly,

$$\begin{aligned} (I_R^{(1)}ua)G_v^{(1)}I_L^{(1)} &= \textstyle\prod_{i=1}^{|v|} f^{(1)}(I_R^{(1)}ua\lambda_i(v), \sigma_i(v), \mu_i(v)I_L^{(1)}) \\ &= \textstyle\prod_{i=|ua|+1}^{|uav|} f^{(1)}(I_R^{(1)}\lambda_i(uav), \sigma_i(uav), \mu_i(uav)\, I_L^{(1)}) \end{aligned}$$

and so (6) holds as well. □

We prove next two other results on morphisms that will become useful in later sections.

Proposition 3.8 *Let $\mathcal{B}^{(1)}$ be an A_1, A_2-bimachine and let $\mathcal{B}^{(2)}$ and $\mathcal{B}'^{(2)}$ be A_2, A_3-bimachines. Let $\varphi^{(2)} : \mathcal{B}^{(2)} \to \mathcal{B}'^{(2)}$ be a morphism. Then there exists a morphism $\varphi^{(21)} : \mathcal{B}^{(2)}\Box\mathcal{B}^{(1)} \to \mathcal{B}'^{(2)}\Box\mathcal{B}^{(1)}$ naturally induced by $\varphi^{(2)}$.*

Moreover, if $\varphi^{(2)}$ is surjective, so is $\varphi^{(21)}$.

Proof. Let $\mathcal{B}^{(i)} = ((I_R^{(i)}, Q_R^{(i)}, S_R^{(i)}), f^{(i)}, (S_L^{(i)}, Q_L^{(i)}, I_L^{(i)}))$ and $\mathcal{B}'^{(i)} = ((I'^{(i)}_R, Q'^{(i)}_R, S'^{(i)}_R), f'^{(i)}, (S'^{(i)}_L, Q'^{(i)}_L, I'^{(i)}_L))$. Let $\varphi^{(2)} = (\varphi_R^{(2)}, \varphi_L^{(2)})$. Write $\mathcal{B}^{(21)} = \mathcal{B}^{(2)}\Box\mathcal{B}^{(1)}$ and $\mathcal{B}'^{(21)} = \mathcal{B}'^{(2)}\Box\mathcal{B}^{(1)}$. We define a mapping $\varphi_R^{(21)} : Q_R^{(21)} \to Q'^{(21)}_R$ by

$$\varphi_R^{(21)}(\gamma, q_R^{(1)}) = (\gamma', q_R^{(1)}),$$

where

$$\gamma'(q_L^{(1)}) = \varphi_R^{(2)}(\gamma(q_L^{(1)})).$$

Note that $\varphi_R^{(21)}$ is surjective if $\varphi_R^{(2)}$ is surjective: given $(\gamma', q_R^{(1)}) \in Q'^{(21)}_R$, there exists some $\gamma \in Q_R^{(2)^{Q_L^{(1)}}}$ such that $\gamma' = \varphi_R^{(2)}\gamma$.

It is routine to check that

$$\varphi_R^{(21)}(I_R^{(21)}) = \varphi_R^{(21)}(\gamma_0, I_R^{(1)}) = (\gamma_0', I_R^{(1)}) = I'^{(21)}_R.$$

Next we show that $\varphi_R^{(21)}$ preserves the action. Let $(\gamma, q_R^{(1)}) \in Q_R^{(21)}$ and $a \in A_1$. We can write

$$\varphi_R^{(21)}((\gamma, q_R^{(1)})a) = \varphi_R^{(21)}(\gamma a \cdot q_R^{(1)} g_a^{(1)}, q_R^{(1)}a) = (\gamma'', q_R^{(1)}a)$$

for some $\gamma'' \in Q'^{(2)^{Q_L^{(1)}}}_R$, while

$$(\varphi_R^{(21)}(\gamma, q_R^{(1)}))a = (\gamma', q_R^{(1)})a = (\gamma' a \cdot q_R^{(1)} g_a^{(1)}, q_R^{(1)}a).$$

It remains to prove that $\gamma'' = \gamma' a \cdot q_R^{(1)} g_a^{(1)}$. For every $q_L^{(1)} \in Q_L^{(1)}$, we have

$$\begin{aligned}\gamma''(q_L^{(1)}) &= \varphi_R^{(2)}((\gamma a \cdot q_R^{(1)} g_a^{(1)})(q_L^{(1)})) = \varphi_R^{(2)}(\gamma(aq_L^{(1)}) \cdot q_R^{(1)} g_a^{(1)} q_L^{(1)}) \\ &= \varphi_R^{(2)}(\gamma(aq_L^{(1)})) \cdot q_R^{(1)} g_a^{(1)} q_L^{(1)} = \gamma'(aq_L^{(1)}) \cdot q_R^{(1)} g_a^{(1)} q_L^{(1)} \\ &= (\gamma' a \cdot q_R^{(1)} g_a^{(1)})(q_L^{(1)})\end{aligned}$$

and so $\varphi_R^{(21)}$ preserves the action.

Now we prove that

$$u_{S_R^{(21)}} = v_{S_R^{(21)}} \Rightarrow u_{S'^{(21)}_R} = v_{S'^{(21)}_R} \tag{7}$$

holds for all $u, v \in A^+$. It is immediate that this is equivalent to have

$$\pi_{S_R^{(2)}} G_u^{(1)} = \pi_{S_R^{(2)}} G_v^{(1)} \Rightarrow \pi_{S'^{(2)}_R} G_u^{(1)} = \pi_{S'^{(2)}_R} G_v^{(1)}.$$

Since $S'^{(2)}_R$ is a quotient of $S_R^{(2)}$, (7) holds and so $\varphi_R^{(21)}$ is a morphism of right A_1-automata.

Similarly, we define a morphism of left A_1-automata $\varphi_L^{(21)} : Q_L^{(21)} \to Q'^{(21)}_L$ by

$$(q_L^{(1)}, \delta)\varphi_L^{(21)} = (q_L^{(1)}, \delta'),$$

where

$$q_R^{(1)} \delta' = (q_R^{(1)} \delta)\varphi_L^{(21)}.$$

Finally, let $u, v \in A_1^+$ and $a \in A_1$. Since $\varphi^{(2)}$ is a morphism, we have

$$\begin{aligned}f^{(21)}(I_R^{(21)} u, a, v I_L^{(21)}) &= f^{(21)}((\gamma_0 u \cdot I_R^{(1)} g_u^{(1)}, I_R^{(1)} u), f^{(1)}(q_R^{(1)}, a, q_L^{(1)}), (v I_L^{(1)}, h_v^{(1)} I_L^{(1)} \cdot v\delta_0)) \\ &= f^{(2)}(I_R^{(2)} \cdot I_R^{(1)} G_u^{(1)}(avI_L^{(1)}), f^{(1)}(q_R^{(1)}, a, q_L^{(1)}), (I_R^{(1)} ua) G_v^{(1)} I_L^{(1)} \cdot I_L^{(2)}) \\ &= f'^{(2)}(I'^{(2)}_R \cdot I_R^{(1)} G_u^{(1)}(avI_L^{(1)}), f^{(1)}(q_R^{(1)}, a, q_L^{(1)}), (I_R^{(1)} ua) G_v^{(1)} I_L^{(1)} \cdot I'^{(2)}_L) \\ &= f'^{(21)}(I'^{(21)}_R u, a, v I'^{(21)}_L),\end{aligned}$$

thus $\varphi^{(21)}$ is a morphism as claimed. □

Proposition 3.9 *Let $\mathcal{B}^{(i)}$ be an A_i, A_{i+1}-bimachine for $i = 1, 2$. Then there exist canonical surjective morphisms*

$$\xi_R^{(21)} : (I_R^{(21)}, Q_R^{(21)}, S_R^{(21)}) \to (I_R^{(1)}, Q_R^{(1)}, S_R^{(1)}),$$

$$\xi_L^{(21)} : (S_L^{(21)}, Q_L^{(21)}, I_L^{(21)}) \to (S_L^{(1)}, Q_L^{(1)}, I_L^{(1)}).$$

Proof. Write $\mathcal{B}^{(i)} = ((I_R^{(i)}, Q_R^{(i)}, S_R^{(i)}), f^{(i)}, (S_L^{(i)}, Q_L^{(i)}, I_L^{(i)}))$ for $i = 1, 2$. Since $S_R^{(1)}$ is a quotient of $S_R^{(21)}$, there is a canonical surjective homomorphism

$$\xi_R^{(21)} : (I_R^{(21)}, Q_R^{(21)}, S_R^{(21)}) \to (I_R^{(1)}, Q_R^{(1)}, S_R^{(1)})$$

defined by

$$\xi_R^{(21)}(\gamma, q_R^{(1)}) = q_R^{(1)}.$$

Similarly, there is a canonical surjective homomorphism

$$\xi_L^{(21)} : (S_L^{(21)}, Q_L^{(21)}, I_L^{(21)}) \to (S_L^{(1)}, Q_L^{(1)}, I_L^{(1)})$$

defined by

$$(q_L^{(1)}, \delta)\xi_L^{(21)} = q_L^{(1)}.$$

□

We end this section by observing by means of an example that the block product of faithful bimachines is not necessarily faithful.

Example 3.10 *There exists a finite faithful A_1, A_2-bimachine $\mathcal{B}^{(1)}$ and a finite faithful A_2, A_3-bimachine $\mathcal{B}^{(2)}$ such that $\mathcal{B}^{(2)} \Box \mathcal{B}^{(1)}$ is not faithful.*

Proof. Let $A_1 = A_2 = A_3 = \{a, b\}$. Let $Q_R^{(1)} = \{I_R^{(1)}\}$ and $S_R^{(1)} = \{a\}$ be trivial. Assume that $Q_R^{(2)} = \{I_R^{(2)}, p_R^{(2)}, q_R^{(2)}\}$ and $S_R^{(2)} = \{a, b\}$ is a two-element semilattice with $a < b$. The action of $S_R^{(2)}$ on $Q_R^{(2)}$ is given by

$$Q_R^{(2)} a = q_R^{(2)} b = q_R^{(2)}, \quad I_R^{(2)} b = p_R^{(2)} b = p_R^{(2)}.$$

We take $Q_L^{(1)} = Q_L^{(2)} = Q_R^{(2)}$ and $S_L^{(1)} = S_L^{(2)} = S_R^{(2)}$ with the same action (left and right actions are essentially the same since the semigroup is commutative). Assume furthermore that $\operatorname{Im} f^{(1)} = \operatorname{Im} f^{(2)} = \{a\}$. It is immediate that

$$\mathcal{B}^{(i)} = ((I_R^{(i)}, Q_R^{(i)}, S_R^{(i)}), f^{(i)}, (S_L^{(i)}, Q_L^{(i)}, I_L^{(i)}))$$

is a faithful finite A_i, A_{i+1}-bimachine for $i = 1, 2$.

We have $a_{S_R^{(1)}} = b_{S_R^{(1)}}$ but $a_{S_L^{(1)}} \neq b_{S_L^{(1)}}$. Then

$$\begin{pmatrix} a_{s_L^{(1)}} & 0 \\ g_a^{(1)} & a_{s_R^{(1)}} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b_{s_L^{(1)}} & 0 \\ g_b^{(1)} & b_{s_R^{(1)}} \end{pmatrix}$$

have the same action on $Q_R^{(21)}$ since $g_a^{(1)} = g_b^{(1)}$ has constant image a and $Q_R^{(2)} a = q_R^{(2)}$. Yet the two matrices are different since $a_{S_L^{(1)}} \neq b_{S_L^{(1)}}$. Therefore $\mathcal{B}^{(2)} \Box \mathcal{B}^{(1)}$ is not faithful. □

The reader should read Section 7 next and then return to Section 4.

4 The quest for associativity

We consider next the product of three bimachines and discuss associativity. Let $\mathcal{B}^{(i)} = ((I_R^{(i)}, Q_R^{(i)}, S_R^{(i)}), f^{(i)}, (S_L^{(i)}, Q_L^{(i)}, I_L^{(i)}))$ be an A_i, A_{i+1}-bimachine for $i = 1, 2, 3$. We shall use the simplified notation

$$\mathcal{B}^{(3(21))} = \mathcal{B}^{(3)} \Box (\mathcal{B}^{(2)} \Box \mathcal{B}^{(1)}), \quad \mathcal{B}^{((32)1)} = (\mathcal{B}^{(3)} \Box \mathcal{B}^{(2)}) \Box \mathcal{B}^{(1)}.$$

The following result shows that we can get associativity at the semigroup level (for three bimachines, but not necessarily for four bimachines!).

Lemma 4.1 $S_R^{(3(21))} \cong S_R^{((32)1)}$ *and* $S_L^{(3(21))} \cong S_L^{((32)1)}$.

Proof. Let $u, v \in A_1^+$. We show that

$$u_{S_R^{(3(21))}} = v_{S_R^{(3(21))}} \Leftrightarrow u_{S_R^{((32)1)}} = v_{S_R^{((32)1)}}. \tag{8}$$

By Lemmas 3.3 and 3.4, $u_{S_R^{(3(21))}} = v_{S_R^{(3(21))}}$ holds if and only if

(A1) $u_{S_R^{(21)}} = v_{S_R^{(21)}}$;

(A2) $u_{S_L^{(21)}} = v_{S_L^{(21)}}$;

(A3) $\pi_{S_R^{(3)}} G_u^{(21)} = \pi_{S_R^{(3)}} G_v^{(21)}$.

Again by Lemmas 3.3 and 3.4, (A1) is equivalent to

(A4) $u_{S_R^{(1)}} = v_{S_R^{(1)}}$;

(A5) $u_{S_L^{(1)}} = v_{S_L^{(1)}}$;

(A6) $\pi_{S_R^{(2)}} G_u^{(1)} = \pi_{S_R^{(2)}} G_v^{(1)}$.

Similarly, (A2) is equivalent to (A4), (A5) and

(A7) $\pi_{S_L^{(2)}} G_u^{(1)} = \pi_{S_L^{(2)}} G_v^{(1)}$.

On the other hand, $u_{S_R^{((32)1)}} = v_{S_R^{((32)1)}}$ holds if and only if (A4) and (A5) and

(A8) $\pi_{S_R^{(32)}} G_u^{(1)} = \pi_{S_R^{(32)}} G_v^{(1)}$

hold. Therefore we may assume that (A4) and (A5) hold, and we must prove that

$$((A3) \wedge (A6) \wedge (A7)) \Leftrightarrow (A8). \tag{9}$$

Assume first that (A8) holds. Let $q_R^{(1)} \in Q_R^{(1)}$ and $q_L^{(1)} \in Q_L^{(1)}$. Write

$$x = q_R^{(1)} G_u^{(1)} q_L^{(1)} = \prod_{i=1}^{|u|} f^{(1)}(q_R^{(1)} \lambda_i(u), \sigma_i(u), \mu_i(u)\, q_L^{(1)}),$$

$$y = q_R^{(1)} G_v^{(1)} q_L^{(1)} = \prod_{i=1}^{|v|} f^{(1)}(q_R^{(1)} \lambda_i(v), \sigma_i(v), \mu_i(v)\, q_L^{(1)}).$$

Note that (similar to the proof of Proposition 2.5)

$$q_R^{(1)} G_{\lambda_i(u)}^{(1)}(\sigma_i(u)\mu_i(u)\, q_L^{(1)}) = \prod_{j=1}^{i-1} f^{(2)}(q_R^{(1)} \lambda_j(u), \sigma_j(u), \mu_j(u) q_L^{(1)}) = \lambda_i(x).$$

Similarly,

$$(q_R^{(1)} \lambda_i(u)\sigma_i(u)) G_{\mu_i(u)}^{(1)} q_L^{(1)} = \mu_i(x).$$

By (A8), $x = y$ holds in $S_R^{(32)}$. Thus $x = y$ holds in both $S_R^{(2)}$ and $S_L^{(2)}$ and so (A6) and (A7) hold.

Given $(\gamma, q_R^{(1)}) \in Q_R^{(21)}$ and $(q_L^{(1)}, \delta) \in Q_L^{(21)}$, we have

$$(\gamma, q_R^{(1)}) G_u^{(21)}(q_L^{(1)}, \delta) = (\gamma(u q_L^{(1)})) G_x^{(1)}((q_R^{(1)} u)\delta). \tag{10}$$

Indeed, it folows from (3) of Section 3 that

$$\begin{aligned}
&(\gamma, q_R^{(1)}) G_u^{(21)}(q_L^{(1)}, \delta) = \textstyle\prod_{i=1}^{|u|} f^{(21)}((\gamma, q_R^{(1)})\lambda_i(u), \sigma_i(u), \mu_i(u)\,(q_L^{(1)}, \delta)) \\
&\quad = \textstyle\prod_{i=1}^{|u|} f^{(21)}((\gamma\lambda_i(u) \cdot q_R^{(1)} g_{\lambda_i(u)}^{(1)}, q_R^{(1)}\lambda_i(u)), \sigma_i(u), (\mu_i(u)\, q_L^{(1)}, h_{\mu_i(u)}^{(1)} q_L^{(1)} \cdot \mu_i(u)\delta)) \\
&\quad = \textstyle\prod_{i=1}^{|u|} f^{(2)}(\gamma(u q_L^{(1)}) \cdot q_R^{(1)} G_{\lambda_i(u)}^{(1)}(\sigma_i(u)\mu_i(u)\, q_L^{(1)}), f^{(1)}(q_R^{(1)}\lambda_i(u), \sigma_i(u), \mu_i(u)\, q_L^{(1)}), \\
&\qquad\quad (q_R^{(1)}\lambda_i(u)\sigma_i(u)) G_{\mu_i(u)}^{(1)} q_L^{(1)} \cdot (q_R^{(1)} u)\delta) \\
&\quad = \textstyle\prod_{i=1}^{|x|} f^{(2)}(\gamma(u q_L^{(1)}) \cdot \lambda_i(x), \sigma_i(x), \mu_i(x) \cdot (q_R^{(1)} u)\delta) \\
&\quad = (\gamma(u q_L^{(1)})) G_x^{(1)}((q_R^{(1)} u)\delta).
\end{aligned}$$

Similarly,

$$(\gamma, q_R^{(1)}) G_v^{(21)}(q_L^{(1)}, \delta) = (\gamma(v q_L^{(1)})) G_y^{(1)}((q_R^{(1)} v)\delta). \tag{11}$$

On the other hand, (A8) holds for $q_R^{(1)}$ and $q_L^{(1)}$ if and only if $x_{S_R^{(32)}} = y_{S_R^{(32)}}$ if and only if

(B1) $x_{S_R^{(2)}} = y_{S_R^{(2)}}$;

(B2) $x_{S_L^{(2)}} = y_{S_L^{(2)}}$;

(B3) $\pi_{S_R^{(3)}} G_x^{(2)} = \pi_{S_R^{(3)}} G_y^{(2)}$.

By (A4) and (A5), we may take

$$q_R^{(2)} = \gamma(uq_L^{(1)}) = \gamma(vq_L^{(1)}), \quad q_L^{(2)} = (q_R^{(1)}u)\delta = (q_R^{(1)}v)\delta$$

in (B3) and by (10) and (11) deduce $\pi_{S_R^{(3)}} G_u^{(21)} = \pi_{S_R^{(3)}} G_v^{(21)}$. Thus (A3) holds, so we have proved that (A8), in the presence of (A4) and (A5), implies (A3) and (A6) and (A7).

Conversely, assume that (A3), (A6) and (A7) hold. Let $q_R^{(1)} \in Q_R^{(1)}$ and $q_L^{(1)} \in Q_L^{(1)}$. Since (B1) and (B2) are equivalent to (A6) and (A7), respectively, it remains to prove that (B3) holds. Let $q_R^{(2)} \in Q_R^{(2)}$ and $q_L^{(2)} \in Q_L^{(2)}$. There exist $\gamma \in {Q_R^{(2)}}^{Q_L^{(1)}}$ and $\delta \in {}^{Q_R^{(1)}}Q_L^{(2)}$ such that $\gamma(uq_L^{(1)}) = q_R^{(2)}$ and $(q_R^{(1)}u)\delta = q_L^{(2)}$. In view of (A4) and (A5), (B3) follows from (A3), (10) and (11) since

$$\pi_{S_R^{(3)}}(q_R^{(2)} G_x^{(2)} q_L^{(2)}) = \pi_{S_R^{(3)}}((\gamma, q_R^{(1)}) G_u^{(21)} (q_L^{(1)}, \delta)) = \pi_{S_R^{(3)}}((\gamma, q_R^{(1)}) G_v^{(21)} (q_L^{(1)}, \delta))$$

$$= \pi_{S_R^{(3)}}(q_R^{(2)} G_y^{(2)} q_L^{(2)}).$$

Thus (9) holds and so does (8). Therefore $S_R^{(3(21))} \cong S_R^{((32)1)}$. Similarly, we can show that $S_L^{(3(21))} \cong S_L^{((32)1)}$. $\square$

Unfortunately the right A_1-automata of $\mathcal{B}_R^{(3(21))}$ and $\mathcal{B}_R^{((32)1)}$ are not in general isomorphic, as one can easily show using a cardinality argument on the states (the first can be strictly larger), and the same goes for the left A_1-automata. However, we can define morphisms. Let $\varphi_R : Q_R^{(3(21))} \to Q_R^{((32)1)}$ be defined as follows. Given

$$(\gamma^{(3(21))}, (\gamma^{(21)}, q_R^{(1)})) \in {Q_R^{(3)}}^{Q_L^{(21)}} \times ({Q_R^{(2)}}^{Q_L^{(1)}} \times Q_R^{(1)}) = Q_R^{(3(21))},$$

we set

$$(\gamma^{(3(21))}, (\gamma^{(21)}, q_R^{(1)}))\varphi_R = (\gamma^{((32)1)}, q_R^{(1)}) \in {Q_R^{(32)}}^{Q_L^{(1)}} \times Q_R^{(1)} = Q_R^{((32)1)},$$

where

$$\gamma^{((32)1)}(q_L^{(1)}) = (\beta_{q_L^{(1)}}, \gamma^{(21)}(q_L^{(1)})) \in {Q_R^{(3)}}^{Q_L^{(2)}} \times Q_R^{(2)} = Q_R^{(32)}$$

and

$$\beta_{q_L^{(1)}}(q_L^{(2)}) = \gamma^{(3(21))}(q_L^{(1)}, \overline{q_L^{(2)}})$$

(notice the order of the variables), where $\overline{q_L^{(2)}} \in {}^{Q_R^{(1)}}Q_L^{(2)}$ is the constant mapping with image $q_L^{(2)}$.

