Matroids, hereditary collections and simplicial complexes having boolean representations*

John Rhodes

Department of Mathematics, University of California, Berkeley, California 94720, U.S.A.
email: rhodes@math.berkeley.edu, BlvdBastille@aol.com

Pedro V. Silva

Centro de Matemática, Faculdade de Ciências, Universidade do Porto, R. Campo Alegre 687, 4169-007 Porto, Portugal
email: pvsilva@fc.up.pt

November 29, 2012

2010 Mathematics Subject Classification: 05E45, 05B35, 05B20, 06B15, 15B34

Keywords: hereditary collection, simplicial complex, matroid, representation, boolean matrix, lattice

ABSTRACT

Inspired by the work of Izakhian and Rhodes, a theory of representation of hereditary collections by boolean matrices is developed. This corresponds to representation by finite $\vee$-generated lattices. The lattice of flats, defined for hereditary collections, lattices and matrices, plays a central role in the theory. The representations constitute a lattice and the minimal and strictly join irreducible elements are studied, as well as various closure operators.

Contents

*This is a preliminary report and the final paper may be amended, extended and perhaps have additional authors.

1 Introduction

The background and prehistory for this paper goes something like the following. In 2006 Zur Izakhian [8] defined the notion of independence for columns (rows) of a matrix with coefficients in a supertropical semiring. Restricting this concept to the superboolean semiring $\mathbb{SB}$ (see Subsection 2.2), and then to the subset of boolean matrices (equals matrices with coefficients 0 and 1), we obtain the notion of independence of columns (rows) of a boolean matrix. This notion has several equivalent formulations (see Subsection 2.2 of this paper and references there), one involving permanent, another being the following: if M is an $m \times n$ boolean matrix, then a subset J of columns of M is *independent* if and only if there exists a subset I of rows of M with $|I| = |J| = k$ and the $k \times k$ submatrix $M[I, J]$ can be put into upper triangular form (1's on the diagonal, 0's strictly above it, and 0's or 1's below it) by independently permuting the rows and columns of $M[I, J]$.

This is the notion of independence for columns of a boolean matrix we will use in this paper. In 2008 the first author suggested that this idea would have application in many branches of Mathematics and especially in Combinatorial Mathematics. In this paper we apply it to hereditary collections (also known as abstract simplicial complexes). For other applications of this notion to lattices, posets and matroids by Izhakian and the first author, see [9, 10, 11]. For applications to finite graphs by the present authors, see [16].

If M is an $m \times n$ boolean matrix with column space C, then the set $\mathcal{H}$ of independent subsets of C satisfies the following axioms (see [9, 10]):

(H) $\mathcal{H}$ is nonempty and closed under taking subsets (making it a *hereditary collection*);

(PR) for all nonempty $J, \{p\} \in \mathcal{H}$, there exists some $x \in J$ such that $(J \setminus \{x\}) \cup \{p\} \in \mathcal{H}$ (the *point replacement* property).

Hereditary collections arising from some boolean matrix M as above are said to be *boolean representable*. The elementary properties of such boolean representable collections were considered in [9, 10, 11] and it was shown in [10] that all matroids have boolean representations.

We describe now the structure and contents of this paper.

In Section 2, we present the basic results we need to deal with lattices, superboolean matrices and hereditary collections. Note that all lattices are finite in this paper, but many results admit extensions to arbitrary lattices.

In Section 3, we establish a bijection between boolean matrices and $\vee$-generated lattices. Moving the idea of c-independent columns of a boolean matrix, via the bijection, over to lattices, we obtain the new idea (to us) that $X \subseteq L$ (L a finite lattice) is c-independent if and only if there exists an ordering $X = \{x_1, \ldots, x_k\}$ ($|X| = k$) such that

$$B < x_1 < (x_1 \vee x_2) < \ldots < (x_1 \vee \ldots \vee x_k).$$

Given an $m \times |E|$ boolean matrix M, $\mathrm{Fl}\, M$ is the closure under all intersections of those subsets of E where the rows of M are zero. Equivalently, for a $\vee$-generated lattice (L, E) and assuming that the bottom element B is not in E, we have $\mathrm{Fl}\,(L, E) = \{\ell\downarrow \cap E \mid \ell \in L\}$.

As mentioned before, we intend to consider hereditary collections (E, H) given by a boolean matrix M of size $n \times |E|$. Equivalently, (E, H) can be described through a finite lattice (L, E), $\vee$-generated by E, with H being the set of c-independent subsets of $L \subseteq 2^E$.

First properties of c-independence are proved in Section 4, where the key result is Proposition 4.2. Thus we can consider (E, H) having boolean representations, or equivalently, lattice representations, in their own right. By the main theorem of [10], this includes all matroids.

A central thesis or viewpoint is that, perhaps, boolean representations should replace matroids as the main object of study in present day matroid theory. More on this at the end of this Introduction.

In the central Section 5, we start by introducing the concept of flat (or closed set) of an arbitrary hereditary collection (E, H) and the lattice $\mathrm{Fl}\,(E, H)$. This is done by generalizing one of the formulae in matroid theory (they are not all equivalent in the general case, see (8) and the paragraph following it). Then (E, H) is boolean representable if and only if, considering the transversals of the partition of successive differences for some chain of $\mathrm{Fl}\,(E, H)$, equals H. See Proposition 4.2 and Section 5.

In Section 6 we use $\vee$-maps to define a natural ordering on all boolean (or lattice) representations of a boolean representable (E, H). This leads to considering minimal boolean representations of (E, H), and also to the $\vee$ operator which corresponds to "stacking" the matrices of the boolean representations.

Even for matroids, the minimal representation is a new idea (to us) and it is important to get all the minimal and sji (strictly join irreducible) representations in the classical case.

The connections between (E, H) and its lattice representations exist at all levels. In Section 7 we relate the closure operator induced by a hereditary collection with the closure operator induced by each of its representations.

In Section 8 we do a few examples. This includes computing all the minimal and sji representations of the Fano matroid (E, H) defined by taking $E = \{1, \ldots, 7\}$ and H equal to all subsets of E with at most 3 elements except $125, 137, 146, 236, 247, 345, 567$.

Given integers $2 \leq a < b$, let $U_{a,b} = (E, H)$ be the uniform (simple) matroid defined by $E = \{1, \ldots, b\}$ and $H = \{X \subseteq E : |X| \leq a\}$. We also compute all the minimal and sji representations of $U_{3,b}$ for $b \geq 5$.

Several other aspects of the theory, intersting enough but not required for the central core of results, are gathered in Sections 9 and 10.¡

To end this Introduction, we would like to outline why, perhaps, boolean representable hereditary collections should replace matroids.

1. All matroids have boolean representations (first proved in [10], an alternative proof is supplied here in Theorem 7.6). The proof follows easily by using the lattice of flats of the matroid, but also smaller lattices can, in general, provide representations. Calculating the minimal lattices representing the matroid is a new important question for matroid theory. Also all the representations of a matroid are endowed with an operation of join through stacking, so a representation theory (*à la* ring theory) begins. Thus the boolean representation theory, even for matroids, is much richer than the field matrix representation theory of matroids.

2. The classical matroid closure operator extends to boolean representable hereditary collections (see Section 5).

3. Strong maps are replaced by $\vee$-maps.

4. Importantly, *a geometry*, like for matroids, is attached to a boolean representable (E, H), see Subsection 10.3. Thus boolean representable hereditary collections are "not too far" from matroids, since geometry controls both.

5. The Tutte idea that "theorems for graphs can be extended to matroids" can be extended to boolean representable hereditary collections.

6. Applications: in near future papers, we plan to consider Coxeter matroids and Bruhat orders [1, 12, 15]. The methods here provide a missing ingredient in [1], namely the definition of boolean representable. See future papers.

2 Preliminaries

2.1 Lattices

A poset $(P, \leq)$ is called a lattice if, for all $p, q \in P$, there exist

$$(p \vee q) = \min\{x \in P \mid x \geq p, q\},$$
$$(p \wedge q) = \max\{x \in P \mid x \leq p, q\}.$$

For the various aspects of lattice theory, the reader is referred to [5, 6, 17].

If only the first (respectively the second) of the above conditions is satisfied, we talk of a $\vee$-*semilattice* (respectively $\wedge$-*semilattice*). We assume also that every $\vee$-semilattice (respectively $\wedge$-semilattice) has a minimum (respectively maximum) element.

All the lattices in this paper are finite, and we just write L instead of $(L, \leq)$ most of the time. If L is a finite lattice, it is immediate that L has a maximum (or *top*) element, which we denote by T, and a minimum (or *bottom*) element, which we denote by B.