Dually, we define $\varphi_L : Q_L^{(3(21))} \to Q_L^{((32)1)}$ as follows. Given

$$((q_L^{(1)}, \delta^{(21)}), \delta^{(3(21))}) \in (Q_L^{(1)} \times {}^{Q_R^{(1)}}Q_L^{(2)}) \times {}^{Q_R^{(21)}}Q_L^{(3)} = Q_L^{(3(21))},$$

we set

$$\varphi_L((q_L^{(1)}, \delta^{(21)}), \delta^{(3(21))}) = (q_L^{(1)}, \delta^{((32)1)}) \in Q_L^{(1)} \times {}^{Q_R^{(1)}}Q_L^{(32)} = Q_L^{((32)1)},$$

where

$$q_R^{(1)}\delta^{((32)1)} = (q_R^{(1)}\delta^{(21)}, \varepsilon_{q_R^{(1)}}) \in Q_L^{(2)} \times {}^{Q_R^{(2)}}Q_L^{(3)} = Q_L^{(32)}$$

and

$$q_R^{(2)}\varepsilon_{q_R^{(1)}} = (\overline{q_R^{(2)}}, q_R^{(1)})\delta^{(3(21))}$$

(notice the order of the variables) where $\overline{q_R^{(2)}} \in {Q_R^{(2)}}^{Q_L^{(1)}}$ is the constant mapping with image $q_R^{(2)}$.

Lemma 4.2 *(i)* $\varphi_R : (I_R^{(3(21))}, Q_R^{(3(21))}, S_R^{(3(21))}) \to (I_R^{((32)1)}, Q_R^{((32)1)}, S_R^{((32)1)})$ *is a surjective morphism of right* A_1*-automata;*

(ii) $\varphi_L : (S_L^{(3(21))}, Q_L^{(3(21))}, I_L^{(3(21))}) \to (S_L^{((32)1)}, Q_L^{((32)1)}, I_L^{((32)1)})$ *is a surjective morphism of left* A_1*-automata.*

Proof. We give a proof for φ_L, the other case being dual. We have

$$\varphi_L(I_L^{(3(21))}) = \varphi_L(I_L^{(21)}, \delta_0^{(3(21))}) = \varphi_L((I_L^{(1)}, \delta_0^{(21)}), \delta_0^{(3(21))}) = (I_L^{(1)}, \delta^{((32)1)}),$$

where

$$q_R^{(1)}\delta^{((32)1)} = (q_R^{(1)}\delta_0^{(21)}, \varepsilon_{q_R^{(1)}}) = (I_L^{(2)}, \varepsilon_{q_R^{(1)}})$$

and

$$q_R^{(2)}\varepsilon_{q_R^{(1)}} = (\overline{q_R^{(2)}}, q_R^{(1)})\delta_0^{(3(21))} = I_L^{(3)}.$$

Thus

$$q_R^{(1)}\delta^{((32)1)} = (I_L^{(2)}, \delta_0^{(32)}) = I_L^{(32)}$$

and so $\delta^{((32)1)} = \delta_0^{((32)1)}$. It follows that $\varphi_L(I_L^{(3(21))}) = I_L^{((32)1)}$.

Next let $((q_L^{(1)}, \delta^{(21)}), \delta^{(3(21))}) \in Q_L^{(3(21))}$, and $a \in A_1$. We have

$$\begin{aligned} a((q_L^{(1)}, \delta^{(21)}), \delta^{(3(21))}) &= (a(q_L^{(1)}, \delta^{(21)}), G_a^{(21)}(q_L^{(1)}, \delta^{(21)}) \cdot a\delta^{(3(21))}) \\ &= ((aq_L^{(1)}, G_a^{(1)}q_L^{(1)} \cdot a\delta^{(21)}), G_a^{(21)}(q_L^{(1)}, \delta^{(21)}) \cdot a\delta^{(3(21))}), \end{aligned}$$

hence

$$\varphi_L(a((q_L^{(1)}, \delta^{(21)}), \delta^{(3(21))})) = (aq_L^{(1)}, \eta^{((32)1)})$$

for $\eta^{((32)1)}$ given by

$$q_R^{(1)}\delta^{((32)1)} = (q_R^{(1)}(G_a^{(1)}q_L^{(1)} \cdot a\delta^{(21)}), \varepsilon'_{q_R^{(1)}}) \in Q_L^{(2)} \times {}^{Q_R^{(2)}}Q_L^{(3)} = Q_L^{(32)}$$

and

$$q_R^{(2)}\varepsilon_{q_R^{(1)}} = (\overline{q_R^{(2)}}, q_R^{(1)})(G_a^{(21)}(q_L^{(1)}, \delta^{(21)}) \cdot a\delta^{(3(21))}).$$

On the other hand,

$$a\varphi_L((q_L^{(1)}, \delta^{(21)}), \delta^{(3(21))}) = a(q_L^{(1)}, \delta^{((32)1)}) = (aq_L^{(1)}, G_a^{(1)}q_L^{(1)} \cdot a\delta^{((32)1)}).$$

Therefore, to show that φ_L preserves the action, we only need to show that

$$\eta^{((32)1)} = G_a^{(1)} q_L^{(1)} \cdot a\delta^{((32)1)}. \tag{12}$$

Let $q_R^{(1)} \in Q_R^{(1)}$. Writing $b = f^{(1)}(q_R^{(1)}, a, q_L^{(1)})$, we have

$$\begin{aligned} q_R^{(1)}(G_a^{(1)} q_L^{(1)} \cdot a\delta^{((32)1)}) &= q_R^{(1)} G_a^{(1)} q_L^{(1)} \cdot (q_R^{(1)} a)\delta^{((32)1)} \\ &= b((q_R^{(1)} a)\delta^{(21)}, \varepsilon_{q_R^{(1)} a}) \\ &= (b \cdot (q_R^{(1)} a)\delta^{(21)}, G_b^{(2)}((q_R^{(1)} a)\delta^{(21)}) \cdot b\varepsilon_{q_R^{(1)} a}) \end{aligned}$$

and

$$q_R^{(1)} \eta^{((32)1)} = (q_R^{(1)}(G_a^{(1)} q_L^{(1)} \cdot a\delta^{(21)}), \varepsilon'_{q_R^{(1)}}).$$

Since

$$q_R^{(1)}(G_a^{(1)} q_L^{(1)} \cdot a\delta^{(21)}) = q_R^{(1)} G_a^{(1)} q_L^{(1)} \cdot (q_R^{(1)} a)\delta^{(21)} = b \cdot (q_R^{(1)} a)\delta^{(21)},$$

(12) will follow from

$$G_b^{(2)}((q_R^{(1)} a)\delta^{(21)}) \cdot b\varepsilon_{q_R^{(1)} a} = \varepsilon'_{q_R^{(1)}}. \tag{13}$$

We have

$$\begin{aligned} q_R^{(2)}[G_b^{(2)}((q_R^{(1)} a)\delta^{(21)}) \cdot b\varepsilon_{q_R^{(1)} a}] &= q_R^{(2)} G_b^{(2)}((q_R^{(1)} a)\delta^{(21)}) \cdot (q_R^{(2)} b)\varepsilon_{q_R^{(1)} a} \\ &= f^{(2)}(q_R^{(2)}, b, (q_R^{(1)} a)\delta^{(21)}) \cdot (\overline{q_R^{(2)} b}, q_R^{(1)} a)\delta^{(3(21))}, \end{aligned}$$

$$\begin{aligned} q_R^{(2)} \varepsilon'_{q_R^{(1)}} &= (\overline{q_R^{(2)}}, q_R^{(1)})(G_a^{(21)}(q_L^{(1)}, \delta^{(21)}) \cdot a\delta^{(3(21))}) \\ &= (\overline{q_R^{(2)}}, q_R^{(1)}) G_a^{(21)}(q_L^{(1)}, \delta^{(21)}) \cdot ((\overline{q_R^{(2)}}, q_R^{(1)}) a)\delta^{(3(21))} \\ &= f^{(21)}((\overline{q_R^{(2)}}, q_R^{(1)}), a, (q_L^{(1)}, \delta^{(21)})) \cdot (\overline{q_R^{(2)}} a \cdot q_R^{(1)} G_a^{(1)}, q_R^{(1)} a)\delta^{(3(21))} \\ &= f^{(2)}(q_R^{(2)}, b, (q_R^{(1)} a)\delta^{(21)}) \cdot (\overline{q_R^{(2)}} a \cdot q_R^{(1)} G_a^{(1)}, q_R^{(1)} a)\delta^{(3(21))}, \end{aligned}$$

thus we only need to show that

$$\overline{q_R^{(2)} b} = \overline{q_R^{(2)}} a \cdot q_R^{(1)} G_a^{(1)}.$$

Indeed, for every $p_L^{(1)} \in Q_L^{(1)}$,

$$\begin{aligned} (\overline{q_R^{(2)}} a \cdot q_R^{(1)} G_a^{(1)})(p_L^{(1)}) &= \overline{q_R^{(2)}}(a p_L^{(1)}) \cdot q_R^{(1)} G_a^{(1)} p_L^{(1)} \\ &= q_R^{(2)} b = \overline{q_R^{(2)} b}(p_L^{(1)}), \end{aligned}$$

hence (13) holds and so does (12). Therefore φ_L preserves the action and so is a morphism of left A_1-automata in view of Lemma 4.1.

To show that φ_L is onto, take

$$(q_L^{(1)}, \eta) \in Q_L^{(1)} \times {}^{Q_R^{(1)}}Q_L^{(32)} = Q_L^{((32)1)}.$$

We define $\delta^{(21)} \in {}^{Q_R^{(1)}}Q_L^{(2)}$ and $\eta_{q_R^{(1)}} \in {}^{Q_R^{(2)}}Q_L^{(3)}$ for each $q_R^{(1)} \in Q_R^{(1)}$ by

$$q_R^{(1)}\eta = (q_R^{(1)}\delta^{(21)}, \eta_{q_R^{(1)}}).$$

Finally, we define $\delta^{(3(21))} \in {}^{Q_R^{(21)}}Q_L^{(3)}$ by

$$(\gamma, q_R^{(1)})\delta^{(3(21))} = (\gamma(q_L^{(1)}))\eta_{q_R^{(1)}}$$

and show that

$$(q_L^{(1)}, \eta) = \varphi_L((q_L^{(1)}, \delta^{(21)}), \delta^{(3(21))}).$$

We have $\varphi_L((q_L^{(1)}, \delta^{(21)}), \delta^{(3(21))}) = (q_L^{(1)}, \delta^{((32)1)})$ with

$$q_R^{(1)}\delta^{((32)1)} = (q_R^{(1)}\delta^{(21)}, \varepsilon_{q_R^{(1)}}), \quad q_R^{(2)}\varepsilon_{q_R^{(1)}} = (\overline{q_R^{(2)}}, q_R^{(1)})\delta^{(3(21))}.$$

We must show that $\delta^{((32)1)} = \eta$, which follows from $\varepsilon_{q_R^{(1)}} = \eta_{q_R^{(1)}}$. Indeed,

$$q_R^{(2)}\varepsilon_{q_R^{(1)}} = (\overline{q_R^{(2)}}, q_R^{(1)})\delta^{(3(21))} = (\overline{q_R^{(2)}}(q_L^{(1)}))\eta_{q_R^{(1)}} = q_R^{(2)}\eta_{q_R^{(1)}}$$

and so φ_L is onto as claimed. $\square$

Theorem 4.3 $(\mathcal{B}^{(3)}\square\mathcal{B}^{(2)})\square\mathcal{B}^{(1)}$ *is a quotient of* $\mathcal{B}^{(3)}\square(\mathcal{B}^{(2)}\square\mathcal{B}^{(1)})$.

Proof. By Proposition 3.7, we have

$$\alpha_{\mathcal{B}^{(3(21))}} = \alpha_{\mathcal{B}^{(3)}}\alpha_{\mathcal{B}^{(21)}} = \alpha_{\mathcal{B}^{(3)}}\alpha_{\mathcal{B}^{(2)}}\alpha_{\mathcal{B}^{(1)}} = \alpha_{\mathcal{B}^{(32)}}\alpha_{\mathcal{B}^{(1)}} = \alpha_{\mathcal{B}^{((32)1)}}.$$

By Proposition 2.3 and Lemma 4.2, $\varphi = (\varphi_R, \varphi_L)$ is an onto morphism from $\mathcal{B}$ to $\mathcal{B}'$. $\square$

5 The trim block product

In general, given two trim bimachines $\mathcal{B}^{(1)}$ and $\mathcal{B}^{(2)}$, their block product is not trim, far from it. Since the lp-mapping defined by a bimachine uses only its trim part, it is a natural idea to consider

$$\mathcal{B}^{(2)}\square_{tr}\mathcal{B}^{(1)} = \mathrm{tr}(\mathcal{B}^{(2)}\square\mathcal{B}^{(1)}),$$

which we call the *trim block product* of $\mathcal{B}^{(2)}$ and $\mathcal{B}^{(1)}$.

Even though we chose to work with the unrestricted block product to make recursion easier, it is interesting to compare results on the two versions of the block product.

Lemma 5.1 *Let $\mathcal{B}^{(i)} = ((I_R^{(i)}, Q_R^{(i)}, S_R^{(i)}), f^{(i)}, (S_L^{(i)}, Q_L^{(i)}, I_L^{(i)}))$ be a trim A_1, A_2-bimachine for $i = 1, 2$. Then there exists a morphism $\varphi : \mathcal{B}^{(1)} \to \mathcal{B}^{(2)}$ if and only if the following conditions hold for all $u, v \in A_1^+$ and $a \in A_1$:*

(i) $I_R^{(1)}u = I_R^{(1)}v \Rightarrow I_R^{(2)}u = I_R^{(2)}v$;

(ii) $uI_L^{(1)} = vI_L^{(1)} \Rightarrow uI_L^{(2)} = vI_L^{(2)}$;

(iii) $u_{S_R^{(1)}} = v_{S_R^{(1)}} \Rightarrow u_{S_R^{(2)}} = v_{S_R^{(2)}}$;

(iv) $u_{S_L^{(1)}} = v_{S_L^{(1)}} \Rightarrow u_{S_L^{(2)}} = v_{S_L^{(2)}}$;

(v) $f^{(1)}(I_R^{(1)}u, a, vI_L^{(1)}) = f^{(2)}(I_R^{(2)}u, a, vI_L^{(2)})$.

Moreover, in this case the morphism is unique and surjective.

Proof. Assume that there exists a morphism $\varphi : \mathcal{B}^{(1)} \to \mathcal{B}^{(2)}$, say $\varphi = (\varphi_R, \varphi_L)$. Suppose that $I_R^{(1)}u = I_R^{(1)}v$. Then

$$\begin{aligned} I_R^{(2)}u &= \varphi_R(I_R^{(1)}) \cdot u = \varphi_R(I_R^{(1)}u) = \varphi_R(I_R^{(1)}v) \\ &= \varphi_R(I_R^{(1)}) \cdot v = I_R^{(2)}v \end{aligned}$$

and so (i) holds. Similarly, (ii) holds and the remaining conditions follow from the definition of morphism.

Conversely, assume that conditions (i) – (v) hold. By (i), and since $\mathcal{B}^{(1)}$ is trim, the mapping

$$\begin{aligned} \varphi_R : Q_R^{(1)} &\to Q_R^{(2)} \\ I_R^{(1)}u &\mapsto I_R^{(2)}u \qquad (u \in A_1^*) \end{aligned}$$

is well defined. It follows that φ_R is a morphism of right A_1-automata in view of (iii). Note that φ_R is the only possible morphism since it must satisfy $\varphi_R(I_R^{(1)}) = I_R^{(2)}$ and preserve the action, and is onto since $\mathcal{B}^{(2)}$ is trim.

Similarly, there exists a unique surjective morphism

$$\begin{aligned} \varphi_L : Q_L^{(1)} &\to Q_L^{(2)} \\ uI_L^{(1)} &\mapsto uI_L^{(2)} \qquad (u \in A_1^*) \end{aligned}$$

and $\varphi = (\varphi_R, \varphi_L)$ is a (unique) surjective morphism by (v). $\square$

By Lemma 5.1, we can define a partial order on the set of all trim A_1, A_2-bimachines (up to isomorphism) by

$$\mathcal{B}^{(1)} \geq \mathcal{B}^{(2)} \Leftrightarrow \text{there exists a morphism } \varphi : \mathcal{B}^{(1)} \to \mathcal{B}^{(2)}.$$

We can show that this partial order is compatible with the trim block product. We start by stating the analogues of Lemmas 3.2 and 3.3 for the trim block product:

Lemma 5.2 *For all $u, v \in A_1^+$, $I_R^{(21)}u = I_R^{(21)}v$ if and only if the following conditions hold:*

(i) $I_R^{(1)}u = I_R^{(1)}v$;

(ii) $\forall z \in A_1^+ \quad I_R^{(2)} \cdot I_R^{(1)} g_u^{(1)}(zI_L^{(1)}) = I_R^{(2)} \cdot I_R^{(1)} g_v^{(1)}(zI_L^{(1)})$.

Lemma 5.3 *For all $u, v \in A_1^+$, the equality $u = v$ holds in $S_R^{(21)}$ if and only if the following conditions hold:*

(i) $u = v$ *holds in* $S_R^{(1)}$;

(ii) $u = v$ *holds in* $S_L^{(1)}$;

(iii) $\forall w, z \in A_1^+$, $(I_R^{(1)}w)g_u^{(1)}(zI_L^{(1)}) = (I_R^{(1)}w)g_v^{(1)}(zI_L^{(1)})$ *holds in* $S_R^{(2)}$.

Proposition 5.4 *Let $\mathcal{B}^{(i)}, \mathcal{B}'^{(i)}$ be trim A_i, A_{i+1}-bimachines for $i = 1, 2$ such that $\mathcal{B}'^{(i)} \leq \mathcal{B}^{(i)}$. Then*

$$\mathcal{B}'^{(2)} \Box_{tr} \mathcal{B}'^{(1)} \leq \mathcal{B}^{(2)} \Box_{tr} \mathcal{B}^{(1)}.$$

Proof. We must show that conditions (i) – (v) of Lemma 5.1 hold for $\mathcal{B}^{(2)} \Box_{tr} \mathcal{B}^{(1)}$ and $\mathcal{B}'^{(2)} \Box_{tr} \mathcal{B}'^{(1)}$. Let $u, v \in A_1^*$.

Assume that $I_R^{(21)}u = I_R^{(21)}v$. By Lemma 5.2, this is equivalent to

$$I_R^{(1)}u = I_R^{(1)}v \ \wedge \ \forall w \in A_1^* \ \ (I_R^{(2)} \cdot I_R^{(1)} g_u^{(1)}(wI_L^{(1)}) = I_R^{(2)} \cdot I_R^{(1)} g_v^{(1)}(wI_L^{(1)})). \tag{14}$$

Similarly, ${I'_R}^{(21)}u = {I'_R}^{(21)}v$ is equivalent to

$${I'_R}^{(1)}u = {I'_R}^{(1)}v \ \wedge \ \forall w \in A_1^* \ \ ({I'_R}^{(2)} \cdot {I'_R}^{(1)} {g'_u}^{(1)}(w{I'_L}^{(1)}) = {I'_R}^{(2)} \cdot {I'_R}^{(1)} {g'_v}^{(1)}(w{I'_L}^{(1)})). \tag{15}$$

Since $\mathcal{B}'^{(1)} \leq \mathcal{B}^{(1)}$, condition (v) of Lemma 5.1 yields

$$f^{(1)}(I_R^{(1)}u', a, v'I_L^{(1)}) = f'^{(1)}({I'_R}^{(1)}u', a, v'{I'_L}^{(1)})$$

for all $u', v' \in A_1^*$ and $a \in A_1$. In view of Lemma 3.4, we obtain

$$(I_R^{(1)}z)g_u^{(1)}(wI_L^{(1)}) = ({I'_R}^{(1)}z){g'_u}^{(1)}(w{I'_L}^{(1)}) \tag{16}$$

for all $z, w \in A_1^*$. Since $\mathcal{B}'^{(i)} \leq \mathcal{B}^{(i)}$ for $i = 1, 2$, condition (i) of Lemma 5.1 yields

$$I_R^{(1)}u = I_R^{(1)}v \Rightarrow {I'_R}^{(1)}u = {I'_R}^{(1)}v \tag{17}$$

and condition (iii) of Lemma 5.1 yields

$$I_R^{(2)}u' = I_R^{(2)}v' \Rightarrow {I'_R}^{(2)}u' = {I'_R}^{(2)}v'. \tag{18}$$

Taking $u' = I_R^{(1)}g_u^{(1)}(wI_L^{(1)})$ and $v' = I_R^{(1)}g_v^{(1)}(wI_L^{(1)})$, it follows from (16), (17) and (18) that (14) implies (15). Thus condition (i) of Lemma 5.1 holds.

Condition (ii) follows by duality.

We next check (iii). We assume that $u = v$ holds in $S_R^{(21)}$. By Lemma 5.3, $u = v$ holds in both S_R^1 and S_L^1, and also $g_u^{(1)} = g_v^{(1)}$. Since $\mathcal{B}'^{(1)} \leq \mathcal{B}^{(1)}$, Lemma 5.1(iii) and (iv) yield

$$u_{S'_R{}^1} = v_{S'_R{}^1}, \quad u_{S'_L{}^1} = v_{S'_L{}^1}.$$

Furthermore, in view of (16), and since the bimachines $\mathcal{B}^{(1)}$ and $\mathcal{B}'^{(1)}$ are trim, $g_u^{(1)} = g_v^{(1)}$ yields $g'_u{}^{(1)} = g'_v{}^{(1)}$ and so $u = v$ holds in $S'_R{}^{(21)}$.

Condition (iv) follows by duality.

Finally, take $u, v \in A_1^*$ and $a \in A_1$. Since $\mathcal{B}'^{(i)} \leq \mathcal{B}^{(i)}$, Lemma 5.1(v) yields

$$f^{(i)}(I_R^{(i)}u', a, v'I_L^{(i)}) = f'^{(i)}(I'_R{}^{(i)}u', a, v'I'_L{}^{(i)})$$

for all $u', v' \in A_i^*$. Together with (16) and its dual, this implies

$$\begin{aligned}
&f^{(21)}(I_R^{(21)}u, a, vI_L^{(21)}) \\
&= f^{(21)}((\gamma_0 u \cdot I_R^{(1)}g_u^{(1)}, I_R^{(1)}u), a, (vI_L^{(1)}, h_v^{(1)}I_L^{(1)} \cdot v\delta_0)) \\
&= f^{(2)}(I_R^{(2)} \cdot I_R^{(1)}g_u^{(1)}(avI_L^{(1)}), f^{(1)}(I_R^{(1)}u, a, vI_L^{(1)}), (I_R^{(1)}ua)h_v^{(1)}I_L^{(1)} \cdot I_L^{(2)}) \\
&= f'^{(2)}(I'_R{}^{(2)} \cdot I'_R{}^{(1)}g'_u{}^{(1)}(avI'_L{}^{(1)}), f'^{(1)}(I'^{(1)}_R u, a, vI'_L{}^{(1)}), (I'_R{}^{(1)}ua)h'_v{}^{(1)}I'_L{}^{(1)} \cdot I'_L{}^{(2)}) \\
&= f'^{(21)}((\gamma'_0 u \cdot I'_R{}^{(1)}g'_u{}^{(1)}, I'_R{}^{(1)}u), a, (vI'_L{}^{(1)}, h'_v{}^{(1)}I'_L{}^{(1)} \cdot v\delta'_0)) \\
&= f'^{(21)}(I'_R{}^{(21)}u, a, vI'_L{}^{(21)})
\end{aligned}$$

and so condition (v) holds as required. □

We consider next the trim block product of three bimachines. *Surprisingly enough, the relation of Theorem 4.3 is reversed.*

We start with a rather technical lemma.

Lemma 5.5 *For all $u \in A_1^+$ and $w, z \in A_1^+$,*

$$(I_R^{(21)}w)G_u^{(21)}(zI_L^{(21)}) = (I_R^{(2)} \cdot I_R^{(1)}G_w^{(1)}(uzI_L^{(1)}))G^{(2)}_{(I_R^{(1)}w)G_u^{(1)}(zI_L^{(1)})}((I_R^{(1)}wu)G_z^{(1)}I_L^{(1)} \cdot I_L^{(2)}).$$

Proof. We have by (3), (4), Lemmas 3.4 and 3.1

$$
\begin{aligned}
&(I_R^{(21)}w)G_u^{(21)}(zI_L^{(21)})\\
&=\textstyle\prod_{i=1}^{|u|} f^{(21)}(I_R^{(21)}w\lambda_i(u),\sigma_i(u),\mu_i(u)zI_L^{(21)})\\
&=\textstyle\prod_{i=1}^{|u|} f^{(21)}((\gamma_0 w\lambda_i(u)\cdot I_R^{(1)}G^{(1)}_{w\lambda_i(u)}, I_R^{(1)}w\lambda_i(u)),\sigma_i(u),\\
&\qquad (\mu_i(u)zI_L^{(1)}, G^{(1)}_{\mu_i(u)z}I_L^{(1)}\cdot\mu_i(u)z\delta_0))\\
&=\textstyle\prod_{i=1}^{|u|} f^{(2)}(I_R^{(2)}\cdot I_R^{(1)}G^{(1)}_{w\lambda_i(u)}(\sigma_i(u)\mu_i(u)zI_L^{(1)}), f^{(1)}(I_R^{(1)}w\lambda_i(u),\sigma_i(u),\mu_i(u)zI_L^{(1)}),\\
&\qquad (I_R^{(1)}w\lambda_i(u)\sigma_i(u))G^{(1)}_{\mu_i(u)z}I_L^{(1)}\cdot I_L^{(2)})\\
&=\textstyle\prod_{i=1}^{|u|} f^{(2)}(I_R^{(2)}\cdot I_R^{(1)}G_w^{(1)}(uzI_L^{(1)})\cdot(I_R^{(1)}w)G^{(1)}_{\lambda_i(u)}(\sigma_i(u)\mu_i(u)zI_L^{(1)}),\\
&\qquad f^{(1)}(I_R^{(1)}w\lambda_i(u),\sigma_i(u),\mu_i(u)zI_L^{(1)}),\\
&\qquad\qquad (I_R^{(1)}w\lambda_i(u)\sigma_i(u))G^{(1)}_{\mu_i(u)}(zI_L^{(1)})\cdot(I_R^{(1)}wu)G_z^{(1)}I_L^{(1)}\cdot I_L^{(2)}).
\end{aligned}
$$

On the other hand, for $x=(I_R^{(1)}w)G_u^{(1)}(zI_L^{(1)})$, we have

$$
\begin{aligned}
&(I_R^{(2)}\cdot I_R^{(1)}G_w^{(1)}(uzI_L^{(1)}))G_x^{(2)}((I_R^{(1)}wu)G_z^{(1)}I_L^{(1)}\cdot I_L^{(2)})\\
&=\textstyle\prod_{i=1}^{|x|} f^{(2)}(I_R^{(2)}\cdot I_R^{(1)}G_w^{(1)}(uzI_L^{(1)})\cdot\lambda_i(x),\sigma_i(x),\mu_i(x)\cdot(I_R^{(1)}wu)G_z^{(1)}I_L^{(1)}\cdot I_L^{(2)}).
\end{aligned}
$$

Since $|x|=|u|$, it suffices to show that

$$\lambda_i(x)=(I_R^{(1)}w)G^{(1)}_{\lambda_i(u)}(\sigma_i(u)\mu_i(u)zI_L^{(1)}), \tag{19}$$

$$\sigma_i(x)=f^{(1)}(I_R^{(1)}w\lambda_i(u),\sigma_i(u),\mu_i(u)zI_L^{(1)}), \tag{20}$$

$$\mu_i(x)=(I_R^{(1)}w\lambda_i(u)\sigma_i(u))G^{(1)}_{\mu_i(u)}(zI_L^{(1)}). \tag{21}$$

By Lemma 3.4 we have

$$x=\prod_{j=1}^{|u|} f^{(2)}(I_R^{(1)}w\lambda_j(u),\sigma_j(u),\mu_j(u)zI_L^{(1)}),$$

hence

$$
\begin{aligned}
\lambda_i(x)&=\textstyle\prod_{j=1}^{i-1} f^{(2)}(I_R^{(1)}w\lambda_j(u),\sigma_j(u),\mu_j(u)zI_L^{(1)})\\
&=\textstyle\prod_{j=1}^{|\lambda_i|} f^{(2)}(I_R^{(1)}w\lambda_j(\lambda_i(u)),\sigma_j(\lambda_i(u)),\mu_j(\lambda_i(u))\sigma_i(u)\mu_i(u)zI_L^{(1)})\\
&=(I_R^{(1)}w)G^{(1)}_{\lambda_i(u)}(\sigma_i(u)\mu_i(u)zI_L^{(1)})
\end{aligned}
$$

and (19) holds.

The proofs for (20) and (21) are completely similar and are therefore omitted. □

Theorem 5.6 *$\mathcal{B}^{(3)}\Box_{tr}(\mathcal{B}^{(2)}\Box_{tr}\mathcal{B}^{(1)})$ is a quotient of $(\mathcal{B}^{(3)}\Box_{tr}\mathcal{B}^{(2)})\Box_{tr}\mathcal{B}^{(1)}$.*

Proof. We must show that conditions (i) – (v) of Lemma 5.1 hold for $(\mathcal{B}^{(3)}\Box_{tr}\mathcal{B}^{(2)})\Box_{tr}\mathcal{B}^{(1)}$ and $\mathcal{B}^{(3)}\Box_{tr}(\mathcal{B}^{(2)}\Box_{tr}\mathcal{B}^{(1)})$. We adapt the notation of Section 4 by writing

$$\begin{aligned}\mathcal{B}^{(3(21))} &= \mathcal{B}^{(3)}\Box_{tr}(\mathcal{B}^{(2)}\Box_{tr}\mathcal{B}^{(1)})\\ &= ((I_R^{(3(21))}, Q_R^{(3(21))}, S_R^{(3(21))}), f^{(3(21))}, (S_L^{(3(21))}, Q_L^{(3(21))}, I_L^{(3(21))}))\end{aligned}$$

and so on.

Let $u, v \in A_1^+$. By Lemma 5.2 and (3), we have $I_R^{((32)1)}u = I_R^{((32)1)}v$ if and only if

$$I_R^{(1)}u = I_R^{(1)}v, \tag{22}$$

$$\forall z \in A_1^+ \qquad I_R^{(32)} \cdot I_R^{(1)} G_u^{(1)}(zI_L^{(1)}) = I_R^{(32)} \cdot I_R^{(1)} G_v^{(1)}(zI_L^{(1)}). \tag{23}$$

Again by Lemma 5.2, (23) is equivalent to

$$\forall z \in A_1^+ \qquad I_R^{(2)} \cdot I_R^{(1)} G_u^{(1)}(zI_L^{(1)}) = I_R^{(2)} \cdot I_R^{(1)} G_v^{(1)}(zI_L^{(1)}), \tag{24}$$

$$\forall z \in A_1^+ \, \forall t \in A_2^+ \qquad I_R^{(3)} \cdot I_R^{(2)} G^{(2)}_{I_R^{(1)} G_u^{(1)}(zI_L^{(1)})}(tI_L^{(2)}) = I_R^{(3)} \cdot I_R^{(2)} G^{(2)}_{I_R^{(1)} G_v^{(1)}(zI_L^{(1)})}(tI_L^{(2)}). \tag{25}$$

On the other hand, $I_R^{(3(21))}u = I_R^{(3(21))}v$ if and only if

$$I_R^{(21)}u = I_R^{(21)}v, \tag{26}$$

$$\forall z \in A_1^+ \qquad I_R^{(3)} \cdot I_R^{(21)} G_u^{(21)}(zI_L^{(21)}) = I_R^{(3)} \cdot I_R^{(21)} G_v^{(21)}(zI_L^{(21)}). \tag{27}$$

By Lemma 5.5, (27) is equivalent to

$$\begin{aligned}\forall z \in A_1^+ \qquad & I_R^{(3)} \cdot (I_R^{(2)} G^{(2)}_{I_R^{(1)} G_u^{(1)}(zI_L^{(1)})}((I_R^{(1)}u) G_z^{(1)} I_L^{(1)} \cdot I_L^{(2)})\\ &= I_R^{(3)} \cdot (I_R^{(2)} G^{(2)}_{I_R^{(1)} G_v^{(1)}(zI_L^{(1)})}((I_R^{(1)}v) G_z^{(1)} I_L^{(1)} \cdot I_L^{(2)}).\end{aligned} \tag{28}$$

Thus we must show that (22), (24) and (25) together imply (26) and (28). Now (22) and (24) imply (26) by Lemma 5.2 and (28) follows from (25) and (22) taking $t = (I_R^{(1)}u) G_z^{(1)} I_L^{(1)}$. Therefore condition (i) of Lemma 5.1 is satisfied. Condition (ii) holds by duality.

By Lemma 5.3, the equality $u = v$ holds in $S_R^{((32)1)}$ if and only if

$$u = v \text{ holds in } S_R^{(1)}, \tag{29}$$

$$u = v \text{ holds in } S_L^{(1)}, \tag{30}$$

$$\forall w, z \in A_1^+ \qquad (I_R^{(1)}w) G_u^{(1)}(zI_L^{(1)}) = (I_R^{(1)}w) G_v^{(1)}(zI_L^{(1)}) \text{ holds in } S_R^{(32)}. \tag{31}$$

Again by Lemma 5.3, (31) is equivalent to

$$\forall w, z \in A_1^+ \qquad (I_R^{(1)}w) G_u^{(1)}(zI_L^{(1)}) = (I_R^{(1)}w) G_v^{(1)}(zI_L^{(1)}) \text{ holds in } S_R^{(2)}, \tag{32}$$

$$\forall w, z \in A_1^+ \qquad (I_R^{(1)} w) G_u^{(1)}(z I_L^{(1)}) = (I_R^{(1)} w) G_v^{(1)}(z I_L^{(1)}) \text{ holds in } S_R^{(2)}, \tag{33}$$

$$\forall w, z \in A_1^+ \ \forall s, t \in A_2^+ \qquad (I_R^{(2)} s) G^{(2)}_{(I_R^{(1)} w) G_u^{(1)}(z I_L^{(1)})}(t I_L^{(2)})$$
$$= (I_R^{(2)} s) G^{(2)}_{(I_R^{(1)} w) G_v^{(1)}(z I_L^{(1)})}(t I_L^{(2)}) \text{ holds in } S_R^{(3)}. \tag{34}$$

On the other hand, the equality $u = v$ holds in $S_R^{(3(21))}$ if and only if

$$u = v \text{ holds in } S_R^{(21)}, \tag{35}$$

$$u = v \text{ holds in } S_L^{(21)}, \tag{36}$$

$$\forall w, z \in A_1^+ \qquad (I_R^{(21)} w) G_u^{(21)}(z I_L^{(21)}) = (I_R^{(21)} w) G_v^{(21)}(z I_L^{(21)}) \text{ holds in } S_R^{(3)}. \tag{37}$$

Thus we must show that (29), (30), (32), (33) and (34) together imply (35), (36) and (37).

By Lemma 5.3, (35) follows from (29), (30) and (32). By its dual, (36) follows from (29), (30) and (33). By Lemma 5.5, (37) follows from (34) by taking

$$s = I_R^{(1)} G_w^{(1)}(u z I_L^{(1)}), \quad t = (I_R^{(1)} w u) G_z^{(1)} I_L^{(1)},$$

since (29) (respectively (30)) implies $I_R^{(1)} w u = I_R^{(1)} w v$ (respectively $u z I_L^{(1)} = v z I_L^{(1)}$).

Thus condition (iii) of Lemma 5.1 is satisfied. Condition (iv) holds by duality.

Clearly, these four conditions are sufficient to imply the existence of morphisms of right and left automata between our bimachines. Now condition (v) can be derived from Proposition 2.3 as in the proof of Theorem 4.3. □

We remark that there is no analogue of Lemma 4.1 here, its arguments being not valid anymore.

Assume now that $\mathcal{B}^{(i)}$ is a trim A_i, A_{i+1}-bimachine for $i = 1, \ldots, t$. Let

$$\mathrm{bra}_{tr}(\mathcal{B}^{(t)}, \ldots, \mathcal{B}^{(1)})$$

be the set of all A_1, A_{t+1}-bimachines obtained by different bracketings of the expression

$$\mathcal{B}^{(t)} \Box_{tr} \mathcal{B}^{(n-1)} \Box_{tr} \ldots \Box_{tr} \mathcal{B}^{(2)} \Box_{tr} \mathcal{B}^{(1)}.$$

Proposition 5.7 *(i)* $(\ldots((\mathcal{B}^{(t)} \Box_{tr} \mathcal{B}^{(n-1)}) \Box_{tr} \mathcal{B}^{(t-2)}) \Box_{tr} \ldots \Box_{tr} \mathcal{B}^{(2)}) \Box_{tr} \mathcal{B}^{(1)}$ *is the maximum element of* $bra_{tr}(\mathcal{B}^{(t)}, \ldots, \mathcal{B}^{(1)})$ *for the partial order* $\leq$*;*

(ii) $\mathcal{B}^{(t)} \Box_{tr} (\mathcal{B}^{(n-1)} \Box_{tr} (\mathcal{B}^{(t-2)} \Box_{tr} \ldots \Box_{tr} (\mathcal{B}^{(2)} \Box_{tr} \mathcal{B}^{(1)}) \ldots))$ *is the minimum element of* $bra_{tr}(\mathcal{B}^{(t)}, \ldots, \mathcal{B}^{(1)})$ *for the partial order* $\leq$*.*

Proof. We prove (i) by induction on t. The case $t = 2$ being trivial, assume that $t \geq 3$ and (i) holds for smaller t. Let $X \in \mathrm{bra}_{tr}(\mathcal{B}^{(t)}, \ldots, \mathcal{B}^{(1)})$. We may write $X = Y \Box_{tr} Z$ with $Y \in \mathrm{bra}_{tr}(\mathcal{B}^{(t)}, \ldots, \mathcal{B}^{(i+1)})$, $Z \in \mathrm{bra}_{tr}(\mathcal{B}^{(i)}, \ldots, \mathcal{B}^{(1)})$ and $i \in \{1, \ldots, t-1\}$. We use a secondary induction on i.

Suppose that $i = 1$. By the induction hypothesis on t, we have

$$Y \leq (\ldots(\mathcal{B}^{(t)} \Box_{tr} \mathcal{B}^{(t-1)}) \Box_{tr} \ldots \Box_{tr} \mathcal{B}^{(3)}) \Box_{tr} \mathcal{B}^{(2)}$$

and so

$$X = Y \Box_{tr} \mathcal{B}^{(1)} \leq (\ldots(\mathcal{B}^{(t)} \Box_{tr} \mathcal{B}^{(t-1)}) \Box_{tr} \ldots \Box_{tr} \mathcal{B}^{(2)}) \Box_{tr} \mathcal{B}^{(1)}$$

by Proposition 5.4.

Suppose now that $i \in \{2, \ldots, t-1\}$ and

$$Y' \Box_{tr} Z' \leq (\ldots(\mathcal{B}^{(t)} \Box_{tr} \mathcal{B}^{(t-1)}) \Box_{tr} \ldots \Box_{tr} \mathcal{B}^{(2)}) \Box_{tr} \mathcal{B}^{(1)}$$

whenever $Y' \in \mathrm{bra}_{tr}(\mathcal{B}^{(t)}, \ldots, \mathcal{B}^{(j+1)})$, $Z' \in \mathrm{bra}_{tr}(\mathcal{B}^{(j)}, \ldots, \mathcal{B}^{(1)})$ and $j \in \{1, \ldots, i-1\}$.

We may write $Z = Z'' \Box_{tr} Z'$ with $Z'' \in \mathrm{bra}_{tr}(\mathcal{B}^{(i)}, \ldots, \mathcal{B}^{(j+1)})$, $Z' \in \mathrm{bra}_{tr}(\mathcal{B}^{(j)}, \ldots, \mathcal{B}^{(1)})$ and $j \in \{1, \ldots, i-1\}$. By Theorem 5.6, we have

$$X = Y \Box_{tr} (Z'' \Box_{tr} Z') \leq (Y \Box_{tr} Z'') \Box_{tr} Z'$$

and so the induction hypothesis on i yields

$$X \leq (Y \Box_{tr} Z'') \Box_{tr} Z' \leq (\ldots(\mathcal{B}^{(t)} \Box_{tr} \mathcal{B}^{(t-1)}) \Box_{tr} \ldots \Box_{tr} \mathcal{B}^{(2)}) \Box_{tr} \mathcal{B}^{(1)}$$

as required.

Condition (ii) follows by duality. $\Box$

We end this section by showing that the trimming operator commutes with the (iterated) block product when we consider bracketing *from left to right*.

Lemma 5.8 *Let $\mathcal{B}^{(i)}$ be an A_i, A_{i+1}-bimachine for $i = 1, 2$. Then*

$$\mathcal{B}^{(2)} \Box_{tr} \mathcal{B}^{(1)} = tr(\mathcal{B}^{(2)}) \Box_{tr} \mathcal{B}^{(1)}.$$

Proof. Write

$$\mathcal{B}^{(i)} = ((I_R^{(i)}, Q_R^{(i)}, S_R^{(i)}), f^{(i)}, (S_L^{(i)}, Q_L^{(i)}, I_L^{(i)}))$$

for $i = 1, 2$. Since trimming a bimachine does not affect the R and L semigroups, it follows that $\mathcal{B}^{(2)} \Box \mathcal{B}^{(1)}$ and $\mathrm{tr}(\mathcal{B}^{(2)}) \Box \mathcal{B}^{(1)}$ share the same R (respectively L) semigroup $S_R^{(21)}$ (respectively $S_L^{(21)}$) and so do $\mathcal{B}^{(2)} \Box_{\mathrm{tr}} \mathcal{B}^{(1)} = \mathrm{tr}(\mathcal{B}^{(2)} \Box \mathcal{B}^{(1)})$ and $\mathrm{tr}(\mathcal{B}^{(2)}) \Box_{\mathrm{tr}} \mathcal{B}^{(1)} = \mathrm{tr}(\mathrm{tr}(\mathcal{B}^{(2)}) \Box \mathcal{B}^{(1)})$.