We say that $E \subseteq L$ is a $\vee$-*generating set* of L if $L = \{\vee X \mid X \subseteq E\}$. Note that, whenever convenient, we may assume that $B \notin E$ since $B = \vee\emptyset$. Following [17, Chapters 6,8,9], we say that $\varphi : L \to L'$ is a $\vee$-map if $(\vee X)\varphi = \vee(X\varphi)$ for every $X \subseteq E$. We denote by FL the category of *finite lattices* together with $\vee$-maps.

We define also the category FLg by taking objects of the form (L, E), where $L \in$ FL and $E \subseteq L \setminus \{B\}$ is a $\vee$-generating set of L. The arrows $\varphi : (L, E) \to (L', E')$ are $\vee$-maps satisfying $E\varphi \subseteq E' \cup \{B\}$.

We recall that an element x of a finite lattice L is said to be *strictly meet irreducible* (smi) if $x = (y \wedge z)$ implies $y = x$ or $z = x$. This is equivalent to saying that x is covered by at most one element of L. Similarly, x is *strictly join irreducible* (sji) if $x = (y \vee z)$ implies $y = x$ or $z = x$. This is equivalent to saying that x covers at most one element of L. See [11, Subsection 3.3] for further details. It is immediate that the sji elements of L constitute the (unique) minimum $\vee$-generating set of L.

In the well-known *boolean semiring* $\mathbb{B} = \{0, 1\}$, addition and multiplication are described respectively by

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

We denote by $\mathcal{M}_n(\mathbb{B})$ the set of all $n \times n$ matrices with entries in $\mathbb{B}$. The standard boolean matrix representation of a poset $(P, \leq)$ is a $(P \times P)$-matrix $S(P)$ defined by

$$S(P)_{x,y} = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

for all $x, y \in P$. If $(L, E) \in$ FLg, then we denote by $S(L, E)$ the restriction of $S(L)$ to $E \times L$. For reasons which will become clear later, we prefer the alternative matrix representation $M(L, E) = ((S(L, E))^c)^t$, where M^c (for a boolean matrix M) denotes M with 0 and 1 interchanged, and M^t is just the transposed matrix of M. Thus, for all $\ell \in L$ and $e \in E$, we have

$$M(L, E)_{\ell,e} = 0 \Leftrightarrow e \leq \ell.$$

Note that $(S(P, \leq))^t = S(P, \geq)$ for every poset $(P, \leq)$. Moreover, $(M^c)^t = (M^t)^c$ for every boolean matrix M.

The following result collects some of the properties of the boolean matrices $M(L, E)$. We shall see later that these properties characterize actually all such matrices.

Proposition 2.1 *Let $(L, E) \in$ FLg and let $M = M(L, E)$. Then:*

(i) the rows of M are all distinct;

(ii) the columns of M are all distinct;

(iii) M contains a row with all entries equal to 0;

(iv) M contains a row with all entries equal to 1;

(v) the rows of M are closed under addition in $\mathcal{B}^{|E|}$.

Proof. (i) Write $M = (m_{\ell e})$. Since $\ell = \vee\{e \in E \mid e \leq \ell\} = \vee\{e \in E \mid m_{\ell,e} = 0\}$ for every $\ell \in L$, the rows of M are all distinct.

(ii) and (iii) Immediate.

(iv) Since $B \notin E$, we have $m_{0,e} = 1$ for every $e \in E$.

(v) Let $k, \ell \in L$. It suffices to show that $m_{k\wedge\ell,e} = m_{k,e} + m_{\ell,e}$ in $\mathbb{B}$ for every $e \in E$. This follows from the equivalence

$$\begin{aligned} m_{k\wedge\ell,e} = 0 &\Leftrightarrow e \leq k \wedge \ell \Leftrightarrow (e \leq k \text{ and } e \leq \ell) \Leftrightarrow (m_{k,e} = 0 \text{ and } m_{\ell,e} = 0) \\ &\Leftrightarrow m_{k,e} + m_{\ell,e} = 0. \end{aligned}$$

□

2.2 Superboolean matrices

Following [9, 10, 11], we may view boolean matrices as matrices over the *superboolean semiring* $\mathbb{SB} = \{0, 1, 1^\nu\}$, where addition and multiplication are described respectively by

$+$	0	1	1^ν
0	0	1	1^ν
1	1	1^ν	1^ν
1^ν	1^ν	1^ν	1^ν

$\cdot$	0	1	1^ν
0	0	0	0
1	0	1	1^ν
1^ν	0	1^ν	1^ν

We denote by $\mathcal{M}_n(\mathbb{SB})$ the set of all $n \times n$ matrices with entries in $\mathbb{SB}$. Note that $\mathcal{M}_n(\mathbb{B})$ is *not* a subsemiring of $\mathcal{M}_n(\mathbb{SB})$ since $1 + 1 = 1^\nu$.

Next we present definitions of independency and rank appropriate in the context of superboolean matrices, introduced in [8] (see also [9]).

We say that vectors $C_1, \ldots, C_m \in \mathbb{SB}^n$ are *dependent* if $\lambda_1 C_1 + \ldots \lambda_m C_m \in \{0, 1^\nu\}$ for some $\lambda_1, \ldots, \lambda_m \in \{0, 1\}$ not all zero. Otherwise, they are said to be *independent*.

Let S_n denote the symmetric group on $\hat{n} = \{1, \ldots, n\}$. The *permanent* of a matrix $M = (m_{ij}) \in \mathcal{M}_n(\mathbb{SB})$ (a positive version of the determinant) is defined by

$$\operatorname{Per} M = \sum_{\sigma \in S_n} \prod_{i=1}^{n} m_{i,i\sigma}.$$

Recall that addition and multiplication take place in the semiring $\mathbb{SB}$ defined above.

Given $I, J \subseteq \hat{n}$, we denote by $M[I, J]$ the submatrix of M with entries m_{ij} $(i \in I, j \in J)$. In particular, $M[\hat{n}, j]$ denotes the jth column vector of M for each $j \in \hat{n}$.

Proposition 2.2 [8, Th. 2.10], [9, Lemma 3.2] *The following conditions are equivalent for every $M \in \mathcal{M}_n(\mathbb{SB})$:*

(i) the column vectors $M[\hat{n}, 1], \ldots, M[\hat{n}, n]$ are independent;

(ii) $\operatorname{Per} M = 1$*;*

(iii) M can be transformed into some lower triangular matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ ? & 1 & 0 & \dots & 0 \\ ? & ? & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ ? & ? & ? & \dots & 1 \end{pmatrix} \tag{1}$$

by permuting rows and permuting columns independently.

A square matrix satisfying the above (equivalent) conditions is said to be *nonsingular*.

Given (equipotent) $I, J \subseteq \hat{n}$, we say that I is a *witness* for J in M if $M[I, J]$ is nonsingular.

Proposition 2.3 [8, Th. 3.11] *The following conditions are equivalent for every $m \times n$ superboolean matrix M and every $J \subseteq \hat{n}$:*

(i) the column vectors $M[\hat{n}, j]$ $(j \in J)$ are independent;

(ii) J has a witness in M.

Proposition 2.4 [8, Th. 3.11] *The following are equal for a given $m \times n$ superboolean matrix M:*

(i) the maximum number of independent column vectors in M;

(ii) the maximum number of independent row vectors in M;

(iii) the maximum size of a subset $J \subseteq \hat{n}$ having a witness in M;

(iv) the maximum size of a nonsingular submatrix of M.

The *rank* of a superboolean matrix M, denoted by $\operatorname{rk} M$, is the number described above. A row of M with n entries is called an *n-marker* if it has one entry 1 and all the remaining entries are 0. The following remark follows from Proposition 2.2:

Corollary 2.5 [9, Cor. 3.4] *If $M \in \mathcal{M}_n(\mathbb{SB})$ is nonsingular, then it has an n-marker.*

2.3 Hereditary collections

Let E be a set and let $H \subseteq 2^E$. We say that (E, H) is a *hereditary collection* if H is nonempty and closed under taking subsets. Hereditary collections are also known as (abstract) *simplicial complexes* (see [13, 18]).

We say that $X \subseteq E$ is *independent* if $X \in H$. A maximal independent subset of E is called a *basis*. Given $k \in \mathbb{N}$, we call $X \subseteq E$ a *k-subset* of E if $|X| = k$. we write $P_k(E) = \{X \subseteq E : |X| \leq k\}$.

The hereditary collection (E, H) is said to be a *matroid* if the following condition (the *exchange property*) holds:

(EP) For all $I, J \in H$ with $|I| = |J| + 1$, there exists some $i \in I \setminus J$ such that $J \cup \{i\} \in H$.

Note that this implies that all basis in a matroid have the same size.

There are many other equivalent definitions of matroid. For details, the reader is referred to [13].

3 Matrices versus lattices

We establish in this section correspondences between boolean matrices and $\vee$-generated lattices, adapting results from [10].