The R initial state of $\mathcal{B}^{(2)} \Box_{\mathrm{tr}} \mathcal{B}^{(1)}$ is $(\gamma_0, I_R^{(1)})$, where $\gamma_0 : Q_L^{(1)} \to Q_R^{(2)}$ is the constant mapping with image $I_R^{(2)}$. This same $(\gamma_0, I_R^{(1)})$, viewing γ_0 as a mapping from $Q_L^{(1)}$ to $I_R^{(2)} S_R^{(2)}$, is the R initial state of $\mathrm{tr}(\mathcal{B}^{(2)}) \Box_{\mathrm{tr}} \mathcal{B}^{(1)}$. Clearly, the action of $S_R^{(21)}$ on $(I_R^{(2)} S_R^{(2)})^{Q_L^{(1)}} \times Q_R^{(1)}$ is a restriction of its action on $Q_R^{(2)^{Q_L^{(1)}}} \times Q_R^{(1)}$. Hence $(\gamma_0, I_R^{(1)}) S_R^{(21)}$ is the same R state set in both $\mathcal{B}^{(2)} \Box_{\mathrm{tr}} \mathcal{B}^{(1)}$ and $\mathrm{tr}(\mathcal{B}^{(2)}) \Box_{\mathrm{tr}} \mathcal{B}^{(1)}$. Analogous results hold for the L states.

Finally, we remark that the output function of $\mathrm{tr}(\mathcal{B}^{(2)}) \Box \mathcal{B}^{(1)}$ is a restriction of the output function of $\mathcal{B}^{(2)} \Box \mathcal{B}^{(1)}$. Since the output function of a trimmed bimachine is obtained by

restriction to the new state sets and these coincide in our case, it follows that $\mathcal{B}^{(2)}\Box_{\mathrm{tr}}\mathcal{B}^{(1)}$ and $\mathrm{tr}(\mathcal{B}^{(2)})\Box_{\mathrm{tr}}\mathcal{B}^{(1)}$ must share the same output function. Therefore

$$\mathcal{B}^{(2)}\Box_{\mathrm{tr}}\mathcal{B}^{(1)} = \mathrm{tr}(\mathcal{B}^{(2)})\Box_{\mathrm{tr}}\mathcal{B}^{(1)}.$$

□

Proposition 5.9 *Let $\mathcal{B}^{(i)}$ be an A_i, A_{i+1}-bimachine for $i = 1, t$. Then*

$$((\ldots(\mathcal{B}^{(t)}\Box_{tr}\mathcal{B}^{(t-1)})\Box_{tr}\mathcal{B}^{(t-2)})\Box_{tr}\ldots)\Box_{tr}\mathcal{B}^{(1)} = tr(((\ldots(\mathcal{B}^{(t)}\Box\mathcal{B}^{(t-1)})\Box\mathcal{B}^{(t-2)})\Box\ldots)\Box\mathcal{B}^{(1)}).$$

Proof. We use induction on t. The case $t = 2$ follows from the definition of $\Box_{\mathrm{tr}}$, so we assume that $t > 2$ and the claim holds for fewer bimachines. Let

$$\mathcal{B} = ((\ldots(\mathcal{B}^{(t)}\Box\mathcal{B}^{(t-1)})\Box\mathcal{B}^{(t-2)})\Box\ldots)\Box\mathcal{B}^{(2)}.$$

By the induction hypothesis, we have

$$\mathrm{tr}(\mathcal{B}) = ((\ldots(\mathcal{B}^{(t)}\Box_{\mathrm{tr}}\mathcal{B}^{(t-1)})\Box_{\mathrm{tr}}\mathcal{B}^{(t-2)})\Box_{\mathrm{tr}}\ldots)\Box_{\mathrm{tr}}\mathcal{B}^{(2)}.$$

Thus Lemma 5.8 yields

$$\begin{aligned}((\ldots(\mathcal{B}^{(t)}\Box_{\mathrm{tr}}\mathcal{B}^{(t-1)})\Box_{\mathrm{tr}}\mathcal{B}^{(t-2)})\Box_{\mathrm{tr}}\ldots)\Box_{\mathrm{tr}}\mathcal{B}^{(1)} &= \mathrm{tr}(\mathcal{B})\Box_{\mathrm{tr}}\mathcal{B}^{(1)} = \mathcal{B}\Box_{\mathrm{tr}}\mathcal{B}^{(1)}\\ &= \mathrm{tr}(\mathcal{B}\Box\mathcal{B}^{(1)}) = \mathrm{tr}(((\ldots(\mathcal{B}^{(t)}\Box\mathcal{B}^{(t-1)})\Box\mathcal{B}^{(t-2)})\Box\ldots)\Box\mathcal{B}^{(1)})\end{aligned}$$

and the claim holds. □

6 Iterating the block product

We intend to compose an arbitrary number of bimachines via block product. Since the block product is not associative, we must choose the bracketing to be considered. Our choice is bracketing from left to right, that is, priority is assumed to hold from left to right. In the case of three bimachines, this means that $(\mathcal{B}^{(3)}\Box\mathcal{B}^{(2)})\Box\mathcal{B}^{(1)}$ is our option. Note that, in view of Proposition 5.9, this allows us to revert to the trim block product at any moment just by trimming the final bimachine.

Let $\mathcal{B}^{(i)} = ((I_R^{(i)}, Q_R^{(i)}, S_R^{(i)}), f^{(i)}, (S_L^{(i)}, Q_L^{(i)}, I_L^{(i)}))$ be an A_i, A_{i+1}-bimachine for $i = 1, \ldots, t$. Then

$$\mathcal{B}^{[t,k]} = ((\ldots(\mathcal{B}^{(t)}\Box\mathcal{B}^{(t-1)})\Box\mathcal{B}^{(t-2)})\Box\ldots)\Box\mathcal{B}^{(k)}.$$

More generally, we shall use $[t, k]$ as a superscript to refer to any of the components associated with the bimachine $\mathcal{B}^{[t,k]}$.

We can prove a one-sided version of Proposition 5.7. Let $\mathcal{B}^{(i)}$ be an A_i, A_{i+1}-bimachine for $i = 1, \ldots, t$. Let

$$\mathrm{bra}(\mathcal{B}^{(t)}, \ldots, \mathcal{B}^{(1)})$$

be the set of all A_1, A_{t+1}-bimachines obtained by different bracketings of the expression

$$\mathcal{B}^{(t)}\Box\mathcal{B}^{(t-1)}\Box\ldots\Box\mathcal{B}^{(2)}\Box\mathcal{B}^{(1)}.$$

Proposition 6.1 *$\mathcal{B}^{[t,1]}$ is a quotient of every element of $bra(\mathcal{B}^{(t)}, \ldots, \mathcal{B}^{(1)})$ for the partial order $\leq$.*

Proof. We use induction on t. The case $t = 2$ being trivial, assume that $t \geq 3$ and the claim holds for smaller t. Let $X \in \mathrm{bra}(\mathcal{B}^{(t)}, \ldots, \mathcal{B}^{(1)})$. We may write $X = Y \Box Z$ with $Y \in \mathrm{bra}(\mathcal{B}^{(t)}, \ldots, \mathcal{B}^{(i+1)})$, $Z \in \mathrm{bra}(\mathcal{B}^{(i)}, \ldots, \mathcal{B}^{(1)})$ and $i \in \{1, \ldots, t-1\}$. We use a secondary induction on i.

Suppose that $i = 1$. By the induction hypothesis on t, $\mathcal{B}^{[t,2]}$ is a quotient of Y and so $X = Y \Box \mathcal{B}^{(1)}$ is a quotient of $\mathcal{B}^{[t,2]} \Box \mathcal{B}^{(1)}$ by Proposition 3.8.

Suppose now that $i \in \{2, \ldots, t-1\}$ and $\mathcal{B}^{[t,1]}$ is a quotient of $Y' \Box Z'$ whenever $Y' \in \mathrm{bra}(\mathcal{B}^{(t)}, \ldots, \mathcal{B}^{(j+1)})$, $Z' \in \mathrm{bra}(\mathcal{B}^{(j)}, \ldots, \mathcal{B}^{(1)})$ and $j \in \{1, \ldots, i-1\}$.

We may write $Z = Z'' \Box Z'$ with $Z'' \in \mathrm{bra}(\mathcal{B}^{(i)}, \ldots, \mathcal{B}^{(j+1)})$, $Z' \in \mathrm{bra}(\mathcal{B}^{(j)}, \ldots, \mathcal{B}^{(1)})$ and $j \in \{1, \ldots, i-1\}$. By Theorem 4.3, we have that $(Y \Box Z'') \Box Z'$ is a quotient of $X = Y \Box (Z'' \Box_{tr} Z')$ and so the induction hypothesis on i yields that $\mathcal{B}^{[t,1]}$ is a quotient of $(Y \Box Z'') \Box Z'$ and therefore of X as required. $\Box$

It will be convenient to develop an alternative characterization of the states in the iterated block product. For all $t \geq 2$ and $k \in \{1, \ldots, t-1\}$, we define

$$P_R^{[t,k]} = Q_R^{(t)^{Q_L^{(k)} \times Q_L^{(k+1)} \times \ldots \times Q_L^{(t-1)}}} \times Q_R^{(t-1)^{Q_L^{(k)} \times Q_L^{(k+1)} \times \ldots \times Q_L^{(t-2)}}}$$

$$\times \ldots \times Q_R^{(k+1)^{Q_L^{(k)}}} \times Q_R^{(k)}.$$

Dually, we define

$$P_L^{[t,k]} = Q_L^{(k)} \times {}^{Q_R^{(k)}} Q_L^{(k+1)} \times \ldots \times {}^{Q_R^{(t-1)} \times \ldots \times Q_R^{(k)}} Q_L^{(t)}.$$

The elements of $P_R^{[t,1]}$ and $P_L^{[t,1]}$ are termed respectively *R-generalized* and *L-generalized n step crossing sequences*.

Intuition for the following material is as follows (see Section 9 for more details). $\mathcal{B}^{[t,1]}$ is already defined, but to further see what it is and what the action is, we first note the states are in one-to-one correspondence with

$$\begin{aligned} P_R^{[t,1]} &= Q_R^{(t)^{Q_L^{(1)} \times \cdots \times Q_L^{(t-1)}}} \times Q_R^{(t-1)^{Q_L^{(1)} \times \cdots \times Q_L^{(t-2)}}} \times \cdots \times Q_R^{(3)^{Q_L^{(1)} \times Q_L^{(2)}}} \times Q_R^{(2)^{Q_L^{(1)}}} \times Q_R^{(1)} \\ &\cong_{(\text{bijection})} (P_R^{[t-2,1]})^{Q_L^{(1)}} \times Q_R^{(1)} \end{aligned}$$

and we write a member of this as $(\gamma, q_R^{(1)})$, where $\gamma : Q_L^{(1)} \to \mathcal{P}_R^{[t-2,1]}$.

Now we use the same formula as we did for the action of $\mathcal{B}^{(2)} \Box \mathcal{B}^{(1)}$ (see the beginning of Section 3), namely for $a \in A$,

$$\begin{aligned} (\gamma, q_R^{(1)}) \cdot a &\cong (\gamma, q_R^{(1)}) \begin{pmatrix} a_L = a_{S_L^{(1)}} & 0 \\ g_a^{(1)} & a_R = a_{S_R^{(1)}} \end{pmatrix} \\ &= (\gamma a_L(\ ) \cdot q_R^{(1)} g_a^{(1)}, q_R^{(1)} a_R), \end{aligned}$$

where $\cdot$ is the inductively defined action on $P_R^{[t,2]}$. It is as simple as that. The details will follow.

Whenever $x = (x_n, \dots, x_2, x_1)$ we shall write $\pi_j(x) = x_j$ for $j = 1, \dots, n$. Dually, for $y = (y_1, \dots, y_n)$ we shall write $y\pi_j = y_j$ for $j = 1, \dots, n$.

Clearly (see the beginning of Section 6),

$$P_R^{[t,t-1]} = Q_R^{[t,t-1]}, \quad P_L^{[t,t-1]} = Q_L^{[t,t-1]}.$$

We define next a natural bijection $\theta_R^{[t,k]} : Q_R^{[t,k]} \to P_R^{[t,k]}$ by induction on k. We take $\theta_R^{[t,t-1]}$ to be the identity maping. Assume that $k \in \{1, \dots, t-2\}$ and $\theta_R^{[t,k+1]}$ is defined. Since the bimachines are arbitrary, $\theta_R^{[t,k]}$ is essentially the same as $\theta_R^{[t-k+1,1]}$, thus we shall assume that $k = 1$ for the sake of notation.

Let

$$(\gamma, \gamma_1) \in {Q_R^{[t,2]}}^{Q_L^{(1)}} \times Q_R^{(1)} = Q_R^{[t,1]}.$$

We proceed to define $\theta_R^{[t,1]}(\gamma, \gamma_1)$ in three steps.

Step 1: We use the bijection $\theta_R^{[t,2]}$ to define a bijection

$$Q_R^{[t,1]} = {Q_R^{[t,2]}}^{Q_L^{(1)}} \times Q_R^{(1)} \to {P_R^{[t,2]}}^{Q_L^{(1)}} \times Q_R^{(1)}.$$

More precisely, we consider $(\gamma, \gamma_1) \mapsto (\widehat{\gamma}, \gamma_1)$ where

$$\widehat{\gamma}(q_L^{(1)}) = \theta_R^{[t,2]}(\gamma(q_L^{(1)})).$$

Step 2: We dissociate $\widehat{\gamma}$ into its components according to the bijection

$${P_R^{[t,2]}}^{Q_L^{(1)}} \to ({Q_R^{(t)}}^{Q_L^{(2)} \times \dots \times Q_L^{(t-1)}})^{Q_L^{(1)}} \times ({Q_R^{(3)}}^{Q_L^{(2)}})^{Q_L^{(1)}} \times (Q_R^{(2)})^{Q_L^{(1)}}.$$

We write $(\widehat{\gamma}, \gamma_1) \mapsto (\widehat{\gamma}_t, \dots, \widehat{\gamma}_2, \gamma_1)$.

Step 3: We use the natural bijections

$$({Q_R^{(j)}}^{Q_L^{(2)} \times \dots \times Q_L^{(j-1)}})^{Q_L^{(1)}} \to {Q_R^{(j)}}^{Q_L^{(1)} \times \dots \times Q_L^{(j-1)}}$$

to define a mapping $(\widehat{\gamma}_t, \dots, \widehat{\gamma}_2, \gamma_1) \mapsto (\gamma_t, \dots, \gamma_2, \gamma_1)$, where

$$\gamma_j(q_L^{(1)}, \dots, q_L^{(j-1)}) = \widehat{\gamma}_j(q_L^{(1)})(q_L^{(2)}, \dots, q_L^{(j-1)}).$$

We make a liberal use of the mappings π_i in the following lemma, that provides an explicit description of the mappings γ_j from γ. Brackets have been omitted for the sake of simplicity, since there is only one possible bracketing interpretation in each case. For instance, we must have

$$\pi_1\pi_2^2\gamma(q_L^{(1)})(q_L^{(2)})(q_L^{(3)}) = \pi_1(\pi_2(\pi_2\gamma(q_L^{(1)}))(q_L^{(2)}))(q_L^{(3)}).$$

Lemma 6.2 *For $j = 2, \dots, t$, we have*

$$\gamma_j(q_L^{(1)}, \dots, q_L^{(j-1)}) = \begin{cases} \pi_1\pi_2^{j-2}\gamma(q_L^{(1)}) \dots (q_L^{(j-1)}) & \text{if } j < t \\ \pi_2^{t-2}\gamma(q_L^{(1)}) \dots (q_L^{(t-1)}) & \text{if } j = t. \end{cases}$$

Proof. If $t = 2$, the lemma holds trivially since $\theta_R^{[2,1]}$ is the identity, hence we assume that $t > 2$.

By definition, we have $\gamma_j = \pi_j \theta_R^{[t,1]}(\gamma, \gamma_1)$. We must show by induction on t that

$$[\pi_j \theta_R^{[t,1]}(\gamma, \gamma_1)](q_L^{(1)}, \dots, q_L^{(j-1)}) = \begin{cases} \pi_1 \pi_2^{j-2} \gamma(q_L^{(1)}) \dots (q_L^{(j-1)}) & \text{if } j < t \\ \pi_2^{t-2} \gamma(q_L^{(1)}) \dots (q_L^{(t-1)}) & \text{if } j = t \end{cases}$$

holds for $j = 2, \dots, t$. We have

$$\begin{aligned} [\pi_j \theta_R^{[t,1]}(\gamma, \gamma_1)](q_L^{(1)}, \dots, q_L^{(j-1)}) &= \gamma_j(q_L^{(1)}, \dots, q_L^{(j-1)}) \\ &= \widehat{\gamma}_j(q_L^{(1)})(q_L^{(2)}, \dots, q_L^{(j-1)}) \\ &= \pi_{j-1} \widehat{\gamma}(q_L^{(1)})(q_L^{(2)}, \dots, q_L^{(j-1)}) \\ &= \pi_{j-1} \theta_R^{[t,2]}(\gamma(q_L^{(1)}))(q_L^{(2)}, \dots, q_L^{(j-1)}) \\ &= \pi_{j-1} \theta_R^{[t,2]}(\pi_2 \gamma(q_L^{(1)}), \pi_1 \gamma(q_L^{(1)}))(q_L^{(2)}, \dots, q_L^{(j-1)}) \end{aligned}$$

Ij $j = 2$, then we get

$$[\pi_2 \theta_R^{[t,1]}(\gamma, \gamma_1)](q_L^{(1)}) = \pi_1 \theta_R^{[t,2]}(\pi_2 \gamma(q_L^{(1)}), \pi_1 \gamma(q_L^{(1)})) = \pi_1 \gamma(q_L^{(1)})$$

as required since $j = 2 < t$.

Otherwise, the induction hypothesis yields

$$[\pi_j \theta_R^{[t,1]}(\gamma, \gamma_1)](q_L^{(1)}, \dots, q_L^{(j-1)}) = \begin{cases} \pi_1 \pi_2^{j-3} (\pi_2 \gamma(q_L^{(1)}))(q_L^{(2)}) \dots (q_L^{(j-1)}) & \text{if } j < t \\ \pi_2^{t-3} (\pi_2 \gamma(q_L^{(1)}))(q_L^{(2)}) \dots (q_L^{(t-1)}) & \text{if } j = t \end{cases}$$

and the lemma follows. □

The action of $S_R^{[t,1]}$ on $Q_R^{[t,1]}$ induces an action $P_R^{[t,1]} \times S_R^{[t,1]} \to P_R^{[t,1]}$ defined by

$$(\theta_R^{[t,1]}(q_R^{[t,1]}))s_R^{[t,1]} = \theta_R^{[t,1]}(q_R^{[t,1]} s_R^{[t,1]}).$$

The next Lemma 6.3, although being of technical nature, unveils some of the properties of the action and may be the most important lemma in this paper.

We note that, for all $q_R^{[t,j]} \in Q_R^{[t,j]}$ $(j < t)$ and $u \in A^+$, we have

$$\pi_1(q_R^{[t,j]} u) = \pi_1((\pi_2(q_R^{[t,j]}), \pi_1(q_R^{[t,j]}))u) = (\pi_1(q_R^{[t,j]}))u \tag{38}$$

since the action on the second component does not depend on the first (see 1.1 of the Introduction).

Lemma 6.3 *Let* $(\gamma_t, \dots, \gamma_1) \in P_R^{[t,1]}$ *and* $u_1 \in A_1^+$. *Then* $(\gamma_t, \dots, \gamma_1)u_1 = (\gamma'_t, \dots, \gamma'_1)$ *with*

$$\gamma'_j(q_L^{(1)}, \dots, q_L^{(j-1)}) = (\gamma_j(u_1 q_L^{(1)}, \dots, u_{j-1} q_L^{(j-1)}))u_j,$$

where the words $u_2, \dots, u_t$ *are defined recursively by*

$$u_{j+1} = (\gamma_j(u_1 q_L^{(1)}, \dots, u_{j-1} q_L^{(j-1)}))g_{u_j}^{(j)} q_L^{(j)} \quad (j = 1, \dots, t-1).$$

Proof. Assume that $(\gamma_t, \dots, \gamma_1) = \theta_R^{[t,1]}(\gamma, \gamma_1)$. Then

$$(\gamma, \gamma_1)u_1 = (\gamma u_1 \cdot \gamma_1 g_{u_1}^{(1)}, \gamma_1 u_1). \tag{39}$$

Since $\gamma'_1 = \gamma_1 u_1$, the lemma holds for $j = 1$. Suppose next that $j \in \{2, \dots, t-1\}$.

Lemma 6.2 and (39) yield

$$\begin{aligned}\gamma'_j(q_L^{(1)}, \dots, q_L^{(j-1)}) &= \pi_1 \pi_2^{j-2}(\gamma u_1 \cdot \gamma_1 g_{u_1}^{(1)})(q_L^{(1)}) \dots (q_L^{(j-1)}) \\ &= \pi_1 \pi_2^{j-2}[(\gamma(u_1 q_L^{(1)}))u_2](q_L^{(2)}) \dots (q_L^{(j-1)}).\end{aligned}$$

We show that

$$\gamma'_j(q_L^{(1)}, \dots, q_L^{(j-1)}) = \pi_1 \pi_2^{j-i}[(\pi_2^{i-2}\gamma(u_1 q_L^{(1)}) \dots (u_{i-1} q_L^{(i-1)}))u_i](q_L^{(i)}) \dots (q_L^{(j-1)}). \tag{40}$$

for $i = 2, \dots, j$ by induction on i.

This was already proved for $i = 2$. Assume that it holds for $i \in \{2, \dots, j-1\}$. Then by Lemma 6.2

$$\begin{aligned}\gamma'_j(q_L^{(1)}, \dots, q_L^{(j-1)}) &= \pi_1 \pi_2^{j-i}[(\pi_2^{i-2}\gamma(u_1 q_L^{(1)}) \dots (u_{i-1} q_L^{(i-1)}))u_i](q_L^{(i)}) \dots (q_L^{(j-1)}) \\ &= \pi_1 \pi_2^{j-i}[(\pi_2^{i-1}\gamma(u_1 q_L^{(1)}) \dots (u_{i-1} q_L^{(i-1)}), \pi_1 \pi_2^{i-2}\gamma(u_1 q_L^{(1)}) \dots (u_{i-1} q_L^{(i-1)}))u_i] \\ &\qquad (q_L^{(i)}) \dots (q_L^{(j-1)}) \\ &= \pi_1 \pi_2^{j-i}[(\pi_2^{i-1}\gamma(u_1 q_L^{(1)}) \dots (u_{i-1} q_L^{(i-1)}), \gamma_i(u_1 q_L^{(1)}, \dots, u_{i-1} q_L^{(i-1)}))u_i] \\ &\qquad (q_L^{(i)}) \dots (q_L^{(j-1)}) \\ &= \pi_1 \pi_2^{j-i-1}[(\pi_2^{i-1}\gamma(u_1 q_L^{(1)}) \dots (u_{i-1} q_L^{(i-1)}))u_i \cdot (\gamma_i(u_1 q_L^{(1)}, \dots, u_{i-1} q_L^{(i-1)}))g_{u_i}^{(i)}] \\ &\qquad (q_L^{(i)}) \dots (q_L^{(j-1)}) \\ &= \pi_1 \pi_2^{j-(i+1)}[\pi_2^{i-1}\gamma(u_1 q_L^{(1)}) \dots (u_i q_L^{(i)}) \cdot (\gamma_i(u_1 q_L^{(1)}, \dots, u_{i-1} q_L^{(i-1)}))g_{u_i}^{(i)} q_L^{(i)}] \\ &\qquad (q_L^{(i+1)}) \dots (q_L^{(j-1)}) \\ &= \pi_1 \pi_2^{j-(i+1)}[(\pi_2^{i-1}\gamma(u_1 q_L^{(1)}) \dots (u_i q_L^{(i)}))u_{i+1}](q_L^{(i+1)}) \dots (q_L^{(j-1)}),\end{aligned}$$

and so (40) holds. In particular, for $i = j$, we obtain

$$\gamma'_j(q_L^{(1)}, \dots, q_L^{(j-1)}) = \pi_1[(\pi_2^{j-2}\gamma(u_1 q_L^{(1)}) \dots (u_{j-1} q_L^{(j-1)}))u_j]$$

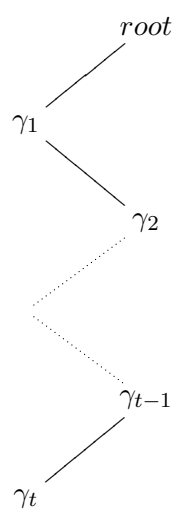
and so

$$\begin{aligned}\gamma'_j(q_L^{(1)}, \dots, q_L^{(j-1)}) &= (\pi_1 \pi_2^{j-2}\gamma(u_1 q_L^{(1)}) \dots (u_{j-1} q_L^{(j-1)}))u_j \\ &= (\gamma_j(u_1 q_L^{(1)}, \dots, u_{j-1} q_L^{(j-1)}))u_j\end{aligned}$$

by (38) and Lemma 6.2.

The case $j = t$ is actually a simplification of the preceding case and can safely be omitted. □

Since $P_R^{[t,1]}$ is a direct product of t factors, we can view it as a tree of depth t having uniform degree *for each depth*. Typically, the state $(\gamma_t, \dots, \gamma_1) \in P_R^{[t,1]}$ is represented in this tree as a path



and can be identified with the corresponding leaf. Naturally, each node of depth $j \in \{0, \dots t-1\}$ has precisely $|Q_R^{(j)^{Q_L^{(1)} \times \dots \times Q_L^{(j-1)}}}|$ sons.

Following the terminology of [24], we say that an *elliptic contraction* Ψ of $P_R^{[t,1]}$ is a depth-preserving endomorphism of the associated tree. More precisely, we view Ψ of a mapping that sends vertices to vertices of same depth (fixing the root in particular) and edges to edges, preserving adjacency.

Alternatively, if $\Psi(\gamma_t, \dots, \gamma_1) = (\gamma'_t, \dots, \gamma'_1)$ and $\Psi(\beta_t, \dots, \beta_1) = (\beta'_t, \dots, \beta'_1)$, then

$$(\gamma_1 = \beta_1, \dots, \gamma_j = \beta_j) \Rightarrow (\gamma'_1 = \beta'_1, \dots, \gamma'_j = \beta'_j)$$

holds for $j = 1, \dots, t$. This amounts to say that $\pi_j \Psi(\gamma_t, \dots, \gamma_1)$ depends on $(\gamma_j, \dots, \gamma_1)$ only (see 1.1 of the Introduction).

Theorem 6.4 *For every $u \in A_1^+$, the right action of u on $P_R^{[t,1]}$ induces an elliptic contraction ν_u of $P_R^{[t,1]}$.*

Proof. By Lemma 6.3, it is clear that whenever $(\gamma_t, \dots, \gamma_1)u = (\gamma'_t, \dots, \gamma'_1)$ then γ'_j depends on $\gamma_j, \dots, \gamma_1$ and u only. □

We can also refer to this property by saying that the right action on $P_R^{[t,1]}$ is *sequential*.

An immediate consequence of Theorem 6.4 is the following result, which will play an important role in going into profinite limits. Note that for $m \leq t$ there exists a natural onto mapping $\pi_{[m,1]} : P_R^{[t,1]} \to P_R^{[m,1]}$ defined by

$$\pi_{[m,1]}(p_R^{[t,1]}) = (\pi_m(p_R^{[t,1]}), \ldots, \pi_1(p_R^{[t,1]})).$$

Corollary 6.5 *For all $p_R^{[t,1]} \in P_R^{[t,1]}$ and $u \in A_1^+$,*

$$\pi_{[m,1]}(p_R^{[t,1]}u) = (\pi_{[m,1]}(p_R^{[t,1]}))u.$$

We consider next the expression of the initial state in the $P_R^{[t,1]}$ description.

Proposition 6.6 *$\theta_R^{[t,1]}(I_R^{[t,1]}) = (\gamma_t, \ldots, \gamma_2, I_R^{(1)})$ with $\gamma_j(q_L^{(1)}, \ldots, q_L^{(j-1)}) = I_R^{(j)}$ for $j = 2, \ldots, t$.*

Proof. We use induction on t. The case $t = 2$ is trivial since $\theta_R^{[2,1]}$ is the identity mapping. Assume that $t > 2$ and the proposition holds for $t - 1$. We have $I_R^{[t,1]} = (\gamma, I_R^{(1)})$ with $\gamma(q_L^{(1)}) = I_R^{[t,2]}$ for every $q_L^{(1)} \in Q_L^{(1)}$. By the induction hypothesis, $(\gamma, I_R^{(1)})$ is taken by $\theta_R^{[t,1]}$ in the first step to $(\widehat{\gamma}, I_R^{(1)})$, where each $\widehat{\gamma}(q_L^{(1)})$ is an $(t-1)$-uple of constant mappings defined by

$$\widehat{\gamma}_j(q_L^{(1)})(q_L^{(2)}, \ldots, q_L^{(j-1)}) = I_R^{(j)}, \quad (j = 2, \ldots, t-1).$$

Thus $\gamma_j(q_L^{(1)}, \ldots, q_L^{(j-1)}) = I_R^{(j)}$ and the lemma holds. $\square$

Naturally, all the results presented in this section for $P_R^{[t,1]}$ have dual versions for $P_L^{[t,1]}$ which will wisely be omitted.

We end this section by computing the output function in terms of the states $P_R^{[t,1]}$ and $P_L^{[t,1]}$. To avoid introducing extra notation, we keep the notation $f^{[t,1]}$ for the function

$$P_R^{[t,1]} \times A_1 \times P_L^{[t,1]} \to A_{t+1}$$

$$(\theta_R^{[t,1]}(q_R^{[t,1]}), a, q_L^{[t,1]}\theta_L^{[t,1]}) \mapsto f^{[t,1]}(q_R^{[t,1]}, a, q_L^{[t,1]})$$

.

Proposition 6.7 *Let $(\gamma_t, \ldots, \gamma_1) \in P_R^{[t,1]}$, $(\delta_1, \ldots, \delta_t) \in P_L^{[t,1]}$ and $a_1 \in A_1$. Let $q_R^{(1)} = \gamma_1$, $q_L^{(1)} = \delta_1$ and define $q_R^{(j)} \in Q_R^{(j)}$, $q_L^{(j)} \in Q_L^{(j)}$ and $a_j \in A_j$ $(j = 2, \ldots t)$ recursively by*

$$q_R^{(j)} = \gamma_j(a_1 q_L^{(1)}, \ldots, a_{j-1} q_L^{(j-1)}), \quad q_L^{(j)} = (q_R^{(j-1)} a_{j-1}, \ldots, q_R^{[1]} a_1)\delta_j,$$

$$a_j = f^{(j-1)}(q_R^{(j-1)}, a_{j-1}, q_L^{(j-1)}).$$

Then

$$f^{[t,1]}((\gamma_t, \ldots, \gamma_1), a_1, (\delta_1, \ldots, \delta_t)) = f^{(t)}(q_R^{(t)}, a_t, q_L^{(t)}).$$

Proof. We show that

$$f^{[t,1]}((\gamma_t,\dots,\gamma_1),a_1,(\delta_1,\dots,\delta_t))$$

$$= f^{[t,j]}((\pi_2^{j-1}\gamma(a_1q_L^{(1)})\dots(a_{j-1}q_L^{(j-1)}),q_R^{(j)}),a_j,(q_L^{(j)},(q_R^{(j-1)}a_{j-1})\dots(q_R^{(1)}a_1)\delta\pi_2^{j-1}))$$

holds for $j=1,\dots,t$.

Since

$$f^{[t,1]}((\gamma_t,\dots,\gamma_1),a_1,(\delta_1,\dots,\delta_t)) = f^{[t,1]}((\gamma,q_R^{[1]}),a_1,(q_L^{(1)},\delta)),$$

the claim holds for $j=1$. Assume that it holds for $j<t-1$. Then

$$f^{[t,1]}((\gamma_t,\dots,\gamma_1),a_1,(\delta_1,\dots,\delta_t))$$

$$= f^{[t,j]}((\pi_2^{j-1}\gamma(a_1q_L^{(1)})\dots(a_{j-1}q_L^{(j-1)}),q_R^{(j)}),a_j,(q_L^{(j)},(q_R^{(j-1)}a_{j-1})\dots(q_R^{(1)}a_1)\delta\pi_2^{j-1}))$$

$$= f^{[t,j+1]}(\pi_2^{j-1}\gamma(a_1q_L^{(1)})\dots(a_jq_L^{(j)}),f^{[j]}(q_R^{(j)},a_j,q_L^{(j)}),(q_R^{(j)}a_j)\dots(q_R^{(1)}a_1)\delta\pi_2^{j-1})$$

$$= f^{[t,j+1]}((\pi_2^{j}\gamma(a_1q_L^{(1)})\dots(a_jq_L^{(j)}),\pi_1\pi_2^{j-1}\gamma(a_1q_L^{(1)})\dots(a_jq_L^{(j)})),a_{j+1},$$

$$((q_R^{(j)}a_j)\dots(q_R^{(1)}a_1)\delta\pi_2^{j-1}\pi_1,(q_R^{(j)}a_j)\dots(q_R^{(1)}a_1)\delta\pi_2^{j}))$$

$$= f^{[t,j+1]}((\pi_2^{j}\gamma(a_1q_L^{(1)})\dots(a_jq_L^{(j)}),q_R^{(j+1)}),a_{j+1},(q_L^{(j+1)},(q_R^{(j)}a_j)\dots(q_R^{(1)}a_1)\delta\pi_2^{j}))$$

since

$$\pi_1\pi_2^{j-1}\gamma(a_1q_L^{(1)})\dots(a_jq_L^{(j)}) = \gamma_{j+1}(a_1q_L^{(1)},\dots,a_jq_L^{(j)}) = q_R^{(j+1)},$$
$$(q_R^{(j)}a_j)\dots(q_R^{(1)}a_1)\delta\pi_2^{j-1}\pi_1 = (q_R^{(j)}a_j,\dots,q_R^{(1)}a_1)\delta_{j+1} = q_L^{(j+1)}$$

by Lemma 6.2 and its dual. It follows that the claim holds for $j+1$ and therefore for $t-1$. Thus

$$f^{[t,1]}((\gamma_t,\dots,\gamma_1),a_1,(\delta_1,\dots,\delta_t))$$

$$= f^{(t,t-1)}((\pi_2^{t-2}\gamma(a_1q_L^{(1)})\dots(a_{t-2}q_L^{(t-2)}),q_R^{(t-1)}),a_{t-1},$$
$$(q_L^{(t-1)},(q_R^{(t-2)}a_{t-2})\dots(q_R^{(1)}a_1)\delta\pi_2^{t-2})$$

$$= f^{(t)}(\pi_2^{t-2}\gamma(a_1q_L^{(1)})\dots(a_{t-1}q_L^{(t-1)}),f^{(t-1)}(q_R^{(t-1)},a_{t-1},q_L^{(t-1)}),$$
$$(q_R^{(t-1)}a_{t-1})\dots(q_R^{(1)}a_1)\delta\pi_2^{t-2}))$$

$$= f^{(t)}(\gamma_t(a_1q_L^{(1)},\dots,a_{t-1}q_L^{(t-1)}),a_t,(q_R^{(t-1)}a_{t-1},\dots,q_R^{(1)}a_1)\delta_t)$$

$$= f^{(t)}(q_R^{(t)},a_t,q_L^{(t)})$$

as required. □

7 Turing machines and bimachines

The aim of this section is to show that the iterated block product of a very simple bimachine can be naturally associated with the behavior of a deterministic Turing machine that halts for all inputs. Moreover, the constructions involved are compatible with the standard concepts of space and time (functions).

We are interested in deterministic Turing machines that halt for all inputs, particularly those that can solve NP-complete problems. In comparison with the most standard model of deterministic Turing machines, the model we shall be considering in this paper presents three particular features:

- the "tape" has unbounded length in *both* directions and has a distinguished cell named *the origin*;
- the origin contains the symbol # until the very last move of the computation, and # appears in no other cell;
- the machine always halts in one of a very restricted set of configurations.

The reasons that took us to mark the origin with a special symbol will become somehow apparent in Section 9. We expect this particular feature to play a significative role in future research prospects.

There are of course many ways of achieving these goals, we shall just choose a particular one.

Our deterministic Turing machine is then a quadruple of the form $\mathcal{M} = (Q, q_0, A, \delta)$ where

- Q is a finite set (set of states) containing the initial state q_0;
- A is a finite set (tape alphabet) containing the special symbols B (blank), Y (yes), N (no) and # (origin);
- δ is a union of total maps

$$Q \times (A \setminus \{\#, Y, N\}) \to Q \times (A \setminus \{\#, B, Y, N\}) \times \{L, R\},$$

$$Q \times \{\#\} \to (Q \times \{\#\} \times \{L, R\}) \cup \{Y, N\}.$$

We write $A^o = A \setminus \{\#, B, Y, N\}$.

For notational convenience the machine is not allowed to write blanks on the tape.

Note that, in the final move of a computation, the control head is removed from the tape and so we allow $\{Y, N\}$ in the image of δ. The symbols Y, N are used to classify the final configurations: for a TM solving a certain problem, Y will stand for *acceptance*, N for *rejection*.

We intend to work exclusively with words, hence we shall soon exchange the classical model of "tape" and "control head" by a purely algebraic formalism. We introduce what we shall call henceforth the extended tape alphabet:

$$A' = A \cup \{a^q \mid a \in A, q \in Q\}.$$

The exponent q on a symbol acknowledges the present scanning of the corresponding cell by the control head, in state q.

We are now naturally led to the concept of *instantaneous description* for $\mathcal{M}$. Informally, instantaneous descriptions are meant to encode all (theoretically) possible configurations of the tape during any sequence of computations. Formally, let tape : $A'^+ \to A^+$ be the homomorphism returning the content of the tape, defined by

$$\text{tape}(a) = a, \quad \text{tape}(a^q) = a \quad (a \in A, q \in Q),$$

and let heads : $A'^+ \to (\mathbb{N}, +)$ be the homomorphism counting the number of heads in the expression, defined by

$$\text{heads}(a) = 0, \quad \text{heads}(a^q) = 1 \quad (a \in A, q \in Q).$$

Then we define

$$ID = B^*\{w \in A'^+ \mid \text{tape}(w) \in (\{1\}\cup B)(A^o)^*(\{\#, Y, N\})(A^o)^*(\{1\}\cup B),\ \text{heads}(w)\ \leq 1\}B^*.$$

As an example, the instantaneous description $BBab\#ab^qaBBB$ indicates that $ab\#aba$ is the content of the tape, the third cell is the origin, the control head is scanning the fifth cell in state q. The presence of some blanks on the left or/and on the right can be useful in some circumstances, in others can be ignored. We should always keep in mind that the tape can always be extended by an unbounded number of blanks $B \ldots BBabc\#abaBB \ldots B$. Note that we may have heads$(w) = 0$ or 1 due to our peculiar stopping conditions. The interdiction of blanks between non-blanks follows of course from the impossibility of writing blanks.

We denote by $\overline{ID}$ the set of all nonempty factors of words in ID.

The Turing machine $\mathcal{M}$ induces a mapping $\beta : \overline{ID} \to \overline{ID}$ (the *one-move mapping*) as follows:

Let $w \in \overline{ID}$. If heads$(w) = 0$, let $\beta(w) = w$. Suppose now that $w = ua^qv$ with $a \in A$ and $q \in Q$.

- if $\delta(q, a) = b \in \{Y, N\}$, let $\beta(w) = ubv$;
- if $\delta(q, a) = (p, b, R)$ and c is the first letter of $v = cv'$, let $\beta(w) = ubc^pv'$;
- if $\delta(q, a) = (p, b, R)$ and $v = 1$, let $\beta(w) = ubB^p$;
- if $\delta(q, a) = (p, b, L)$ and c is the last letter of $u = u'c$, let $\beta(w) = u'c^pbv$;
- if $\delta(q, a) = (p, b, R)$ and $u = 1$, let $\beta(w) = B^pbv$.

Given $w \in ID$, it should be clear that the sequence $(\beta^t(w))_t$ is eventually constant if and only if $\mathcal{M}$ stops after finitely many moves if and only if $\beta^m(w) \in A^+$ for some $m \in \mathbb{N}$. In this case, we write

$$\lim_{t\to\infty} \beta^t(w) = \beta^m(w).$$

More generally, given any eventually constant sequence $(u_t)_t$, we shall use $\lim_{t\to\infty} u_t$ with the obvious meaning.

We say that our deterministic Turing machine (TM) is *normalized* if

- $(\beta^t(w))_t$ is eventually constant for every $w \in ID$;
- $\lim_{t\to\infty} \beta^t(w) \in B^*(A^o)^*\{Y,N\}(A^o)^*B^*$ for every $w \in ID$.

In view of our stopping conventions, this implies in particular that a symbol Y or N must be precisely at the origin. Normalized TMs will be used as models for solving a certain problem, and we can now be more precise with respect to the possible final configurations: $\lim_{t\to\infty} \beta^t(w) \in B^*(A^o)^*Y(A^o)^*B^*$ will correspond to *acceptance*, $\lim_{t\to\infty} \beta^t(w) \in B^*(A^o)^*N(A^o)^*B^*$ to *rejection*.

The *space* and *time* functions for the normalized TM $\mathcal{M}$ can be naturally defined by

$$\begin{aligned} s_{\mathcal{M}} : ID &\to \mathbb{N} \\ w &\mapsto |\lim_{t\to\infty} \beta^t(w)|, \end{aligned}$$

$$\begin{aligned} t_{\mathcal{M}} : ID &\to \mathbb{N} \\ w &\mapsto \min\{m \in \mathbb{N} : \beta^m(w) = \lim_{t\to\infty} \beta^t(w)\}. \end{aligned}$$

Indeed, we are assuming that our TM halts after finitely many moves, and the length of the limit gives precisely the number of cells that have been used in the computation (if we include all the cells occupied by w). On the other hand, since each iteration of β corresponds to one move of $\mathcal{M}$, the time function computes the number of moves needed to get to a terminal configuration. It follows from known results (see [3]) that any deterministic (multi-tape) Turing machine solving a problem with space and time complexities of order $s(n)$ and $t(n)$ (not less than linear) can be turned into a normalized TM with space and time functions of order $s(n)$ and $(t(n))^2$, respectively. In particular, if $s(n)$ and $t(n)$ are polynomial, we remain within the realm of polynomial complexity.