Let E be a finite set. Given $\mathcal{Z} \subseteq 2^E$, it is easy to see that

$$\widehat{\mathcal{Z}} = \{\cap S \mid S \subseteq \mathcal{Z}\}$$

is the $\wedge$-subsemilattice of $(2^E, \subseteq)$ generated by $\mathcal{Z}$. Note that $\cap\mathcal{Z} = \min\widehat{\mathcal{Z}}$, and also $E = \cap\emptyset = \max\widehat{\mathcal{Z}}$. In fact, $(\widehat{\mathcal{Z}}, \subseteq)$ is itself a lattice with the *determined join*

$$(P \vee Q) = \cap\{X \in \mathcal{Z} \mid P \cup Q \subseteq X\}.$$

However, $(\widehat{\mathcal{Z}}, \subseteq)$ is not in general a sublattice of $(2^E, \subseteq)$ since the determined join $P \vee Q$ (in $(\widehat{\mathcal{Z}}, \subseteq)$) needs not to coincide with $P \cup Q$ (see [5, 17]).

Let $M = (m_{ij})$ be an $m \times n$ boolean matrix and let $E = \hat{n}$ denote the set of columns of M. We may assume that all the rows of M are distinct. For $i \in \hat{m}$, write $Z_i = \{j \in \hat{n} \mid m_{ij} = 0\}$ and define

$$\mathcal{Z}(M) = \{Z_1, \ldots, Z_m\} \subseteq 2^E.$$

The *lattice of flats* of M is then the lattice $\mathrm{Fl}\, M = (\widehat{\mathcal{Z}(M)}, \subseteq)$ (with the determined join).

Now assume that M has no zero columns. This is equivalent to saying that $\emptyset \in \mathrm{Fl}\, M$. For $j \in \hat{n}$, define also $Y_j = \cap\{Z_i \mid m_{ij} = 0\}$ and let

$$\mathcal{Y}(M) = \{Y_1, \ldots, Y_n\} \subseteq \mathrm{Fl}\, M.$$

Note that $Y_j = \cap\{Z_i \mid j \in Z_i\}$ and so $j \in Y_j$ for every j.

Lemma 3.1 *Let $M = (m_{ij})$ be an $m \times n$ boolean matrix without zero columns. Then $(\mathrm{Fl}\, M, \mathcal{Y}(M)) \in \mathrm{FLg}$.*

Proof. First note that Y_j can never be the bottom element $\emptyset$ since $j \in Y_j$. Hence it suffices to show that

$$Z_{i_1} \cap \ldots \cap Z_{i_k} = \vee\{Y_j \mid j \in Z_{i_1} \cap \ldots \cap Z_{i_k}\}. \tag{2}$$

holds for all $i_1, \ldots, i_k \in \hat{m}$.

Indeed, take $j \in Z_{i_1} \cap \ldots \cap Z_{i_k}$. On the one hand, we have $m_{i_1 j} = \ldots = m_{i_k j} = 0$ and so $Y_j \subseteq Z_{i_1} \cap \ldots \cap Z_{i_k}$. Thus $\vee\{Y_j \mid j \in Z_{i_1} \cap \ldots \cap Z_{i_k}\} \subseteq Z_{i_1} \cap \ldots \cap Z_{i_k}$.

On the other hand, since $j \in Y_j$ for every j, we get

$$Z_{i_1} \cap \ldots \cap Z_{i_k} \subseteq \cup\{Y_j \mid j \in Z_{i_1} \cap \ldots \cap Z_{i_k}\} \subseteq \vee\{Y_j \mid j \in Z_{i_1} \cap \ldots \cap Z_{i_k}\}$$

and so (2) holds as required. $\square$

Hence $M \mapsto (\mathrm{Fl}\, M, \mathcal{Y}(M))$ defines an operator from the set of boolean matrices without zero columns into FLg.

We can relate this operator with the matrix representation defined in Subsection 2.1. Given a lattice L and $\ell \in L$, let $\ell{\downarrow} = \{x \in L \mid x \leq \ell\}$. We start with the following remark:

Lemma 3.2 *Let $(L, E) \in \mathrm{FLg}$ and let $M = M(L, E) = (m_{\ell e})$. Then $Z_\ell = \ell{\downarrow} \cap E$ for every $\ell \in L$.*

Proof. Indeed,

$$Z_\ell = \{e \in E \mid m_{\ell e} = 0\} = \{e \in E \mid e \leq \ell\} = \ell\downarrow \cap E.$$

□

Next we establish that the lattice of flats of the matrix representation of a lattice gives back the original lattice:

Proposition 3.3 *Let* $(L, E) \in \mathrm{FLg}$ *and let* $M = M(L, E) = (m_{\ell e})$. *Then* $(\mathrm{Fl}\, M, \mathcal{Y}(M)) \cong (L, E)$.

Proof. Let $\varphi : L \to \mathrm{Fl}\, M$ be defined by $\ell\varphi = Z_\ell$. Since E is a $\vee$-generating set of L, it follows from Lemma 3.2 that

$$Z_k \subseteq Z_\ell \Leftrightarrow k \leq \ell \tag{3}$$

holds for all $k, \ell \in L$. Thus φ is a poset embedding. On the other hand, $e \leq (k \wedge \ell)$ if and only if $e \leq k$ and $e \leq \ell$, hence $Z_k \cap Z_\ell = Z_{k\wedge\ell}$ for all $k, \ell \in L$. This immediately generalizes to

$$Z_{\ell_1} \cap \ldots \cap Z_{\ell_n} = Z_{\ell_1 \wedge \ldots \wedge \ell_n} \tag{4}$$

for all $\ell_1, \ldots, \ell_n \in L$, hence φ is surjective. Thus φ is an isomorphism of posets and therefore of lattices.

It remains to show that $\mathcal{Y}(M) = \{Z_e \mid e \in E\}$. It suffices to prove that $Y_e = Z_e$ for every $e \in E$. Indeed,

$$Y_e = \cap\{Z_\ell \mid m_{\ell e} = 0\} = \cap\{Z_\ell \mid e \leq \ell\} = Z_e$$

and we are done. □

We shall refer to $\mathrm{Fl}\,(L, E) = \mathrm{Fl}\, M(L, E)$ as the *lattice of flats* of $(L, E) \in \mathrm{FLg}$.

Given matrices M and M', we say that M and M' are *congruent* and write $M \cong M'$ if M' can be obtained from M by permuting rows and permuting columns (independently!). Given a boolean matrix M without zero columns, we write $M^\nu = M(\mathrm{Fl}\, M, \mathcal{Y}(M))$. In view of Proposition 2.1, it is not true that $M^\nu \cong M$ in general. However, we can get a correspondence by focusing our attention on the set $\mathcal{M}$ of all boolean matrices satisfying conditions (i)-(v) of Proposition 2.1:

Proposition 3.4 *Let* $M \in \mathcal{M}$. *Then* $M^\nu \cong M$.

Proof. Assume that $M = (m_{ij})$ is an $m \times n$ matrix in $\mathcal{M}$. Since M satisfies conditions (iii) and (v) of Proposition 2.1, we have $\mathrm{Fl}\, M = \{Z_1, \ldots, Z_m\}$. Since M satisfies condition (i) of Proposition 2.1, these elements are all distinct. Note that, since M satisfies condition (iv) of Proposition 2.1, has no zero columns and so $(\mathrm{Fl}\, M, \mathcal{Y}(M)) \in \mathrm{FLg}$ by Lemma 3.1.

Therefore $M^\nu = (m'_{Z_i Y_j})$ is also a boolean matrix with m rows. To complete the proof, it suffices to show that $m'_{Z_i Y_j} = m_{ij}$ for all $i \in \hat{m}$ and $j \in \hat{n}$. In view of condition (ii) of Proposition 2.1, M^ν is then an $m \times n$ matrix in $\mathcal{M}$ and we shall be done.

Indeed, $m'_{Z_i Y_j} = 0$ if and only if $Y_j \subseteq Z_i$. Since $j \in Y_j$, this implies $j \in Z_i$. Conversely, $j \in Z_i$ implies $Y_j \subseteq Z_i$ and so

$$m'_{Z_i Y_j} = 0 \Leftrightarrow Y_j \subseteq Z_i \Leftrightarrow j \in Z_i \Leftrightarrow m_{ij} = 0.$$

Therefore $m'_{Z_i Y_j} = m_{ij}$ and so $M^\nu \cong M$. □

Now it is easy to establish a correspondence between the set $\mathrm{FLg}/\cong$ of isomorphism classes of FLg and the set $\mathcal{M}/\cong$ of congruence classes of $\mathcal{M}$:

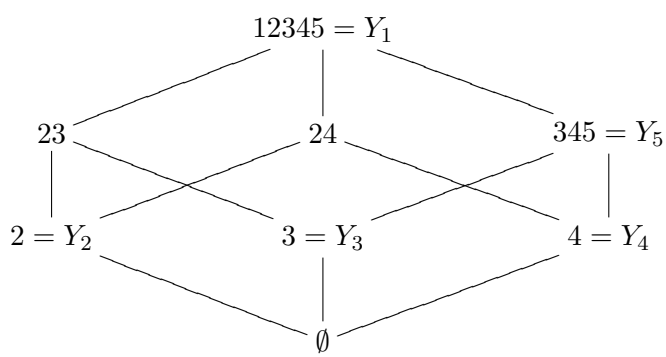
Corollary 3.5 *The mappings* $\mathcal{M} \to \mathrm{FLg}$: $M \mapsto (\mathrm{Fl}\, M, \mathcal{Y}(M))$ *and* $\mathrm{FLg} \to \mathcal{M} : (L, E) \mapsto M(L, E)$ *induce mutually inverse bijections between* $\mathcal{M}/\cong$ *and* $\mathrm{FLg}/\cong$.

Proof. It follows easily from the definitions that the above operators induce mappings between $\mathcal{M}/\cong$ and $\mathrm{FLg}/\cong$. These mappings are mutually inverse by Propositions 3.3 and 3.4. $\square$

Example 3.6 *Let* M *be the matrix*

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Omitting brackets and commas, and identifying the elements $Y_1, \ldots, Y_5$ *of* $\mathcal{Y}(M)$*, the lattice of flats* $\mathrm{Fl}\, M$ *can be represented as*



Finally, M^ν *is the matrix*

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

The above example illustrates a simple remark: if all the columns of M are distinct and nonzero, if all its rows are distinct, then M^ν can be obtained from M by adding a zero row and any new rows obtained by adding the rows of M in $\mathcal{B}^{|E|}$.

4 c-independence and subset closure

From now on, if we mention a lattice L without specifying a $\vee$-generating set E, it is assumed that $E = L \setminus \{B\}$. We also assume that $B \neq T$ in L. Its *height*, denoted by $\mathrm{ht}\, L$,

is the maximum length of a chain in $\mathcal{L}$. In view of the matrix representation $M(L)$, we say that $\ell_1, \ldots, \ell_k \in L$ are *c-independent* if the the column vectors of $M(L)$ corresponding to $\ell_1, \ldots, \ell_k$ are independent (over $\mathbb{SB}$). Note that, if $(L, E) \in \mathrm{FLg}$ and $X \subseteq E$, this is equivalent to saying that the column vectors of $M(L, E)$ defined by the elements of X are independent (over $\mathbb{SB}$).

The next result generalizes Theorem 3.6 of [10]:

Proposition 4.1 *Let* $(L, E) \in \mathrm{FLg}$. *Then* $\mathrm{rk}\, M(L, E) = \mathrm{rk}\, M(L) = \mathrm{ht}\, L$.

Proof. The second equality follows from [10, Theorem 3.6] because the omitted column corresponding to 0 contains only zeros and is therefore irrelevant to the computation of the c-rank.

Since $M(L, E)$ is a submatrix of $M(L)$, we have $\mathrm{rk}\, M(L, E) \leq \mathrm{rk}\, M(L)$. To prove the opposite inequality, it suffices to show that

$$\{x \vee y\} \cup Z \text{ c-independent} \qquad \Rightarrow \qquad \{x\} \cup Z \text{ or } \{y\} \cup Z \text{ c-independent.} \tag{5}$$

Indeed, if (5) holds, we can start with a c-independent subset $\{\ell_1, \ldots, \ell_k\}$ of L and by successive application of (5) replace it by a c-independent subset of E with the same number of elements.

Assume that $\{x \vee y, z_1, \ldots, z_k\}$ is c-independent. By Proposition 2.3, and permuting columns and rows if necessary, we may assume that $M(L)$ has a submatrix of the form (1), where the columns correspond to $z_1, \ldots, z_j, x \vee y, z_{j+1}, \ldots, z_k$ ($j \in \{0, \ldots, k\}$) and the rows correspond to $\ell_1, \ldots, \ell_j, \ell, \ell_{j+1}, \ldots, \ell_k$. Hence $(x \vee y) \leq \ell_1, \ldots, \ell_j$ and $(x \vee y) \not\leq \ell$. It follows that $x \not\leq \ell$ or $y \not\leq \ell$. On the other hand, we get $x, y \leq (x \vee y) \leq \ell_1, \ldots, \ell_j$ and so $\ell_1, \ldots, \ell_j, \ell, \ell_{j+1}, \ldots, \ell_k$ is a witness for at least one of the sets $\{x, z_1, \ldots, z_k\}$, $\{y, z_1, \ldots, z_k\}$. Therefore (5) holds as required. $\square$

We can use the lattice of flats $\mathrm{Fl}\,(L, E)$ to define a closure operator (see Subsection 10.1 of the Appendix) in the lattice $(2^E, \subseteq)$: given $X \subseteq E$, let

$$\mathrm{Cl}_L X = \cap\{Z \in \mathrm{Fl}\,(L, E) \mid X \subseteq Z\}.$$

Recalling the notation from Section 3 and Lemma 3.2, it is easy to see that

$$\mathrm{Cl}_L X = Z_{\vee X} = (\vee X)\downarrow \cap E. \tag{6}$$

Indeed, we have $X \subseteq Z_{\vee X} \in \mathrm{Fl}\,(L, E)$, and the equivalence

$$X \subseteq Z_\ell \quad \Leftrightarrow \quad \forall x \in X\ x \leq \ell \quad \Leftrightarrow \quad \vee X \leq \ell \quad \Leftrightarrow \quad Z_{\vee X} \subseteq Z_\ell$$

follows from (3), hence (6) holds.

Note that $X \in \mathrm{Fl}\,(L, E)$ if and only if $\mathrm{Cl}_L X = X$, and Cl_L is indeed a closure operator in the lattice $(2^E, \subseteq)$.

We say that $X = \{x_1, \ldots, x_k\} \subseteq E$ is a *transversal* of the partition of the successive differences for the chain $Y_0 \supset \ldots \supset Y_k$ in $\mathrm{Fl}\,(L, E)$ if $x_i \in Y_{i-1} \setminus Y_i$ for $i = 1, \ldots, k$. A subset of a transversal is a *partial transversal*.

By adapting the proofs of [10, Lemmas 3.4 and 3.5], we can prove the following:

Proposition 4.2 *Let* $(L, E) \in \mathrm{FLg}$ *and* $X \subseteq E$. *Then the following conditions are equivalent:*

(i) X *is c-independent;*

(ii) X *admits an enumeration* $x_1, \ldots, x_k$ *such that*

$$(x_1 \vee \ldots \vee x_k) > (x_2 \vee \ldots \vee x_k) > \ldots > (x_{k-1} \vee x_k) > x_k; \tag{7}$$

(iii) X *admits an enumeration* $x_1, \ldots, x_k$ *such that*

$$\mathrm{Cl}_L(x_1, \ldots, x_k) \supset \mathrm{Cl}_L(x_2, \ldots, x_k) \supset \ldots \supset \mathrm{Cl}_L(x_k);$$

(iv) X *admits an enumeration* $x_1, \ldots, x_k$ *such that*

$$x_i \notin \mathrm{Cl}_L(x_{i+1}, \ldots, x_k) \qquad (i = 1, \ldots, k-1);$$

(v) X *is a transversal of the partition of successive differences for some chain of* $\mathrm{Fl}(L, E)$*;*

(vi) X *is a partial transversal of the partition of successive differences for some maximal chain of* $\mathrm{Fl}(L, E)$.

Proof. (i) $\Rightarrow$ (ii). If X is c-independent, then $M(L)$ admits a submatrix of the form (1), with the columns labelled, say, by $x_1, \ldots, x_k$. It is a simple exercise to show that (7) holds.

(ii) $\Rightarrow$ (iii). By (3) and (6).

(iii) $\Rightarrow$ (iv). If $x_i \in \mathrm{Cl}_L(x_{i+1}, \ldots, x_k)$, then $\mathrm{Cl}_L(x_i, \ldots, x_k) = \mathrm{Cl}_L(x_{i+1}, \ldots, x_k)$.

(iv) $\Rightarrow$ (ii). Clearly, $(x_i \vee \ldots \vee x_k) \geq (x_{i+1} \vee \ldots \vee x_k)$, and equality would imply $\mathrm{Cl}_L(x_i, \ldots, x_k) = \mathrm{Cl}_L(x_{i+1}, \ldots, x_k)$ by (6).

(ii) $\Rightarrow$ (i). If (7) holds, we build a nonsingular submatrix of $M(L)$ of the form (1) by taking rows labelled by $\ell_1, \ldots, \ell_k \in L$, where $\ell_i = (x_{i+1} \vee \ldots \vee x_k)$ $(i = 1, \ldots, k)$.