We note that, for every $w \in ID$, $\beta(w)$ either has the same length or is one letter longer than w (when $\beta(w)$ is of the form $w'B^p$ or B^pw'). The *one-move lp-mapping* $\beta_0 : ID \to ID$ is defined by

$$\beta_0(w) = \begin{cases} \beta(w) & \text{if } |\beta(w)| = |w| \\ w' & \text{if } |\beta(w)| = |w| + 1 \text{ and } \beta(w) = w'B^p; \\ w' & \text{if } |\beta(w)| = |w| + 1 \text{ and } \beta(w) = B^pw'. \end{cases}$$

Alternatively, we can say that $\beta_0(w)$ is obtained from $\beta(BwB)$ by removing the first and the last letter (see 1.2 of the Introduction). Similarly, we can consider the extension $\beta_0 : \overline{ID} \to \overline{ID}$ which is also an lp-mapping.

We remark that, if $|\beta(w)| = |w| + 1$, then $\beta_0(w) \in A^+$ and so we cannot deduce $\beta(w)$ from $\beta_0(w)$. This loss of information is only apparent: it is a fact for a particular input, but not if we consider the *full domain of instantaneous descriptions*: indeed, we can deduce $\beta(w)$ from $\beta_0(BwB)$ and, more generally, $\beta^t(w)$ from $\beta_0^t(B^twB^t)$. It follows that:

Theorem 7.1 *The behavior of a normalized deterministic Turing machine $\mathcal{M}$ is fully determined by its associated one-move lp-mapping $\beta_0 : ID \to ID$ through*

$$\beta^t(w) = \beta_0^t(B^twB^t).$$

Let $\iota_B : ID \to (A' \setminus \{B\})^+$ be the mapping that removes all blanks from a given $w \in ID$. We see now how the space and time functions of $\mathcal{M}$ can be recovered from β_0.

Lemma 7.2 *Let $\mathcal{M}$ be a normalized TM with one-move mapping β. Let $w \in \overline{ID}$ be such that tape$(w) \in (A \setminus \{B\})^+$. Then*

(i) $\lim_{t\to\infty} \beta^t(w) = \lim_{t\to\infty} \iota_B(\beta_0^t(B^t w B^t))$;

(ii) $s_{\mathcal{M}}(w) = |\lim_{t\to\infty} \iota_B(\beta_0^t(B^t w B^t))|$;

(iii) $t_{\mathcal{M}}(w) = min\{m \in \mathbb{N} : \iota_B(\beta_0^m(B^m w B^m)) = \lim_{t\to\infty} \iota_B(\beta_0^t(B^t w B^t))\}$.

Proof. It is immediate that $\beta^t(B^t w B^t) = \beta_0^t(B^t w B^t)$ for every $t \in \mathbb{N}$, and so

$$\iota_B(\beta_0^t(B^t w B^t)) = \iota_B(\beta^t(B^t w B^t)) = \beta^t(w).$$

Thus the result follows from $\mathcal{M}$ being normalized and the definitions of space and time functions. $\square$

We prefer to extend $\beta_0 : \overline{ID} \to \overline{ID}$ to an lp-mapping $\beta_0 : A'^+ \to \overline{ID}$ for formal reasons. Since we are not really interested in non-IDs, we may consider some arbitrary lp-mapping $\Delta : A'^+ \to ID$ fixing every $w \in \overline{ID}$ and then take the composition $\beta_0\Delta$. We can take for instance, for some fixed $a \in A$,

$$\Delta(w) = \begin{cases} w & \text{if } w \in \overline{ID} \\ Na^{|w|-1} & \text{otherwise.} \end{cases}$$

So far, we have associated with the normalized TM $\mathcal{M}$ an lp-mapping β_0 encoding the full computational power of $\mathcal{M}$ with space and time functions equivalent to those of $\mathcal{M}$. We proceed now to define a canonical finite bimachine of very simple nature matching β_0 in $\overline{ID}$.

The A', A'-bimachine

$$\mathcal{B}_{\mathcal{M}} = ((I_R, Q_R, S_R), f, (S_L, Q_L, I_L))$$

is defined as follows:

- $Q_R = A' \cup \{I_R\}$, $Q_L = A' \cup \{I_L\}$;
- $S_R = A'$ is a right zero semigroup ($ab = b$);
- $S_L = A'$ is a left zero semigroup ($ab = a$);
- the action $Q_R \times S_R \to Q_R$ is defined by $q_R a = a$;
- the action $S_L \times Q_L \to Q_L$ is defined by $a q_L = a$

For the output function, let us write $I'_R = B$ and $q'_R = q_R$ for every $q_R \in Q_R \setminus \{I_R\}$. Similarly, we define q'_L. Given $q_R \in Q_R$, $a \in A'$ and $q_L \in Q_L$, let

$$f(q_R, a, q_L) = \beta_0(q'_R, a, q'_L) = \sigma_{|q'_R|+1}\beta_0(q'_R a q'_L)$$

(recall the convention in the beginning of Section 2). If $q_R a q_L \in \overline{ID}$, then $q_R a q_L$ will encode the situation of three consecutive tape cells at a certain moment. Then $f(q_R, a, q_L)$ describes the situation of the middle cell after one move of $\mathcal{M}$. If $q_R a q_L \notin \overline{ID}$, then $f(q_R, a, q_L)$ will have some pretty arbitrary meaning, depending on the choice of Δ previously taken.

Lemma 7.3 *Let $\mathcal{M}$ be a normalized TM with one-move lp-mapping β_0. Then $\alpha_{\mathcal{B}_\mathcal{M}}(w) = \beta_0(w)$ for every $w \in \overline{ID}$.*

Proof. Write $\alpha = \alpha_{\mathcal{B}_\mathcal{M}}$. Let $w \in \overline{ID}$ and write $w = uav$ with $a \in A'$. We must show that $\alpha(u, a, v) = \beta_0(u, a, v)$. Now

$$\alpha(u, a, v) = f(I_R u, a, v I_L) = f(q_R, a, q_L)$$

where

$$q_R = \begin{cases} I_R & \text{if } u = 1 \\ \text{last letter of } u & \text{otherwise,} \end{cases}$$

$$q_L = \begin{cases} I_L & \text{if } v = 1 \\ \text{first letter of } v & \text{otherwise.} \end{cases}$$

By definition, we have

$$f(q_R, a, q_L) = \beta_0(q'_R, a, q'_L).$$

On the other hand, we certainly have

$$\beta_0(u, a, v) = \beta_0(q_R, a, q_L) = \beta_0(q'_R, a, q'_L)$$

if $u, v \neq 1$. If $u = 1$ and $v \neq 1$, then

$$f(q_R, a, q_L) = \beta_0(B, a, q_L) = \beta_0(u, a, v)$$

since both B and u are irrelevant to the computation. The other cases being of course similar, the result follows. $\square$

We can now prove the main result of the section, establishing the bridge between Turing machines and bimachines.

Theorem 7.4 *The behavior of a normalized deterministic Turing machine $\mathcal{M}$ is determined by the iterated block product of the finite A', A'-bimachine $\mathcal{B}_\mathcal{M}$.*

Proof. By Theorem 7.1, the behavior of $\mathcal{M}$ is fully determined by its associated one-move lp-mapping $\beta_0 : ID \to ID$ through $\beta^t(w) = \beta_0^t(B^t w B^t)$.

By Lemma 7.3, β_0 and the lp-mapping $\alpha_{\mathcal{B}_\mathcal{M}}$ defined by $\mathcal{M}$ coincide on ID. Hence $\beta^t(w) = (\alpha_{\mathcal{B}_\mathcal{M}})^t(B^t w B^t)$ for every $w \in ID$. Let $\mathcal{B}_\mathcal{M}^{[t]}$ denote the block product

$$((\ldots(\mathcal{B}_\mathcal{M} \square \mathcal{B}_\mathcal{M}) \square \mathcal{B}_\mathcal{M}) \square \ldots) \square \mathcal{B}_\mathcal{M}$$

of t copies of $\mathcal{B}_\mathcal{M}$. By Proposition 3.7, we get

$$\beta^t(w) = (\alpha_{\mathcal{B}_\mathcal{M}})^t(B^t w B^t) = \alpha_{\mathcal{B}_\mathcal{M}^{[t]}}(B^t w B^t)$$

and the theorem holds. $\square$

We consider next some sort of converse statement for our assignment of a finite bimachine to a normalized TM. We assume that A is an alphabet with four symbols singled out (playing the roles of $B, \#, Y, N$).

Let $\mathcal{B}$ be a finite A, A-bimachine and write $\alpha = \alpha_{\mathcal{B}}$. Assume that

(1) $\alpha(u, a, v) \neq B$ if $a \neq B$;

(2) $(\alpha^t(w))_t$ is eventually constant for every $w \in A^+$.

As a consequence,

(3) there exists a space function $s_{\mathcal{B}} : A^+ \to \mathbb{N}$ defined by

$$s_{\mathcal{B}}(w) = |\lim_{t\to\infty} \iota_B(\alpha^t(B^t w B^t))|;$$

(4) there exists a time function $t_{\mathcal{B}} : A^+ \to \mathbb{N}$ defined by

$$t_{\mathcal{B}}(w) = \min\{m \in \mathbb{N} : \iota_B(\alpha^m(B^m w B^m)) = \lim_{t\to\infty} \iota_B(\alpha^t(B^t w B^t))\},$$

where the limits are taken for eventually constant sequences. Then there exists some normalized TM $\mathcal{M}$ with $s_{\mathcal{M}}(w) = s_{\mathcal{B}}(w)$ and $t_{\mathcal{M}}(w) = \mathcal{O}((s_{\mathcal{B}}(w)t_{\mathcal{B}}(w))^2)$ that computes $\lim_{t\to\infty} \iota_B(\alpha^t(B^t w B^t))$. Moreover, if $s_{\mathcal{B}}$ and $t_{\mathcal{B}}$ are polynomially bounded, so are $s_{\mathcal{M}}$ and $t_{\mathcal{M}}$.

In fact, let $w \in A^+$. Each time we perform an iteration of α on $B^m w B^m$ (for the smallest m we need to obtain the space and time limits), we perform at most $s_{\mathcal{B}}(w)$ changes of symbols. Therefore the limit can be reached within a maximum of $s_{\mathcal{B}}(w)t_{\mathcal{B}}(w)$ elementary operations, that can be assumed to have constant cost since they can be computed by the finite bimachine. In view of Church's thesis and [3], this yields a deterministic Turing machine with the claimed time bound (it is obvious for space), and the subroutines to make it normalized can be afforded at the same level of complexity.

8 A profinite fractal equation

We use fractal in the sense similar to [2]. Also see the comments on fractal at the end of this section.

We assume from now on that $\mathcal{B}^{(1)} = \mathcal{B}^{(2)} = \mathcal{B}^{(3)} = \ldots$ are countably many copies of the A', A'-bimachine $\mathcal{B}$ defined in the preceding section for the one-move lp-mapping of a normalized TM.

Let $m, t \geq 1$ with $m < t$. We can extend the canonical surjective homomorphism

$$\xi_R^{[t,m]} : (I_R^{[t,m]}, Q_R^{[t,m]}, S_R^{[t,m]}) \to (I_R^{(m)}, Q_R^{(m)}, S_R^{(m)})$$

given by Proposition 3.9 to a morphism

$$\overline{\xi_R}^{[t,m]} : (I_R^{[t,1]}, Q_R^{[t,1]}, S_R^{[t,1]}) \to (I_R^{[m,1]}, Q_R^{[m,1]}, S_R^{[m,1]})$$

by successive application of Proposition 3.8. Similarly, we define a morphism

$$\overline{\xi_L}^{[t,m]} : (S_L^{[t,1]}, Q_L^{[t,1]}, I_L^{[t,1]}) \to (S_L^{[m,1]}, Q_L^{[m,1]}, I_L^{[m,1]}).$$

Let $\overline{\xi}_R^{[t,t]}$ and $\overline{\xi}_L^{[t,t]}$ be the obvious identity mappings.

It is straightforward that if we choose to represent the states in the $P_R^{[t,1]}$, $P_L^{[t,1]}$ versions, then

$$\overline{\xi_R}^{[t,m]}(\gamma_t, \ldots, \gamma_1) = (\gamma_m, \ldots, \gamma_1), \quad (\delta_1, \ldots, \delta_t)\overline{\xi_L}^{[t,m]} = (\delta_1, \ldots, \delta_m). \tag{41}$$

In fact, these are the mappings considered in Section 6 before Corollary 6.5.

We recall the definition of projective system and projective limit. A set $\{P_n \mid n \geq 1\}$ of algebras and morphisms $\{\pi_{ij} : P_i \to P_j \mid i \geq j\}$ is said to be a *projective system* if

- π_{nn} is the identity mapping for every $n \in \mathbb{N}$;
- $\pi_{ij}\pi_{jk} = \pi_{ik}$ whenever $i \geq j \geq k$.

Its *projective limit* is defined as

$$P = \{(a_n)_n \in \prod_{n=1}^{\infty} P_n \mid a_i\pi_{ij} = a_j \text{ whenever } i \geq j\}.$$

Lemma 8.1 $\{(I_R^{[t,1]}, Q_R^{[t,1]}, S_R^{[t,1]}) \mid t \geq 1\}$ *and the morphisms* $\overline{\xi_R}^{[t,m]}$ $(t \geq m)$ *constitute a projective system of right* A', A'*-automata.*

Proof. We must show that $\overline{\xi_R}^{[m,k]}\overline{\xi_R}^{[t,m]} = \overline{\xi_R}^{[t,k]}$ whenever $t \geq m \geq k$. This follows immediately from (41). □

We denote by

$$(I_R^\omega, Q_R^\omega, S_R^\omega)$$

the projective limit of the projective system defined above. If we represent the states in the $P_R^{[t,1]}$, $P_L^{[t,1]}$ versions, it is routine to see that

$$\ldots \times Q_R^{(t)^{Q_L^{(1)} \times \ldots \times Q_L^{(t-1)}}} \times Q_R^{(t-1)^{Q_L^{(1)} \times \ldots \times Q_L^{(t-2)}}} \times \ldots \times Q_R^{(1)}$$

provides a representation of Q_R^ω. Moreover, the initial state I_R^ω coresponds to

$$(\ldots, \overline{I_R^{(3)}}, \overline{I_R^{(2)}}, I_R^{(1)}),$$

where $\overline{I_R^{(t)}}$ is the constant mapping with image $I_R^{(t)}$.

In view of Theorem 6.4 and Corollary 6.5, it should be clear that the action of A'^+ on P_R^ω is fully determined by the action on $P_R^{[t,1]}$ in the obvious way.

We have dual L-versions of these definitions and results that lead to a projective limit

$$(S_L^\omega, Q_L^\omega, I_L^\omega)$$

and a representation

$$Q_L^\omega = Q_L^{(1)} \times \ldots \times {}^{Q_R^{(t-2)} \times \ldots \times Q_R^{(1)}} Q_L^{(t-1)} \times {}^{Q_R^{(t-1)} \times \ldots \times Q_R^{(1)}} Q_L^{(t)} \times \ldots$$

We define now an A', A'-bimachine

$$\mathcal{B}^\omega = ((I_R^\omega, Q_R^\omega, S_R^\omega), f^\omega, (S_L^\omega, Q_L^\omega, I_L^\omega))$$

as follows. Given $u, v \in A'^+$ and $a \in A'$, we define

$$f^\omega(I_R^\omega u, a, v I_L^\omega) = \lim_{t\to\infty} f^{[t,1]}(I_R^{[t,1]} B^t u, a, v B^t I_L^{[t,1]}).$$

If either q_R^ω or q_L^ω is not accessible, $f^\omega(q_R^\omega, a, q_L^\omega)$ is arbitrary (say for simplicity $f^\omega(q_R^\omega, a, q_L^\omega) = a$).

We show that $f^\omega(I_R^\omega u, a, v I_L^\omega)$ is well defined. Indeed

$$f^{[t,1]}(I_R^{[t,1]} B^t u, a, v B^t I_L^{[t,1]}) = \alpha_{\mathcal{B}^{[t,1]}}(B^t u, a, v B^t) = \beta_0^t(B^t u, a, v B^t).$$

If $uav \in \overline{ID}$, we have that

$$\lim_{t\to\infty} \beta^t(uav) = \lim_{t\to\infty} \iota_B(\beta_0^t(B^t uav B^t))$$

by Lemma 7.2 and so $(\beta_0^t(B^t u, a, v B^t))_t$ and therefore $(f^{[t,1]}(I_R^{[t,1]} B^t u, a, v B^t I_L^{[t,1]}))_t$ is eventually constant. If $uav \notin \overline{ID}$, then

$$\beta_0^t(B^t uav B^t) = B^t N^{|uav|} B^t$$

and $(f^{[t,1]}(I_R^{[t,1]} B^t u, a, v B^t I_L^{[t,1]}))_t$ is also eventually constant. Thus f^ω is well defined.

We show now that the bimachine $\mathcal{B}^\omega$ satisfies the following property, referred to as the *fractal equation*.

Theorem 8.2 $\mathcal{B}^\omega \cong \mathcal{B}^\omega \Box \mathcal{B}$.

Proof. Write $\mathcal{B}^{(\omega,1)} = B^\omega \Box \mathcal{B}$. Since

$$Q_R^\omega = \ldots \times {Q_R}^{(Q_L)^{t-1}} \times {Q_R}^{(Q_L)^{t-2}} \times \ldots \times Q_R,$$

we define

$$\zeta_R : Q_R^\omega \to Q_R^{(\omega,1)} = {Q_R^\omega}^{Q_L} \times Q_R$$

by

$$\zeta_R(\ldots, \gamma_2, \gamma_1) = (\widehat{\gamma}, \gamma_1),$$

where

$$\widehat{\gamma} = (\ldots, \widehat{\gamma}_2, \widehat{\gamma}_1)$$

and $\widehat{\gamma}_t(q_L^{(1)}) \in {Q_R}^{(Q_L)^{t-1}}$ is defined for $t \geq 1$ by

$$\widehat{\gamma}_t(q_L^{(1)})(q_L^{(2)}, \ldots, q_L^{(t)}) = \gamma_{t+1}(q_L^{(1)}, \ldots, q_L^{(t)}).$$

Given $(\widehat{\gamma}, \gamma_1) \in Q_R^{(\omega,1)}$ with $\widehat{\gamma} = (\ldots, \widehat{\gamma}_2, \widehat{\gamma}_1)$, define $(\ldots, \gamma_2, \gamma_1) \in Q_R^\omega$ by

$$\gamma_{t+1}(q_L^{(1)}, \ldots, q_L^{(t)}) = \widehat{\gamma}_t(q_L^{(1)})(q_L^{(2)}, \ldots, q_L^{(t)}).$$

It is immediate that $(\widehat{\gamma}, \gamma_1) = \zeta_R(\ldots, \gamma_2, \gamma_1)$, hence ζ_R is surjective. It is simple routine to check that ζ_R is injective and preserves the initial state.

We show next that ζ_R preserves the action. Let $u \in A_1^+$. We may write

$$(\ldots, \gamma_2, \gamma_1)u_1 = (\ldots, \gamma_2', \gamma_1')$$

with the γ_t' defined as in Lemma 6.3. By Theorem 6.4, the action of A'^+ on $Q_R^{[t,1]}$ is sequential and so must be the action of A'^+ on Q_R^ω. Thus it suffices to remark that the mapping

$$\begin{array}{c} \zeta_R^{(t)} : P_R^{[t,1]} \to P_R^{[t,2]^{Q_L}} \times Q_R \\ (\gamma_t, \ldots, \gamma_1) \mapsto ((\widehat{\gamma}_{t-1}, \ldots, \widehat{\gamma}_1), \gamma_1) \end{array}$$

preserves the action. This is essentially the identity mapping on $Q_R^{[t,1]}$ with different representations of the states. Since the action is the same in

- $Q_R^{[t,2]}$ and $P_R^{[t,2]}$,
- $Q_R^{[t,1]} = Q_R^{[t,2]^{Q_L}} \times Q_R$ and $P_R^{[t,1]}$,

it follows that ζ_R preserves the action.

We show next that

$$S_R^{(\omega,1)} \cong S_R^\omega. \tag{42}$$

Let $u, v \in A'^+$. We have

$$\begin{aligned} u_{S_R^\omega} = v_{S_R^\omega} &\Leftrightarrow \forall t \geq 1 \; u_{S_R^{[t,1]}} = v_{S_R^{[t,1]}} \\ &\Leftrightarrow u_{S_R} = v_{S_R} \wedge \forall t \geq 2 \; \forall q_R \in Q_R \; \forall q_L \in Q_L \; (q_R g_u q_L)_{S_R^{[t,2]}} = (q_R g_v q_L)_{S_R^{[t,2]}} \\ &\Leftrightarrow u_{S_R} = v_{S_R} \wedge \forall t \geq 1 \; \forall q_R \in Q_R \; \forall q_L \in Q_L \; (q_R g_u q_L)_{S_R^{[t,1]}} = (q_R g_v q_L)_{S_R^{[t,1]}} \\ &\Leftrightarrow u_{S_R} = v_{S_R} \wedge \forall q_R \in Q_R \; \forall q_L \in Q_L \; (q_R g_u q_L)_{S_R^\omega} = (q_R g_v q_L)_{S_R^\omega} \\ &\Leftrightarrow u_{S_R^{(\omega,1)}} = v_{S_R^{(\omega,1)}}, \end{aligned}$$

thus (42) holds and therefore ζ_R is an isomorphism of right A'-automata.