(iv) $\Leftrightarrow$ (v) $\Leftrightarrow$ (vi). Immediate. $\square$

It is easy to characterize c-independence for small numbers of vectors:

Proposition 4.3 *Let* $(L, E) \in \mathrm{FLg}$ *and let* $X \subseteq E$.

(i) If $|X| \leq 2$*, then* X *is c-independent.*

(ii) If X *is c-independent and* $\vee X < 1$*, then* $X \cup \{p\}$ *is c-independent for some* $p \in E \setminus X$.

Proof. (i) The case $|X| \leq 1$ is immediate, hence we may assume that $X = \{x_1, x_2\}$ and $x_1 \not\leq x_2$. Then $(x_1 \vee x_2) > x_2$ and so X is c-independent by Proposition 4.2.

(ii) Since $1 = \vee E$, there exists some $p \in E$ such that $((\vee X) \vee p) > (\vee X)$ and so $X \cup \{e\}$ is a c-independent subset of E by Proposition 4.2 (using the characterization in (ii)). $\square$

We discuss now the c-independence of 3-subsets.

Proposition 4.4 *Let $(L, E) \in \mathrm{FLg}$ and let X be a 3-subset of E. Then the following conditions are equivalent:*

(i) X is c-independent;

(ii) X admits an enumeration x_1, x_2, x_3 such that

$$(x_1 \vee x_2 \vee x_3) > (x_2 \vee x_3) > x_3;$$

(iii) X admits an enumeration x_1, x_2, x_3 such that $x_1 \notin \mathrm{Cl}_L(x_2, x_3)$;

(iv) X is contained in some c-independent 4-subset of E or there exists some $x \in X$ such that $\vee(X \setminus \{x\}) < \vee X = 1$.

Proof. (i) $\Leftrightarrow$ (ii). By Proposition 4.2.

(i) $\Rightarrow$ (iii). By Proposition 4.2.

(iii) $\Rightarrow$ (i). Since $x_2 \neq x_3$, we may assume that $x_2 \not\leq x_3$. Hence $x_2 \notin Z_{x_3}$ and so $x_2 \notin \mathrm{Cl}_L(x_3)$ in view of (6). By Proposition 4.2, X is c-independent.

(ii) $\Rightarrow$ (iv). Assume that $(x_1 \vee x_2 \vee x_3) > (x_2 \vee x_3) > x_3$ with $X = \{x_1, x_2, x_3\}$. The case $(x_1 \vee x_2 \vee x_3) = 1$ is immediate and the case $(x_1 \vee x_2 \vee x_3) < 1$ follows from Proposition 4.3(ii).

(iv) $\Rightarrow$ (i). Clearly, c-independent sets are closed under inclusion, hence we may assume that $X = \{x, y, z\}$ and $(y \vee z) < \vee X = 1$. Since we may assume that $y \not\leq z$, then $z < (z \vee y)$ and so X is c-independent by Proposition 4.2. $\square$

5 Representation of hereditary collections

Let (E, H) be a hereditary collection. We say that $X \subseteq E$ is *closed* (or a *flat*) if

$$\forall I \in H \cap 2^X \ \forall p \in E \setminus X \qquad I \cup \{p\} \in H.$$

The set of all flats of (E, H) is denoted by $\mathrm{Fl}\,(E, H)$.

An alternative characterization is provided through the notion of *circuit*: $C \subseteq E$ is said to be a circuit of (E, H) if $C \notin H$ but all proper subsets of C are in H.

Proposition 5.1 *Let (E, H) be a hereditary collection and let $X \subseteq E$. Then the following conditions are equivalent:*

(i) X is closed;

(ii) if $p \in C \subseteq X \cup \{p\}$ for some circuit C, then $p \in X$.

Proof. (i) $\Rightarrow$ (ii). Suppose that there exist a circuit C and $p \in C \subseteq X \cup \{p\}$ such that $p \notin X$. Then $C = I \cup \{p\}$ for some $I \subseteq X$. It follows that $I \in H \cap 2^X$ and $p \in E \setminus X$, however $I \cup \{p\} \notin H$. Therefore X is not closed.

(ii) $\Rightarrow$ (i). Suppose that X is not closed. Then there exist $I \in H \cap 2^X$ and $p \in E \setminus X$ such that $I \cup \{p\} \notin H$. Let $I_0 \subseteq I$ be minimal for the property $I_0 \cup \{p\} \notin H$. Since $I_0 \in H$ due to $I_0 \subseteq I \in H$, it follows that $I_0 \cup \{p\}$ is a circuit by minimality of I_0. Thus condition (ii) fails for $C = I_0 \cup \{p\}$ and we are done. $\square$

Note that condition (ii) is the standard characterization of flats for matroids.

The following result summarizes some straightforward properties of $\mathrm{Fl}\,(E,H)$. We say that (E,H) is *simple* if $P_2(E) \subseteq H$. A 1-subset of E is also called a *point*.

Proposition 5.2 *Let (E,H) be a hereditary collection.*

(i) If $Y \subseteq \mathrm{Fl}\,(E,H)$, then $\cap Y \in \mathrm{Fl}\,(E,H)$.

(ii) If $P_k(E) \subseteq H$ with $k \geq 1$, then $P_{k-1}(E) \subseteq \mathrm{Fl}\,(E,H)$.

(iii) If (E,H) is simple, then the points of E are closed.

Proof. (i) We have $E = \cap\emptyset \in \mathrm{Fl}\,(E,H)$ trivially, hence it suffices to show that $X_1, X_2 \in \mathrm{Fl}\,(E,H)$ implies $X_1 \cap X_2 \in \mathrm{Fl}\,(E,H)$.

Let $I \in H \cap 2^{X_1 \cap X_2}$ and $p \in E \setminus (X_1 \cap X_2)$. Then $p \notin X_1$ or $p \notin X_2$. Hence $X_1 \cap X_2 \subseteq X_j$ and $p \in E \setminus X_j$ for some $j \in \hat{2}$. Since $X_j \in \mathrm{Fl}\,(E,H)$, we get $I \cup \{p\} \in H$ and so $X_1 \cap X_2 \in \mathrm{Fl}\,(E,H)$.

(ii) Immediate.

(iii) By part (ii). $\square$

Similarly to Section 4, we can use the lattice of flats $\mathrm{Fl}\,(E,H)$ to define a closure operator in $(2^E, \subseteq)$: given $X \subseteq E$, let

$$\mathrm{Cl}\,X = \cap\{Z \in \mathrm{Fl}\,(E,H) \mid X \subseteq Z\}.$$

Note that $X \subseteq E$ is closed if and only if $\mathrm{Cl}\,X = X$. We can also make the following remark:

Proposition 5.3 *Let (E,H) be a hereditary collection and let $X \subseteq E$ be a basis. Then $\mathrm{Cl}\,X = E$.*

Proof. Suppose that $p \in E \setminus \mathrm{Cl}\,X$. Since $X \subseteq \mathrm{Cl}\,X$ and $\mathrm{Cl}\,X$ is closed, we get $X \cup \{p\} \in H$, contradicting X being a basis. Thus $\mathrm{Cl}\,X = E$. $\square$

In the matroid case (see Proposition 5.1), the concept of circuit allows a more constructive perspective of the closure:

Proposition 5.4 [13, Proposition 1.4.10(ii)] *Let (E,H) be a matroid and let $X \subseteq E$. Then*

$$\mathrm{Cl}\,X = X \cup \{p \in E \setminus X \mid p \in C \subseteq X \cup \{p\} \textit{ for some circuit } C\}. \tag{8}$$

In the general case, it is still true that all such elements p must be in the closure, but they may not be sufficient. We may then have to iterate the construction. Eventually, iteration will gives us all the elements of $\mathrm{Cl}\,X$.

The subsets of independent column vectors of a given (superboolean) matrix, which include the empty subset and are closed for subsets, constitute an important example of a hereditary collection. In fact, *every* hereditary subset is given this way, see [9]. On the other hand, a *boolean representation* of a hereditary collection (E,H) is a boolean matrix M with column space E such that a subset $X \subseteq E$ of column vectors of M is independent (over $\mathbb{SB}$) if and only if $X \in H$. Obviously, we can always assume that the rows in such a matrix are distinct: the representation is then said to be *reduced*. Note also that by permuting rows in a reduced representation of (E,H) we get an alternative reduced representation of

(E, H). The number of rows in a boolean representation M of (E, H) is said to be its *degree* and is denoted by $\deg M$. We denote by $\operatorname{mindeg}(E, H)$ the minimum degree of a boolean representation of (E, H).

We remark also that if (E, H) admits a boolean representation, then (E, H) satisfies (PR) (see [9, Theorem 5.3]).

Given an $R \times E$ boolean matrix $M = (m_{re})$ and $r \in R$, we recall the notation $Z_r = \{e \in E \mid m_{re} = 0\}$ introduced in Section 3.

Proposition 5.5 *Let (E, H) be a hereditary collection and let M be an $R \times E$ boolean matrix. If M is a boolean representation of (E, H), then $Z_r \in \operatorname{Fl}(E, H)$ for every $r \in R$.*

Proof. Let $r \in R$ and $J \subseteq Z_r$. Suppose that $J \in H$ and $p \in E \setminus Z_r$. Since $M = (m_{re})$ is a boolean representation of (E, H), then the column vectors $M[R, j]$ $(j \in J)$ are independent and so there exists some $I \subseteq R$ such that $M[I, J]$ is of the form (1), for a suitable ordering of I and J. Since $J \subseteq Z_r$, the row vector $M[r, J]$ contains only zeros. On the other hand, since $p \notin Z_r$, we have $m_{rp} = 1$ and so $M[I \cup \{r\}, J \cup \{p\}]$ is also of the form (1) and therefore nonsingular. Thus $J \cup \{p\}$ defines an independent set of columns of M. Since M is a boolean representation of (E, H), it follows that $J \cup \{p\} \in H$ and so $Z_r \in \operatorname{Fl}(E, H)$. $\square$

Let (E, H) be a hereditary collection. In view of Proposition 5.2(i), we can view $(\operatorname{Fl}(E, H), \subseteq)$ as a lattice with $(X \wedge Y) = X \cap Y$ and the determined join $(X \vee Y) = \cap\{Z \in \operatorname{Fl}(E, H) \mid X \cup Y \subseteq Z\} = \operatorname{Cl}(X \cup Y)$. If (E, H) is simple, by identifying $e \in E$ with $\{e\}$ and E with $\{\{e\} \mid e \in E\}$ we may write $(\operatorname{Fl}(E, H), E) \in \mathrm{FLg}$. Indeed, for every $X \in \operatorname{Fl}(E, H)$, we have $X = \vee\{e \mid e \in X\}$. Recalling the representations of $\vee$-generated lattices from Subsection 2.1, we can prove the following:

Lemma 5.6 *Let (E, H) be a simple hereditary collection and let $X \subseteq E$ be c-independent with respect to $M(\operatorname{Fl}(E, H), E)$. Then $X \in H$.*

Proof. We use induction on $|X|$. Since (E, H) is simple, the case $|X| \leq 1$ is trivial. Hence we assume that $|X| = m > 1$ and the claim holds for $|X| = m - 1$.

By Proposition 2.3, X has a witness P in $M = M(\operatorname{Fl}(E, H), E)$. We may assume that $X = \{e_1, \ldots, e_m\}$, $P = \{P_1, \ldots, P_m\}$ and $M[P, X]$ is of the form (1), with the rows (respectively the columns) ordered by $P_1, \ldots, P_m$ (respectively $e_1, \ldots, e_m$). The first row yields $e_1 \notin P_1$ and $e_2, \ldots, e_m \in P_1$.

Now, since $e_2, \ldots, e_m$ is c-independent, it follows from the induction hypothesis that $\{e_2, \ldots, e_m\} \in H$. Together with $\{e_2, \ldots, e_m\} \subseteq P_1 \in \operatorname{Fl}(E, H)$ and $e_1 \notin P_1$, this yields $X = \{e_2, \ldots, e_m\} \cup \{e_1\} \in H$ as required. $\square$

Given matrices M_1 and M_2 with the same number of columns, we define $M_1 \oplus_b M_2$ to be the matrix obtained by concatenating the matrices M_1 and M_2 by

$$\begin{pmatrix} M_1 \\ M_2 \end{pmatrix}$$

and removing repeated rows (leaving only the first occurrence from top to bottom, say). We refer to this matrix as M_1 *stacked over* M_2.

Proposition 5.7 *Let (E, H) be a simple hereditary collection.*

(i) If M_1 and M_2 are reduced boolean representations of (E, H), so is $M_1 \oplus_b M_2$.

(ii) If M is a reduced boolean representation of (E, H) and we add/erase a row which is the sum of other rows in $\mathbb{B}^{|E|}$, we get a matrix M' which is also a reduced boolean representation of (E, H).

Proof. (i) Since M_1 and M_2 have both space of columns E, the matrix $M = M_1 \oplus_b M_2$ is well-defined and has no repeated rows by definition. Let R be the row space of M and let $X \subseteq E$. We show that

$$X \text{ is c-independent with respect to } M \;\Leftrightarrow\; X \in H \tag{9}$$

by induction on $|X|$. The case $|X| = 0$ being trivial, assume that $|X| > 0$ and (9) holds for smaller values of $|X|$.

Suppose that X is c-independent with respect to M. By permuting rows of $M_1 \oplus_b M_2$ if necessary, and using the appropriate ordering of E, we may say that there exists some $P \subseteq R$ such that $B[P, X]$ is of the form (1). Let p_1 (respectively x_1) denote the first element of P (respectively X) for these orderings, so $M[P \setminus \{p_1\}, X \setminus \{x_1\}]$ is the submatrix of $M[P, X]$ obtained by deleting the first row and the first column. Since reduced boolean representations are closed under permuting rows, we may assume without loss of generality that the row $M[p_1, E]$ came from the matrix M_1. On the other hand, since the column vectors $M[R, x]$ ($x \in X \setminus \{x_1\}$) are independent, it follows from the induction hypothesis that $X \setminus \{x_1\} \in H$ and so (since M_1 is a boolean representation of (E, H)) $X \setminus \{x_1\}$ is c-independent with respect to M_1. Hence M_1 has a singular submatrix of the form $M_1[P', X \setminus \{x_1\}]$. Now $M_1[P' \cup \{p_1\}, X]$ is still a nonsingular matrix because the unique nonzero entry in the row $M_1[p_1, X]$ is $M_1[p_1, x_1]$. Hence X is c-independent with respect to M_1 and so $X \in H$.

Conversely, if $X \in H$, then X is c-independent with respect to M_1 and so X is c-independent with respect to M as well. Thus (9) holds and so M is a reduced boolean representation of (E, H) as claimed.

(ii) The claim is obvious when we add a row, so consider the case when a row of the described form is erased. It is easy to see that if a k-marker u is the sum of some vectors in $\mathbb{B}^k$, then one of them is equal to u. Therefore, if the sum row occurs in some nonsingular submatrix of M, we can always replace it by one of the summand rows. □

Proposition 5.7(i) immediately implies that if (E, H) admits a reduced boolean representation, then there exists a unique maximal one. The main theorem of this section provides a more concrete characterization:

Theorem 5.8 *Let (E, H) be a simple hereditary collection. Then the following conditions are equivalent:*

(i) (E, H) has a boolean representation;

(ii) $M(\mathrm{Fl}\,(E, H), E)$ is a reduced boolean representation of (E, H).

Moreover, in this case any other reduced boolean representation of (E, H) is congruent to a submatrix of $M(\mathrm{Fl}\,(E, H), E)$.

Proof. (i) $\Rightarrow$ (ii). Write $M = M(\mathrm{Fl}\,(E,H),E)$. Suppose that (E,H) has a boolean representation $N = (n_{re})$. Then we may assume that the $R \times E$ matrix N is reduced. By Proposition 5.5, we have $Z_r \in \mathrm{Fl}\,(E,H)$ for every $r \in R$. For every $e \in E$, we have

$$n_{re} = 0 \Leftrightarrow e \in Z_r \Leftrightarrow \{e\} \subseteq Z_r \Leftrightarrow M[Z_r, e] = 0,$$

hence N is (up to permutation of rows) a submatrix of M.

We claim that M is also a boolean representation of (E,H). Indeed, let $X \subseteq E$. If $X \in H$, then X is c-independent with respect to N since N is a boolean representation of (E,H), hence X is c-independent with respect to M since N is a submatrix of M. The converse implication follows from Lemma 5.6, hence M is a boolean representation of (E,H). Naturally, every representation arising from a $\vee$-generated lattice is reduced.

(ii) $\Rightarrow$ (i). Trivial. $\square$

6 The lattice of lattice representations

We define a quasi-order on FLg by

$$(L,E) \geq (L',E) \qquad \text{if there exists some } \vee\text{-map } \varphi : L \to L' \text{ such that } \varphi|_E = id.$$

Note that such φ is necessarily onto: if $\ell' \in L'$, we may write $\ell' = (e_1 \vee \ldots \vee e_k)$ in L' for some $e_1, \ldots, e_k \in E$, hence

$$\ell' = (e_1 \vee \ldots \vee e_k) = (e_1\varphi \vee \ldots \vee e_k\varphi) = (e_1 \vee \ldots \vee e_k)\varphi \in L\varphi.$$

Recall that $\mathrm{Fl}\,(L,E) = \{Z_\ell \mid \ell \in L\}$, and $Z_\ell = \ell{\downarrow} \cap E$ by Lemma 3.2.