Similarly, the mapping

$$\zeta_L : Q_L^\omega \to Q_L^{(\omega,1)} = Q_L \times {}^{Q_R}Q_L^\omega$$

defined by

$$(\delta_1, \delta_2, \ldots)\zeta_L = (\delta_1, \widehat{\delta}),$$

where

$$\widehat{\delta} = (\widehat{\delta}_1, \widehat{\delta}_2, \ldots)$$

and $q_R^{(1)}\widehat{\delta}_t \in {}^{(Q_R)^{t-1}}Q_L$ is defined for $t \geq 1$ by

$$(q_R^{(t)}, \ldots, q_R^{(2)})(q_R^{(1)}\widehat{\delta}_t) = (q_R^{(t)}, \ldots, q_R^{(1)})\delta_{t+1},$$

is an isomorphism of left A'-automata.

Finally, we must show that

$$f^{(\omega,1)}(I_R^{(\omega,1)}u, a, vI_L^{(\omega,1)}) = f^{\omega}(I_R^{\omega}u, a, vI_L^{\omega})$$

holds for all $u, v \in A'^{+}$. Indeed,

$$\begin{aligned}
&f^{(\omega,1)}(I_R^{(\omega,1)}u, a, vI_L^{(\omega,1)}) \\
&\quad = f^{(\omega,1)}((\gamma_0, I_R)u, a, v(I_L, \delta_0)) \\
&\quad = f^{(\omega,1)}((\gamma_0 u \cdot I_R g_u, I_R u), a, (vI_L, h_v I_L \cdot v\delta_0)) \\
&\quad = f^{\omega}(I_R^{\omega} \cdot I_R g_u(vI_L), f(I_R u, a, vI_L), (I_R u)h_v I_L \cdot I_L^{\omega}) \\
&\quad = \lim_{t\to\infty} f^{[t,1]}(I_R^{[t,1]} \cdot I_R g_u(vI_L), f(I_R u, a, vI_L), (I_R u)h_v I_L \cdot I_L^{[t,1]}) \\
&\quad = \lim_{t\to\infty} f^{[t+1,2]}(I_R^{[t+1,2]} \cdot I_R g_u(vI_L), f(I_R u, a, vI_L), (I_R u)h_v I_L \cdot I_L^{[t+1,2]}) \\
&\quad = \lim_{t\to\infty} f^{[t+1,1]}((\gamma_0 u \cdot I_R g_u, I_R u), a, (vI_L, h_v I_L \cdot v\delta_0)) \\
&\quad = \lim_{t\to\infty} f^{[t+1,1]}((\gamma_0, I_R)u, a, v(I_L, \delta_0)) \\
&\quad = f^{\omega}(I_R^{\omega}u, a, vI_L^{\omega})
\end{aligned}$$

as required. □

Back to our tree model, we remark next that the elliptic contractions induced by the letters are constant at a given depth for our bimachine $\mathcal{B}$.

For every $u \in A'^{+}$, consider the elliptic contraction

$$\begin{aligned}
\nu_u : P_R^{[t,1]} &\to P_R^{[t,1]} \\
(\gamma_t, \ldots, \gamma_1) &\mapsto (\gamma_t, \ldots, \gamma_1)u.
\end{aligned}$$

Given $(\gamma_{t-1}, \ldots, \gamma_1) \in P_R^{[t-1,1]}$, let

$$\nu_{u,\gamma_{t-1},\ldots,\gamma_1} : Q_R{}^{(Q_L^{(1)})^{t-1}} \to Q_R{}^{(Q_L^{(1)})^{t-1}}$$

be the mapping defined by

$$\nu_{u,\gamma_{t-1},\ldots,\gamma_1}(\gamma_t) = \pi_t \nu_u(\gamma_t, \ldots, \gamma_1).$$

Proposition 8.3 *The mapping $\nu_{u,\gamma_{t-1},\ldots,\gamma_1}$ is constant for all $u \in A'^{+}$ and $(\gamma_{t-1}, \ldots, \gamma_1) \in P_R^{[t-1,1]}$.*

Proof. Write $u_1 = u \in A'^+$ and let $(\gamma_t, \ldots, \gamma_1) \in P_R^{[t,1]}$. By Lemma 6.3, we have $(\gamma_t, \ldots, \gamma_1)u_1 = (\gamma'_t, \ldots, \gamma'_1)$ with

$$\gamma'_j(q_L^{(1)}, \ldots, q_L^{(j-1)}) = (\gamma_j(u_1 q_L^{(1)}, \ldots, u_{j-1} q_L^{(j-1)}))u_j, \tag{43}$$

where the words $u_2, \ldots, u_t$ are defined recursively by

$$u_{j+1} = (\gamma_j(u_1 q_L^{(1)}, \ldots, u_{j-1} q_L^{(j-1)}))g_{u_j}^{(j)} q_L^{(j)} \quad (j = 1, \ldots, t-1). \tag{44}$$

By (43), and since the action is right zero, it is enough to show that the last letter of the word u_t is independent from γ_t. Now by (44) and Lemma 3.4, this last letter is of the form

$$f((\gamma_{t-1}(u_1 q_L^{(1)}, \ldots, u_{t-2} q_L^{(t-2)}))u'_{t-1}), u''_{t-1}, q_L^{(t-1)}),$$

where $u_{t-1} = u'_{t-1}u''_{t-1}$ and $u''_{t-1} \in A'$, therefore $\nu_{u,\gamma_{t-1},\ldots,\gamma_1}$ is constant. $\square$

It follows from Proposition 8.3 and finite semigroup theory [26] that $S_R^{[t,m]}$ and $S_L^{[t,m]}$ are finite nilpotent extensions of a rectangular band. It also follows from standard profinite compactness arguments [26, Chapter 3] that the idempotent B^ω, with B the blank symbol, is well defined. B^ω will be analysed in later papers [31, 32]. See also Section 10.

Fractal has a usual meaning of repeatedly finding substructures isomorphic to the whole. In our case Theorem 8.2 implies that if we take inputs of the form Bu_1B, where u_1 is an input of $\mathcal{B}^\omega$, $\gamma_1 = B$ and γ_j $(j > 1)$ restricted so that γ_j does not depend on $Q_L^{(1)}$, then this sub-bimachine is isomorphic to $\mathcal{B}^\omega$.

9 Bimachines and computation tables

We first want to give some further explanation, extension and intuition to the results already stated and proved in Sections 6 to 8. For example, we explain the terminology R-generated n-step crossing sequence. We also examine the relation between iterated bimachines of Section 6 and 7 and the computation table of the Turing machine. See [21].

The following is from [27] and for categories see [18].

Let $\mathcal{B}$ be an A, B-bimachine. The *representing category* of $\mathcal{B}$, $\mathrm{Cat}(\mathcal{B})$, is the category with objects $Q_R \times Q_L$ and arrows $Q_R \times A^* \times Q_L$ so that the arrow (q_R, w, q_L) is written

$$(q_R, \ ) \xrightarrow{w} (\ , q_L)$$

and determines

$$(q_R, wq_L) \xrightarrow{w} (q_R w, q_L).$$

The multiplication in the category is

$$(q_R, wq_L = wuq'_L) \xrightarrow{w} (q_R w, q_L = uq'_L) \xrightarrow{u} (q_R wu, q'_L)$$

equals

$$(q_R, wq_L = wuq'_L) \xrightarrow{wu} (q_R wu, q'_L)$$

plus identity arrows. The generating arrows $Q_R \times A \times Q_L$ come equipped with an output function $(q_R, a, q_L) \to f(q_R, a, q_L)$. Later, we may identify some coterminous (same domain and range) arrows.

Now this category can be looked at as a "generalized action" generalizing the right action $q_R \xrightarrow{w} q_R w$ and the left action $uq_L \xleftarrow{u} q_L$: given arrow (q_R, a, q_L), consider

$$(q_R,\ ) \xrightarrow{a} (\ , q_L),$$

then fill in by the previous actions to

$$(q_R, aq_L) \xrightarrow{a} (q_R a, q_L),$$

and similarly for $w \in A^+$ as follows: if $w = a_1 \dots a_n \in A^+$, then the arrow (q_R, w, q_L) decomposes as follows. (Take $n = 3$ for ease of exposition.) Starting with (q_R, w, q_L), we have

$$(q_R,\ ) \xrightarrow{w=a_1a_2a_3} (\ , q_L),$$

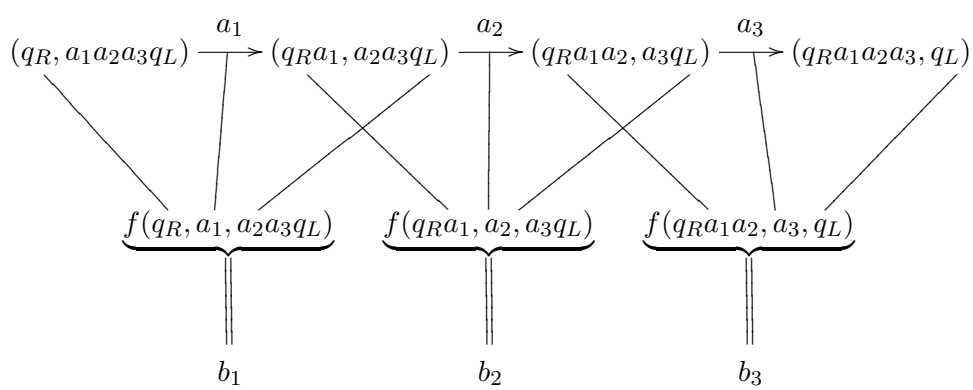
which decomposes as

$$\begin{aligned}(q_R,\ ) \xrightarrow{a_1} (q_R a_1,\ ) \xrightarrow{a_2} (q_R a_1 a_2,\ ) \xrightarrow{a_3} (q_R a_1 a_2 a_3,\ ), \text{ and}\\ (\ , a_1a_2a_3q_L) \xrightarrow{a_1} (\ , a_2a_3q_L) \xrightarrow{a_2} (\ , a_3q_L) \xrightarrow{a_3} (\ , q_L),\end{aligned}$$

and combining yields the factorization in $\mathrm{Cat}(\mathcal{B})$ as

$$(q_R, a_1a_2a_3q_L) \xrightarrow{a_1} (q_R a_1, a_2a_3q_L) \xrightarrow{a_2} (q_R a_1 a_2, a_3q_L) \xrightarrow{a_3} (q_R a_1 a_2 a_3, q_L).$$

And then the output of $\mathcal{B}$ is given by



So $\mathrm{Cat}(\mathcal{B})$ equipped with the output function on the basic arrows $Q_R \times A \times Q_L$ determine $\mathcal{B}$ (and hence $\alpha_{\mathcal{B}}$) and conversely.

If we start with L and R automata (I, S^I, S, A) and (A, S, S^I, I) for some semigroup S with generators A, then the Cat of this is the 2-sided Cayley graph denoted $\mathrm{Kayley}(S)$ (notice the "K").

So looking at $\mathrm{Cat}(\mathcal{B}^{[t,1]})$, for generating arrows with $a \in A_1$, we have

$$
\begin{tikzcd}
(q_R^{(1)} \arrow[rr, "a = a_1 \in A_1"] \arrow[d, dash] & & q_L^{(1)}) \arrow[d, dash] \\
\underbrace{f^{(1)}(q_R^{(1)}, a_1, q_L^{(1)})} \arrow[d, equal, "a_2"] & & \\
(q_R^{(2)} \arrow[rr] \arrow[d, dash] & & q_L^{(2)}) \arrow[d, dash] \\
\underbrace{f^{(2)}(q_R^{(2)}, a_2, q_L^{(2)})} \arrow[d, equal, "a_3"] & & \\
\vdots \arrow[d, dotted, "a_t"] & & \\
(q_R^{(t)} \arrow[rr] \arrow[d, dash] & & q_L^{(t)}) \arrow[d, dash] \\
f^{(t)}(q_R^{(t)}, a_t, q_L^{(t)}) = a_{t+1} & &
\end{tikzcd}
$$

Now (first reviewing Section 7 and 1.2 of the Introduction) each $f^{(i)}$ represents one move of the Turing machine (see Section 7) if we assume $\alpha_{\mathcal{B}^{(i)}} = \beta_0$ for all i. So in this Turing machine situation, it takes t computation steps to go from a_1 to a_{t+1}. So in the bimachine $\mathcal{B}^{[t,1]}$ consider to be taking *all* of A_1^+ as inputs (see also 1.1 of the Introduction)

$$
\begin{array}{l}
\xrightarrow{\mathcal{A}_R^{(t)}} \; a_i \; \xleftarrow{\mathcal{A}_L^{(t)}} \\
a_1 \ldots\ldots a_i \ldots\ldots a_n \\
\alpha_{\mathcal{B}^{[t,1]}} \quad \downarrow \\
b_1 \ldots\ldots b_i \ldots\ldots b_n
\end{array}
$$

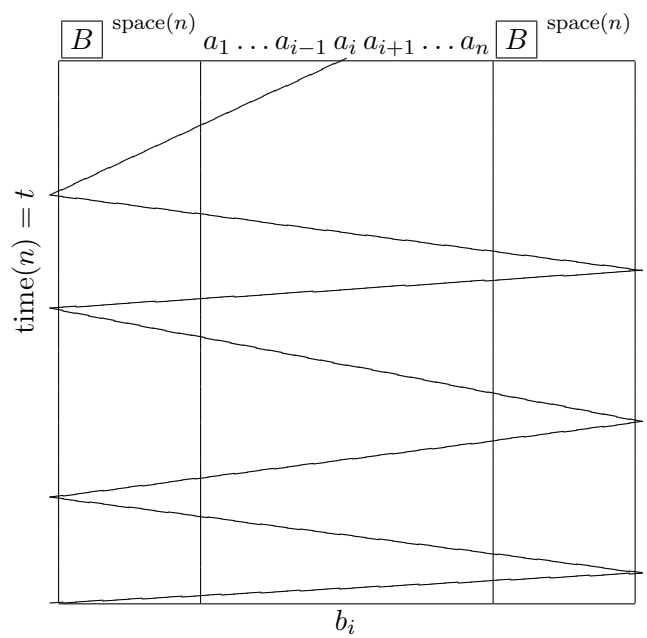
For a sequence of t bimachines $(\mathcal{B}^{[t]}, \ldots, \mathcal{B}^{[1]})$, applied to an input w of length n, the computation time is defined to be tn. So if an input string $a_1 \ldots a_{i-1}$ is fed into $\mathcal{A}_R^{(t)}$ (the right automaton of $\mathcal{B}^{[t,1]}$), then in time $O((i-1) \cdot t)$, the output $\tilde{b}_{i-1}$

$$\begin{array}{c} a_1 \ldots\ldots\ldots a_{i-1} \\ \mathcal{A}_R^{(t)} \\ b_1 \ldots\ldots\ldots b_{i-1} \end{array}$$

can be computed.

Let us look at this in more detail in the Turing machine context of $\alpha_{\mathcal{B}^{(i)}} = \beta_0$ (see Section 7 and 1.2 of the Introduction).

In the following diagram the columns before a_1 and after a_n represent R (first crossing to the left) and L (first crossing to the right) crossing sequences denoted by $(\gamma_1, \ldots, \gamma_t)$ and $(\delta_1, \ldots, \delta_t)$. The importance is that when the reading head crosses over the column moving toward the edge, only the state value matters there, and does not change till the reading head pass over again going in the other direction.



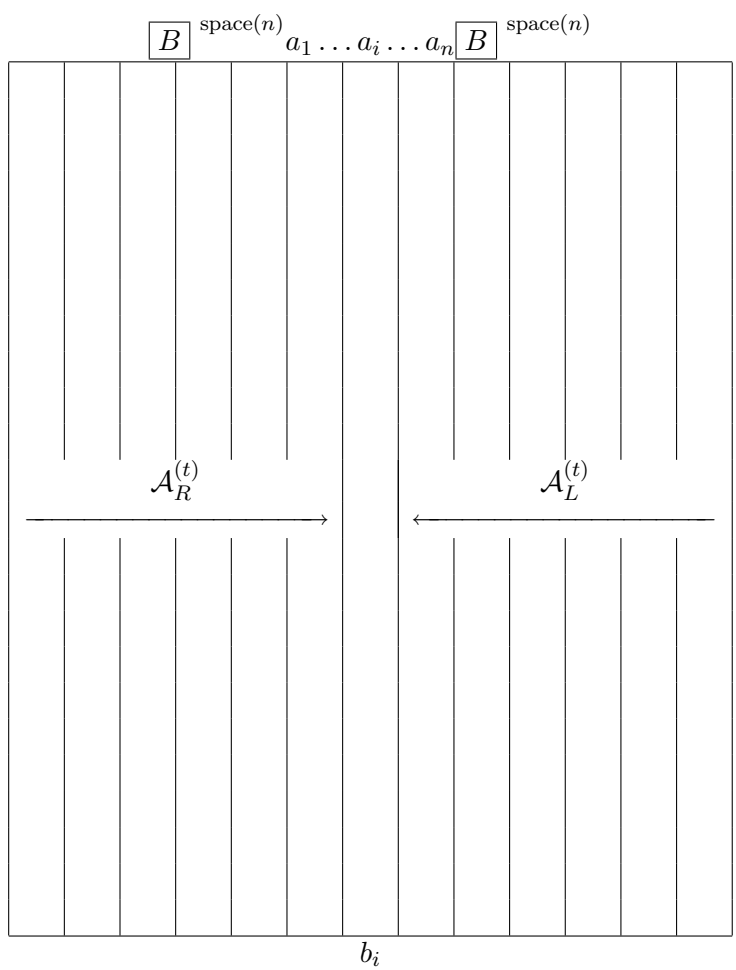
(Riding the reading head)

Or better, following the IDs and not the reading head:

$$\boxed{B}^{\text{space}(n)} a_1 \ldots a_i \ldots a_n \boxed{B}^{\text{space}(n)} \quad (ID_1)$$
$$(ID_2)$$
$$(ID_y)$$
$$\vdots$$
$$\vdots$$
$$(ID_t)$$

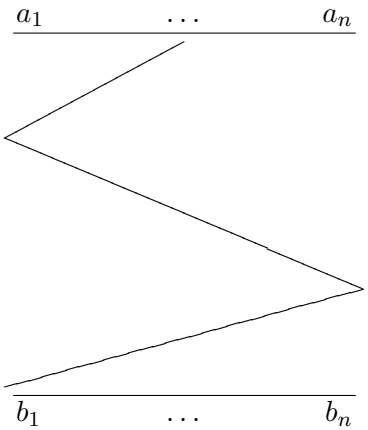
(Computation table, [21], p. 167.)

Now consider columns instead of rows and call this the dual table:



This is the dual table, showing the bimachine action. Now that CIRCUIT VALUE is P-complete can be read off from the computation table (see [21, pp.165-172]), so the bimachines give a *moving algebraic picture* (with semigroups and the representation as elliptic maps, see [24] and the arguments before Theorem 6.4) of the computation table.

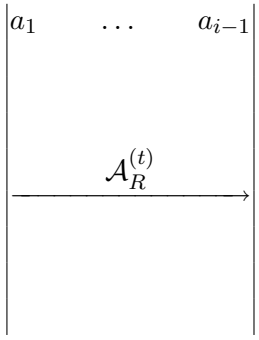
But how can



be turned into

$$\xrightarrow{\mathcal{A}_R^{(t)}} a_i \xleftarrow{\mathcal{A}_L^{(t)}}$$

with the times of the bimachine sweeps being $0(tn)$? How can such $\mathcal{A}_R^{(t)}$ and $\mathcal{A}_L^{(t)}$ exist?



must know the "future" (i. e., the other side L), hence

$$\begin{pmatrix} q_R^{(1)} & \xrightarrow{a_1 \ldots a_{i-1}} & q_L^{(1)} \\ \vdots & & \vdots \\ q_R^{(t-1)} & & q_L^{(t-1)} \\ q_R^{(t)} & & \end{pmatrix}$$

$\mathcal{A}_R^{(t)}$, for input $a_1 \ldots a_n$, must "guess the future"

$$\left. \begin{matrix} q_L^{(1)} \\ \vdots \\ q_L^{(t-1)} \end{matrix} \right)$$

then flow it back to the left to

$$\begin{array}{c} a_i \\ \Big| \\ \Big| \\ \Big| \\ . \end{array}$$

Papadimitriou calls this *quite subtle* ([21], p. 54). See J. C. Shepherdson [28]. For a precise formulation see Lemma 6.3.

Note that we can apply Proposition 2.5 to the dual table and replace $\mathcal{B}^{[t,1]}$ by

$$\alpha_\beta(\boxed{B}^{\,\mathrm{space}(n)}, a_1 \ldots a_n, \boxed{B}^{\,\mathrm{space}(n)}),$$

showing the power of generalized crossing sequences.

We next want to consider nondeterminism and Cook's Theorem in circuit form. Using a choice sequence (choices $c_1, \ldots, c_t$) is the nicest way of introducing nondeterminism (and it's often done that way). However, it is important to consider this as a way of introducing nondeterminism, and not as a way of avoiding nondeterminism. The bimachine itself (viewed without input) remains deterministic, but it now has an additional "input" (the choice sequence); the whole machine (deterministic bimachine + choice sequence input) is a nondeterministic bimachine. The same can be done with any kind of automata, in order to make them nondeterministic. For finite automata the choice sequence would have to be read in parallel with the normal input (e.g., on a second tape). Choice sequences are also known as "certificates". See [8] and also [21, Proposition 9.1 and Note 9.5.1].

To pass from the proof that CIRCUIT VALUE is P-complete to the proof that SAT is NP-complete, Steve Cook's Theorem in circuit form, is not difficult (quoting [21, p.172]):

"Since the computation of nondeterministic Turing machines proceeds in parallel paths, there is no simple notion of computation table that captures all of the behavior of such a machine on an input. If, however, we fix a sequence of choices $c = (c_0, c_1, c_2, \ldots, c_{|x|^{k-1}})$, then the computation is effectively deterministic (at the ith step we take choice c_i), and thus we can define the computation table $T_{i-1,j-1}$, $T_{i-1,j}$ and $T_{i-1,j+1}$ and the choice c_{i-1} at the previous step."

This is the relationship of our approach to the proof of Cook's Theorem in circuit form. It is the usual proof but with *more algebra introduced* via block/double semidirect products and elliptic actions. See Section 10 for suggested applications of this additional algebra.

Here is some *intuition* and pictures for Lemma 6.3.

$$(q_R^{(i)} \xrightarrow{\overrightarrow{v}_i} q_L^{(i)})$$

$$\xrightarrow{\overrightarrow{v}_{i+1}}$$

This denotes an input string $\overrightarrow{v}_i \in A_i^+$ and an output string $\overrightarrow{v}_{i+1} \in A_{i+1}^+$. If $\mathcal{B}^{(i)}$ is started in states $q_R^{(i)}$ and $q_L^{(i)}$ (see Proposition 2.5), then $\overrightarrow{v}_i$ is lp-mapped to $\overrightarrow{v}_{i+1}$, i. e.,

$$\alpha_{\mathcal{B}}^{(i)}(u, \overrightarrow{v}_i, w) = \overrightarrow{v}_{i+1}$$

with $I_R^{(i)} u = q_R^{(i)}$ and $q_L^{(i)} = w I_L^{(i)}$. See Proposition 2.5.

Then, for some $\overrightarrow{v} \in A_1^+$,

$$\begin{pmatrix} q_R^{(1)} & \xrightarrow{\overrightarrow{v}=\overrightarrow{v}_1} & q_L^{(1)} \\ q_R^{(2)} & & q_L^{(2)} \\ \vdots & & \vdots \\ q_R^{(t-1)} & & q_L^{(t-1)} \\ q_R^{(t)} & & \end{pmatrix} = \begin{pmatrix} q_R^{(1)} \cdot \overrightarrow{v}_1 \\ q_R^{(2)} \cdot \overrightarrow{v}_2 \\ \vdots \\ q_R^{(t-1)} \cdot \overrightarrow{v}_{t-1} \\ q_R^{n} \cdot \overrightarrow{v}_t \end{pmatrix}$$

with $\overrightarrow{v}_1 = \overrightarrow{v}$ and

$$(q_R^{(i)} \xrightarrow{\overrightarrow{v}_i} q_L^{(i)})$$
$$\xrightarrow{\overrightarrow{v}_{i+1}}$$

and $q_R^{(i)} \cdot \overrightarrow{v}_i$ denoting action in $(Q_R^{(i)}, S_R^{(i)}, A_i)$.

There is also the dual formulation

$$\begin{matrix} \overleftarrow{v}_1 \cdot q_L^{(1)} \\ \vdots \\ \overleftarrow{v}_{t-1} \cdot q_L^{(t-1)} \\ \overleftarrow{v}_t \cdot q_L^{(t)} \end{matrix} = \begin{pmatrix} q_R^{(1)} & \xleftarrow{\overleftarrow{v}} & q_L^{(1)} \\ \vdots & & \vdots \\ q_R^{(t-1)} & & q_L^{(t-1)} \\ & & q_L^{(t)} \end{pmatrix}.$$

Now the two-sided semidirect multiplication is seen to be

$$\begin{pmatrix} q_R^{(1)} & \xrightarrow{\overrightarrow{v}_1} & \xrightarrow{\overrightarrow{b}_1} & q_L^{(1)} \\ \vdots & & & \vdots \\ \vdots & & & q_L^{(t-1)} \\ q_R^{(t)} & & & \end{pmatrix} =$$

$$\begin{pmatrix} q_R^{(1)} & \xrightarrow{\overrightarrow{v}_1} & \overleftarrow{w}_1 \cdot q_L^{(1)} \\ \vdots & & \vdots \\ q_R^{(t-1)} & & \overleftarrow{w}_{t-1} \cdot q_L^{(t-1)} \\ q_R^{(t)} & & \end{pmatrix} \cdot \begin{pmatrix} q_R^{(1)} \cdot \overrightarrow{v}_1 & \xrightarrow{b_1} & q_L^{(1)} \\ \vdots & & \vdots \\ q_R^{(t-1)} \cdot \overrightarrow{v}_{t-1} & & q_L^{(t-1)} \\ q_R^{(t)} \cdot \overrightarrow{v}_t & & \end{pmatrix}.$$

Another way to look at this is the following.

Proposition 9.1

(a) Given $\overrightarrow{v} \in A_1^+$ and $q_L^{(1)}, \ldots, q_L^{(t-1)}$ (notice the $t-1$), this gives a member of the wreath product, see [9, vol. B], [24] or [26]:

$$(Q_R^{(t)}, S_R^{(t)}) \circ \cdots \circ (Q_R^{(1)}, S_R^{(1)})$$

(notice the t), denoted

$$\begin{matrix} \xrightarrow{\overrightarrow{v}} q_L^{(1)} \\ \vdots \\ q_L^{(t-1)} \end{matrix} \Bigg).$$

(b)

$$\xrightarrow{\vec{v}} \left.\begin{matrix} q_L^{(1)} \\ \vdots \\ q_L^{(i)} \end{matrix}\right)$$

for $i \leq t-1$ is given by projecting

$$\xrightarrow{\vec{v}} \left.\begin{matrix} q_L^{(1)} \\ \vdots \\ q_L^{(t)} \end{matrix}\right)$$

from

$$(Q_R^{(t)}, S_R^{(t)}) \circ \cdots \circ (Q_R^{(1)}, S_R^{(1)})$$

to

$$(Q_R^{(i)}, S_R^{(i)}) \circ \cdots \circ (Q_R^{(1)}, S_R^{(1)}).$$

Proof. The proof is straighforward, but the statement is subtle. □

We want to give an additional interpretation of $\left(\gamma, q_R^{(1)}\right)$ and why it is termed an R-generalized 2-step crossing sequence. The *interpretation* of $\left(\gamma, q_R^{(1)}\right)$ is first given by a picture

$$\begin{aligned} (I_R^{(1)},\ ) &\xrightarrow{w} (I_R^{(1)} w, q_L^{(1)}) \\ (I_R^{(2)},\ ) &\longrightarrow (\gamma(q_L^{(1)}),\ ) \end{aligned}$$

Here, (,) represents a member of $Q_R^{(i)} \times Q_L^{(i)}$ for $i = 1, 2$, i.e. a member of $\mathrm{Cat}(\mathcal{B}^{(i)})$, and $w \in A_1^+$ is an input.

Now in words, in considering $\mathcal{B}^{(2)} \square \mathcal{B}^{(1)}$, if w is an input (i. e., a member of A_1^+) which takes the initial R-state of $\mathcal{B}^{(2)} \square \mathcal{B}^{(1)}$ to $(\gamma, q_R^{(1)})$ under the action, then if $\mathcal{B}^{(2)} \square \mathcal{B}^{(1)}$ is started in the initial R-state with the first left automaton started in $q_L^{(1)}$, then the second right state will be $\gamma(q_L^{(1)})$ (which does not depend on $q_L^{(2)}$, the second state of the left machine).

For an informal proof (for a precise proof see Lemma 6.3 and Proposition 6.6), we must check that this interpretation persists under an application of a letter $a \in A_1$. Well,

$$\begin{aligned} \begin{pmatrix} \gamma & q_R^{(1)} \end{pmatrix} \cdot a &= \begin{pmatrix} \gamma & q_R^{(1)} \end{pmatrix} \begin{pmatrix} a_{S_L^{(1)}} & 0 \\ g_a^{(1)} & a_{S_R^{(1)}} \end{pmatrix} \\ &= \begin{pmatrix} \gamma a_L(\ ) \cdot q_R^{(1)} g_a^{(1)}(\ ) & q_R^{(1)} a_R \end{pmatrix} \qquad \begin{aligned} &(\text{with } a_L \equiv a_{S_L^{(1)}}, \\ &\quad a_R \equiv a_{S_R^{(1)}}) \end{aligned} \end{aligned}$$

with ( ) standing for the variable $(q_L^{(1)})$.

But now, looking at the following pictures, we see this gives the new correct interpretation.

$$
\begin{tikzcd}
(I_R^{(1)},\ ) \arrow[r, "w"] & (I_R^{(1)}w, a_L q_L^{(1)}) \arrow[r, "a"] \arrow[ddr, dash] & (I_R^{(1)}wa_R, q_L^{(1)}) \arrow[ddl, dash] \\
(I_R^{(2)},\ ) \arrow[r] & (\gamma(a_L q_L^{(1)}),\ ) & (x,\ )
\end{tikzcd}
$$

(with $x = \gamma(a_L q_L^{(1)}) \cdot f^{(1)}(I_R^{(1)}w, a, q_L^{(1)})$.)

Note this interpretation of $(\gamma, q_R^{(1)})$ and its dual also motivates the definition of the output function $f^{(21)}$. Also, see Subsection 1.2 of the Introduction and Section 6.

Also, an interpretation can be given to $(\gamma_t, \ldots, \gamma_1) \in P_R^{[t,1]}$ similar to the case $t = 2$ just given, justifying why $(\gamma_t, \ldots, \gamma_1)$ is called an R-generalized t-step crossing sequence, namely the following.

The *interpretation* of $(\gamma_t, \ldots, \gamma_1) \in P_R^{[t,1]}$ is first given by a picture

$$
\begin{aligned}
(I_R^{(1)},\ ) &\xrightarrow{w_1} (I_R^{(1)}\gamma_1, q_L^{(1)}) \\
(I_R^{(2)},\ ) &\xrightarrow{w_2} (I_R^{(2)}\gamma_2(q_L^{(1)}), q_L^{(2)}) \\
&\vdots \\
(I_R^{(t-1)},\ ) &\xrightarrow{w_{t-1}} (I_R^{(t-1)}\gamma_{n-1}(q_L^{(1)}, \ldots, q_L^{(t-2)}), q_L^{(t-1)}) \\
(I_R^{(t)},\ ) &\xrightarrow{w_t} (I_R^{(t)}\gamma_n(q_L^{(1)}, \ldots, q_L^{(t-1)}), -)
\end{aligned}
$$

Here, $(\ ,\ )$ represents a member of $Q_R^{(i)} \times Q_L^{(i)}$, i.e. a member of $\mathrm{Cat}(\mathcal{B}^{(i)})$, for $i = 1, \ldots, t$. Now in words, in considering $\mathcal{B}^{[t,1]}$, if $w \in A_1^+$ is an input which takes the initial R-state of $\mathcal{B}^{[t,1]}$ to $(\gamma_t, \ldots, \gamma_2, \gamma_1) \in P_R^{[t,1]}$ under the action with the first, second, third, $\ldots$, $(t-1)^{\text{st}}$ left automaton states being assigned arbitrarily to $q_L^{(1)}$, $q_L^{(2)}$, $\ldots$, $q_L^{(t-1)}$ (*no* $q_L^{(t)}$ is given), then the final right states will be $I_R^{(1)}\gamma_1$, $I_R^{(2)}\gamma_2(q_L^{(1)})$, $\ldots$, $I_R^{(t-1)}\gamma_{n-1}(q_L^{(1)}, \ldots, q_L^{(t-2)})$, $I_R^{(t)}\gamma_t(q_L^{(1)}, \ldots, q_L^{(t-1)})$. The proof is similar to the $t = 2$ case.

10 The next papers and future approach to P $\neq$ NP

Existence implies feedback and is prior to understanding. That is, things exist, like cells, children, massive computer programs using inductive loops, ecological systems with complex *feedback*, etc., but we may not or do not understand them.

Understanding comes later in the form of *introducing coordinates*, i. e., science, describing the system in question with time and space movements in *sequential form*. Thus, if the coordinates $(\ldots, x_n, \ldots, x_1)$ (finite or infinite) describe a system at time and space $c(t, p)$, then if the input π is a change in time or space, then coordinates

$$(\ldots, x_n, \ldots, x_1)\pi = (\ldots, y_n, \ldots, y_1).$$

Understanding implies *sequential form* which means that y_n depends only on $\pi, x_1, \ldots, x_n$ and not on $x_{n+1}, x_{n+2}, \ldots$. In practice this means elliptic non-invertible contractions acting on infinite trees. (See elliptic contractions above Theorem 6.4, or [24] with erratum to diagram p. 274, or [25]).

Elliptic contractions which are invertible (bijections) form the basis of Ukranian Group Theory [2, 13, 14, 20]. We need to generalize this to non-invertible mappings. Our viewpoint toward P vs. NP is that it is obviously true that P does not equal NP, but we need to get more sophisticated relevant mathematics involved to prove there are no polynomial-time programs for NP-complete problems.

The new mathematical perspective is profinite as in the profinite limit of finite bimachines in Section 8 and in Section 6 above 6.4. For example, in the compact profinite setting the idempotent B^ω exists since the powers of any element in a compact semigroup have a subsequence which limits to an idempotent as in the proof of van der Waerden's Theorem due to Furstenberg [11, 12].

Profinite limits should be mathematically powerful enough to prove P is not equal to NP but are difficult to use in general (e. g., Fermat's Last Theorem: see [7, 4]), thus in our opinion the objections of [22] do not apply.

This is the viewpoint of this paper. At this time we do not have a proof but an approach. Our approach may take 2 to 25 years. For excellent references for the standard material on P vs. NP, see [21] and [29].

Let M be a deterministic Turing machine always halting and solving problem P; e. g., P is finding a Hamiltonian Cycle in a graph. Let M_t be M running for t steps. Then M_t is "some kind of finite-state machine," and the limit of $M_t \to M$ is "some kind of limit." We next describe what "sort of finite-state machine" and what "sort of limit." M_t is a finite-state length-preserving bimachine (see Section 2), and M_t is the t-fold iteration or composition of M_1 considered as a bimachine (see Sections 3 to 8). The limit is the projective or *profinite* limit of $(M_1)^t = M_t$ (see Sections 6 and 8) converging to some machine M and the languages accepted by the M_t converge to the NP-complete problem P. See item 2 on Subsection 1.2 for a precise formulation.

Things get more interesting in the future following papers [31, 32], but the material herein is necessary to understand them. Papers [31, 32] will present respectively results on profinite bimachines and their Cauchy sequences followed by material on random walks on semigroups and Turing machines. The following gives the flavor.

The idea is to generalize the Ukrainian group theory of the R. I. Grigorchuk school, L. Bartholdi, V. Nekrashevych, A. Zuk and others to random walks on non-invertible finitely generated infinite semigroups of elliptic maps. See above Theorem 6.4 and [2, 13, 14, 20, 24] and the references there.

Let S be a semigroup finitely generated by A and let S act to the right of the set X, not necessarily faithfully, denoted (X, S, A). In the following, a knowledge of the paper [19] is useful.

First for each $a \in A$ we can consider the $X \times X$ matrix with entry (x, xa) equal to 1 and all other entries equal to zero. The entries could be in any semiring, but we will consider them in the real or complex field. We denote this matrix by $\mathrm{op}(\cdot a)$ (operator of right action $\cdot$ of a).

We denote the transpose of $\mathrm{op}(\cdot a)$ by $(\mathrm{op}(\cdot a))^*$; note that $(\mathrm{op}(\cdot a))^* = \mathrm{op}(a^{-1}\cdot)$.

Now the *adjacency matrix* for (X, S, A), denoted $\mathrm{Adj}(X, S, A)$, is by definition

$$\sum_{a \in A} \mathrm{op}(\cdot a) + \mathrm{op}(a^{-1}\cdot),$$

a self-adjoint matrix or operator on the suitable Hilbert space with nonnegative integer entries. See [19].

The 2-sided simple random walk (2SRW) on (X, S, A) is

$$\begin{aligned}&\text{transition}\left(\sum_{a\in A}\text{op}(\cdot a)+\text{op}(a^{-1}\cdot)\right)\\ \equiv&\,\text{transition}(\text{Adj}(X,S,A)).\end{aligned}$$

Here transition(M), where M is a matrix with nonnegative entries, is the matrix obtained by multiplying row x by the inverse of the sum of the entries in row $x \equiv 1/\sum x$, so we must assume $\sum x < \infty$ for all $x \in X$.

So the 2SRW (assuming it exists) is a self-adjoint operator on the Hilbert space $l_2(X)$ and is a stochastic matrix with nonnegative entries and with row sums 1. See [19]. So compute its spectrum, eigenvalues, spectral radius, etc. The norm of the operator is ≤ 1. The first question is what is the 2SRW of (A^+, A^+, A)? See [6]. Even for $|A| = 1$, the 2SRW becomes the well known

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

an operator analysed in [15].

The 2SRW of (X, S, A) is well defined if X is finite. Now in our situation for Turing machines from Section 6 we have a finite number of elliptic contractions (one for each input symbol) all operating on a fixed symmetric tree (see [24]), so in this case, the 2SRW is well defined by restricting the action to those vertices distance $\leq t$ from the root and then taking their 2SRW and limiting (i. e., obtaining the operator for the 2SRW as the limit of the finite operators for each t).

Notice by going to the ends ∂ with product measure μ, the adjacency matrix of the finite number of elliptic contractions $A = \{a_1, \dots, a_k\}$ can be defined by considering $L^2(\partial, \mu)$ and then considering the operator

$$(\ )f \to (\cdot a)f$$

which corresponds to $\text{op}(a^{-1}\cdot)$. So the adjoint of this operator corresponds to $\text{op}(\cdot a)$, so the 2SRW is a row normalization of this Adj operator, with $\text{Adj} = \sum_{a\in A}(\text{op}(a^{-1}\cdot) + \text{op}(\cdot a))$.

In the computation table of the deterministic Turing machine (see Section 9) the (i, j) position is determined by the three positions $(i-1, j-1), (i-1, j), (i-1, j+1)$. *This is very much like the Laplacian operator of random walks so computing the Laplacian operator of the random walk gives important information about the progression of the computation table which is directly related to the computational speed of the Turing machine.* See Section 9.

Now given $\alpha : A^+ \to B^+$ an lp-mapping, we can go to the minimal bimachine $\mathcal{B}_\alpha$ (Proposition 2.4) calculating α and obtain the right and left A-automata and take their 2SRWs, denoted R2SRW and L2SRW (they are well-defined since the minimal bimachime semigroup S_R acts on the right on the right state set Q_R, and dually the minimal bimachime

semigroup S_L acts on the left on the left state set Q_L). Thus, given an NP-complete problem P, we can take the bimachine of the problem P which takes each input string to a string of the same length with a single Y or a single N, and we can obtain two 2SRWs: R2SRW(P) and L2SRW(P). We are essentially decomposing the minimal bimachine into its right and left automata.

Also, given a deterministic Turing machine $\mathcal{M}_P$ solving the problem P, via Section 7, we obtain for the right and left automata of the profinite bimachine that is the limit of the block product of $\mathcal{B}_{\mathcal{M}_P}$ (see Theorem 7.4) a finite number of elliptic contractions on a symmetric tree, so the 2SRW is defined, which we denote R2SRW($\mathcal{M}_P$) (and L2SRW($\mathcal{M}_P$)). Then we want to understand how $\mathcal{M}_P$ and R2SRW(P) are related. (We need limit theorems for spectra, spectral radii, Laplacian, etc.)

R. I. Grigorchuk suggested (personal communication) that expanders could be quite important in the above context. See [17].

Another approach is possible using the Brown/Steinberg method utilizing triangular complex matrices. See [30], [5].

This is essentially a simplified version of the random walk approach.

Since the bimachine approach adds algebra to the usual Boolean circuit analysis of the computation table, this algebra should be utilized. For example, the semigroups coming up from the nth iteration of the bimachine of the Turing machine are nilpotent extensions of rectangular bands, see [26] (they could be more complicated if we used different bimachines to iterate). See [26]. Let $S_R^{(n)}$ (respectively $S_L^{(n)}$) be the R (respectively L) semigroup of the nth iteration of the one-move bimachine associated with the Turing machine. These are both nilpotent extensions of rectangular bands [26] and hence have three important numerical invariants attached to them. If S is a finite semigroup which is a nilpotent extension of a rectangular band, then $c(S)$ is the smallest positive integer m so for all $x_j \in S$, $x_1 \dots x_m$ is an idempotent. By definition, $a(S)$ (respectively $b(S)$) is the number of $\mathcal{L}$-equivalent (respectively $\mathcal{R}$-equivalent) idempotents in the unique minimal two-sided ideal of S. Define $c_n = \max\{c(S_R^{(n)}), c(S_L^{(n)})\}$, and similarly define a_n and b_n. The first thing to do is to analyse using Brown/Steinberg the limits of a_n and b_n. Question: given an NP-complete problem P, does the sequence $(a_n + b_n)$ always have a subsequence converging to some $n_0 > 1$? See the next papers [31, 32].

Acknowledgement

Both authors acknowledge generous support from C.M.U.P., financed by F.C.T. (Portugal) through the programmes POCTI and POSI, with national and European Community structural funds, and the European Science Foundation program AutoMathA.

The first author thanks the University of California, Berkeley Math Department for its many kindnesses, and we thank the Math Department, the Berkeley Academic Senate, and Dr. and Mrs. Richard Morland for some financial support.

We thank Jean-Camille Birget for his help with this paper.

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