Proposition 6.1 *Let* $(L,E),(L',E) \in$ FLg. *Then*

$$(L',E) \leq (L,E) \Leftrightarrow \mathrm{Fl}\,(L',E) \subseteq \mathrm{Fl}\,(L,E).$$

Proof. Assume first that $(L',E) \leq (L,E)$. Then there exists some $\vee$-map $\varphi : L \to L'$ such that $\varphi|_E = id$. We show that $\mathrm{Fl}\,(L',E) \subseteq \mathrm{Fl}\,(L,E)$. Indeed, we claim that

$$Z_{\ell'} = Z_{\vee(\ell'\varphi^{-1})} \tag{10}$$

holds for every $\ell' \in L'$. Let $e \in Z_{\ell'}$. Then $e \leq \ell'$. Write $\ell = \vee(\ell'\varphi^{-1})$. Since φ is onto and a $\vee$-map, we have $\ell\varphi = \ell'$. Moreover, $(\ell \vee e)\varphi = (\ell\varphi \vee e\varphi) = (\ell' \vee e) = \ell'$, hence $(\ell \vee e) \in \ell'\varphi^{-1}$ and so $(\ell \vee e) \leq \max(\ell'\varphi^{-1}) = \ell$. Thus $e \leq \ell$ and so $Z_{\ell'} \subseteq Z_{\vee(\ell'\varphi^{-1})}$.

Conversely, assume that $e \in Z_{\vee(\ell'\varphi^{-1})}$. Then $e \leq \vee(\ell'\varphi^{-1})$ and so $e = e\varphi \leq \ell'$. Hence $Z_{\vee(\ell'\varphi^{-1})} \subseteq Z_{\ell'}$ and so (10) holds. Therefore $\mathrm{Fl}\,(L',E) \subseteq \mathrm{Fl}\,(L,E)$.

Conversely, assume that $\mathrm{Fl}\,(L',E) \subseteq \mathrm{Fl}\,(L,E)$. We build a map $\varphi : L \to L'$ as follows. Let $\vee'$ denote join in L'. For every $\ell \in L$, we set $\ell\varphi = \vee'\{e \in E \mid e \leq \ell\}$.

It is immediate that φ is order-preserving. Hence, given $\ell_1, \ell_2 \in L$, we have $\ell_i\varphi \leq (\ell_1 \vee \ell_2)\varphi$ for $i = 1,2$ and so

$$(\ell_1\varphi \vee \ell_2\varphi) \leq (\ell_1 \vee \ell_2)\varphi. \tag{11}$$

Moreover, for every $e \in E$, we have $e\varphi = \vee'\{f \in E \mid f \leq e \text{ in } L\}$. Since $e \leq e$, we get $e \leq e\varphi$ in L'.

Now take $e \in E$ such that $e \leq (\ell_1 \vee \ell_2)$. Since $\mathrm{Fl}\,(L', E) \subseteq \mathrm{Fl}\,(L, E)$, we have $Z_{\ell_1\varphi \vee \ell_2\varphi} = Z_k$ for some $k \in L$. Let $f \in E$. Since φ is order-preserving, $f \leq \ell_i$ implies $f \leq f\varphi \leq \ell_i\varphi \leq (\ell_1\varphi \vee \ell_2\varphi)$ and so $f \in Z_{\ell_1\varphi \vee \ell_2\varphi} = Z_k$. Hence $\ell_i \leq k$ for $i = 1, 2$ and so $(\ell_1 \vee \ell_2) \leq k$. Thus $e \in Z_k = Z_{\ell_1\varphi \vee \ell_2\varphi}$ and so

$$(\ell_1 \vee \ell_2)\varphi = \vee'\{e \in E \mid e \leq (\ell_1 \vee \ell_2)\} \leq (\ell_1\varphi \vee \ell_2\varphi).$$

Together with (11), this implies that φ is a $\vee$-map.

It remains to be proved that $e\varphi \leq e$ holds in L' for every $e \in E$. Since $\mathrm{Fl}\,(L', E) \subseteq \mathrm{Fl}\,(L, E)$, we have

$$\{f \in E \mid f \leq e \text{ in } L'\} = \{f \in E \mid f \leq m \text{ in } L\}$$

for some $m \in L$. Hence $e \leq m$ holds in L. It follows that, for every $f \in E$, $f \leq e$ in L implies $f \leq m$ in L and therefore $f \leq e$ in L'. Hence $e\varphi = \vee'\{f \in E \mid f \leq e \text{ in } L\} = e$ and so $(L', E) \leq (L, E)$. $\square$

Recall that, if $E' = \{Z_e \mid e \in E\}$, then $(\mathrm{Fl}\,(L, E), E') \cong (L, E)$ holds for every $(L, E) \in \mathrm{FLg}$ by Proposition 3.3. We identify E' with E to simplify notation.

Now $\mathrm{Fl}\,(E, H)$ is closed under intersection by Proposition 5.2(i). We say that a $\cap$-semilattice F of $(\mathrm{Fl}\,(E, H), \subseteq)$ is *full* if $\emptyset, E \in F$. Let $\mathrm{FISFl}\,(E, H)$ denote the set of all full $\cap$-subsemilattices of $(\mathrm{Fl}\,(E, H), \subseteq)$. Then $(\mathrm{FISFl}\,(E, H), \subseteq)$ is a poset closed under intersection, hence a $\wedge$-semilattice and therefore a lattice with the determined join

$$(F_1 \vee F_2) = \cap\{F \in \mathrm{FISFl}\,(E, H) \mid F_1 \cup F_2 \subseteq F\}.$$

We say that $(L, E) \in \mathrm{FLg}$ is a boolean representation of a simple hereditary collection (E, H) if $M(L, E)$ is a boolean representation of (E, H). Let $\mathrm{BR}(E, H)$ denote the class of all $(L, E) \in \mathrm{FLg}$ which are boolean representations of (E, H). We restrict to $\mathrm{BR}(E, H)$ the quasi-order previously defined on FLg. If $(L, E) \in \mathrm{BR}(E, H)$, then by Proposition 5.5 we have $Z_\ell \in \mathrm{Fl}\,(E, H)$ for every $\ell \in L$. By Theorem 5.8 and Proposition 6.1, if (E, H) is boolean representable, then $(\mathrm{Fl}\,(E, H), E) \geq (L, E)$ for every $(L, E) \in \mathrm{BR}(E, H)$.

It is easy to check that

$$\begin{aligned} \theta : (\mathrm{BR}(E, H), \leq) &\to (\mathrm{FISFl}\,(E, H), \leq) \\ (L, E) &\mapsto \mathrm{Fl}\,(L, E) \end{aligned}$$

is a well-defined map. Indeed, let $(L, E) \in \mathrm{BR}(E, H)$. Then $\mathrm{Fl}\,(L, E) \subseteq \mathrm{Fl}\,(E, H)$ by Proposition 5.5, and it follows from (4) that $\mathrm{Fl}\,(L, E)$ is a $\cap$-subsemilattice of $\mathrm{Fl}\,(E, H)$. Note that $(L, E) \in \mathrm{FLg}$ implies $E \subseteq L \setminus \{B\}$. Since $\emptyset = Z_B$ and $E = Z_T$, we have $\mathrm{Fl}\,(L, E) \in \mathrm{FISFl}\,(E, H)$ and so θ is a well-defined map.

Our next goal is to turn θ into an isomorphism. A first obstacle is the fact that θ is not onto: not every $F \in \mathrm{FISFl}\,(E, H)$ is rich enough to represent (E, H). However, we claim that $\mathrm{FISFl}\,(E, H) \setminus \mathrm{Im}\,\theta$ is an order ideal of $\mathrm{FISFl}\,(E, H)$.

Since every $F \in \mathrm{FISFl}\,(E, H)$, being a $\cap$-subsemilattice of $\mathrm{Fl}\,(E, H)$, constitutes a lattice of its own right with the determined join, then, in view of Lemma 5.6, the question is whether the matrix arising from F produces enough witnesses to recognize all the independent

subsets in H. Therefore, if $F' \supseteq F$, every witness arising from F can also be obtained from F' and so $\mathrm{FISFl}\,(E,H) \setminus \mathrm{Im}\,\theta$ is an ideal of $\mathrm{FISFl}\,(E,H)$. Let $\mathrm{FISFl}_0(E,H)$ denote the Rees quotient (see Subsection 10.1 of the Appendix) of $\mathrm{FISFl}\,(E,H)$ by the above ideal. By Proposition 10.3, $\mathrm{FISFl}_0(E,H) = \mathrm{Im}\,\theta \cup \{B\}$ is a lattice.

On the other hand, adding a bottom element B to $\mathrm{BR}(E,H)$, we get a quasi-ordered set $\mathrm{BR}_0(E,H) = \mathrm{BR}(E,H) \cup \{B\}$ and we can extend θ to an onto map $\theta_0 : \mathrm{BR}_0(E,H) \to \mathrm{FISFl}_0(E,H)$ by setting $B\theta_0 = B$. Clearly, Proposition 6.1 immediately yields:

Corollary 6.2 *For all $R, S \in BR_0(E,H)$, $R \leq S$ if and only if $R\theta_0 \subseteq S\theta_0$.*

Let ρ be the equivalence in $\mathrm{BR}_0(E,H)$ defined by $\rho = (\leq \cap \geq)$. Clearly, two representations $(L,E), (L',E)$ are ρ-equivalent if there exists some lattice isomorphism $\varphi : L \to L'$ which is the identity on E. Then the quotient $\mathrm{BR}_0(E,H)/\rho$ becomes a poset and by Corollary 6.2, the induced mapping $\overline{\theta_0} : \mathrm{BR}_0(E,H)/\rho \to \mathrm{FISFl}_0(E,H)$ is a poset isomorphism. Since we have already remarked that $\mathrm{FISFl}_0(E,H)$ is a lattice (with the determined join), we have proved the following theorem:

Theorem 6.3 *Let (E,H) be a simple boolean representable hereditary collection. Then $\overline{\theta_0} : \mathrm{BR}_0(E,H)/\rho \to \mathrm{FISFl}_0(E,H)$ is a lattice isomorphism.*

An *atom* of a lattice is an element covering the bottom element B. The atoms of $\mathrm{BR}_0(E,H)$ determine the *minimal* boolean representations of (E,H), and the sji elements of $\mathrm{BR}_0(E,H)$ determine the sji boolean representations. Clearly, meet is given by intersection in $\mathrm{FISFl}_0(E,H)$, collapsing into the bottom B if it does not correspond anymore to a representation of (E,H). But how can join be characterized in this lattice?

Proposition 6.4 *Let (E,H) be a simple boolean representable hereditary collection. Let $F, F' \in \mathrm{FISFl}_0(E,H)$. Then:*

(i) $(F \vee F') = F \cup F' \cup \{Z \cap Z' \mid Z \in F,\ Z' \in F'\}$.

(ii) If $(L,E)\theta = F$, $(L',E)\theta = F'$ and $(L'',E)\theta = (F \vee F')$, then $M(L'',E)$ is the closure of $M(L,E) \oplus_b M(L',E)$ under row sum in $\mathbb{B}^{|E|}$.

Proof. (i) Clearly, the right hand side is the (full) $\cap$-subsemilattice of $\mathrm{Fl}\,(E,H)$ generated by $F \cup F'$.

(ii) Recall the isomorphism from Proposition 3.3. The rows r_Z of $M(L,E)$ (respectively $M(L',E)$, $M(L'',E)$) are determined then by the flats Z in F (respectively F', $F \vee F'$). It is immediate that $r_{Z \cap Z'} = r_Z + r_{Z'}$ in $\mathbb{B}^{|E|}$, hence $M(L'',E)$ must be, up to permutation of rows, the stacking of $M(L,E)$ and $M(L',E)$, to which we add (if needed) rows which are the sum in $\mathbb{B}^{|E|}$ of rows in $M(L',E)$ and $M(L'',E)$. $\square$

Next we introduce the notion of *boolean sum* in $\mathrm{BR}(E,H)$. Given $(L,E), (L',E) \in \mathrm{BR}(E,H)$, let $(L,E) \oplus_b (L',E)$ denote the $\vee$-subsemilattice of the direct product $L \times L'$ $\vee$-generated by the diagonal $E_d = \{(e,e) \mid e \in E\} \subseteq L \times L'$. Taking the determined meet, and identifying E_d with E as expectable, it follows that $(L,E) \oplus_b (L',E) \in \mathrm{FLg}$. In fact, since the projection $(L,E) \oplus_b (L',E) \to (E,L)$ is a $\vee$-map which is the identity on E, it follows easily that $(L,E) \oplus_b (L',E) \in \mathrm{BR}(E,H)$. But we can prove more:

Proposition 6.5 *Let (E,H) be a simple hereditary collection and let $(L,E), (L',E) \in \mathrm{BR}(E,H)$. Then*

$$(L,E)\rho \vee (L',E)\rho = ((L,E) \oplus_b (L',E))\rho \tag{12}$$

holds in $BR_0(E,H)/\rho$. Moreover, $M((L,E)\rho\vee(L',E)\rho)$ is the closure of the stacking matrix $M(L,E)\oplus_b M(L',E)$ under row sum in $\mathbb{B}^{|E|}$.

Proof. By the preceding comment, we have $(L,E)\leq(L,E)\oplus_b(L',E)$ and also $(L',E)\leq(L,E)\oplus_b(L',E)$, hence

$$(L,E)\rho\vee(L',E)\rho\leq((L,E)\oplus_b(L',E))\rho.$$

Now let $(L'',E)\in\mathrm{BR}(E,H)$ and suppose that $(L,E),(L',E)\leq(L'',E)$. We must show that also $(L,E)\oplus_b(L',E)\leq(L'',E)$. Indeed, there exist $\vee$-maps $\varphi:L\to L''$ and $\varphi':L'\to L''$ which fix E. Let $\varphi'':L\times L'\to L''$ be defined by $(\ell,\ell')\varphi''=(\ell\varphi\vee\ell'\varphi')$. Since $((\ell_1,\ell_1')\vee(\ell_2,\ell_2'))=(\ell_1\vee\ell_2,\ell_1'\vee\ell_2')$ in $L\times L'$, it follows easily that φ'' is a $\vee$-map. Moreover, since both φ and φ' fix the elements of E, so does φ''. Since the restriction of a $\vee$-map to a $\vee$-subsemilattice is still a $\vee$-map, it follows that $(L,E)\oplus_b(L',E)\leq(L'',E)$ and (12) holds.

The last claim follows from Proposition 6.4. □

We can now state the following straightforward consequence:

Corollary 6.6 *Let (E,H) be a simple hereditary collection and consider $(L,E),(L',E)\in\mathrm{BR}(E,H)$. Then:*

(i) (L,E) can be decomposed as a boolean sum (equivalently, stacking matrices and closing under row sum) of sji representations;

(ii) this decomposition is not unique in general, but becomes so if we take a maximal decomposition by taking all the sji representations below (L,E).

Examples shall be provided in Section 8.

Remark 6.7 *Let (E,H) be a simple hereditary collection and let $(L,E),(L',E)\in\mathrm{BR}(E,H)$. It is reasonable to identify (L,E) and (L',E) if some bijection of E induces a $\vee$-bijection $L\to L'$, and list only up to this identification in examples. However, for purposes of boolean sum decompositions, the bijection on E must be the identity.*

So we shall devote particular attention to minimal/sji boolean representation of (E,H). How do these concepts relate to the flats in $\mathrm{FISFl}\,(E,H)$ and to the matrices representing them?

If L is a lattice, we denote by $g_\vee(L)$ (respectively $g_\wedge(L)$) the unique minimum set of $\vee$-generators (respectively $\wedge$-generators) of L. Clearly, $g_\vee(L)$ equals the set of all sji elements of $L\setminus\{B\}$. Similarly, $g_\wedge(L)$ equals the set of all smi elements of $L\setminus\{T\}$.

Given $(L,E)\in\mathrm{BR}(E,H)$, we may view $\mathrm{Fl}\,(L,E)$ as a lattice with the determined join and define $\widehat{Z}(L,E)=g_\wedge(\mathrm{Fl}\,(L,E))$. That is, $\widehat{Z}(L,E)$ consists of all the smi flats in $\mathrm{Fl}\,(L,E)\setminus\{E\}$, i.e. which cannot be nontrivially expressed as intersections of other flats in $\mathrm{Fl}\,(L,E)$ (note that $E=\cap\emptyset$, hence $E\notin\widehat{Z}(L,E)$). In view of Proposition 3.3 (and particularly (4)), we have

$$\widehat{Z}(L,E)=\{Z_\ell\mid\ell\in g_\wedge(L)\}. \tag{13}$$

If we transport these concepts into $M(L,E)$, then $\widehat{Z}(L,E)$ corresponds to the submatrix $\widehat{M}(L,E)$ determined by the rows which are not sums of other rows in $\mathbb{B}^{|E|}$, excluding also

the row with just zeroes. By Proposition 5.7(ii), $\widehat{M}(L,E)$ is still a boolean representation of (E,H). Note that, if we consider $\mathbb{B}$ ordered by $0<1$ and the direct product partial order in $\mathbb{B}^{|E|}$, then the rows in $\widehat{M}(L,E)$ are precisely the sji rows of $M(L,E)$ for this partial order.

In the following three key propositions, we shall use the concept of MPS and Proposition 10.7, which the reader can find in Subsection 10.1 of the Appendix. We shall use the notation $\mathring{L}=L\setminus\{T,B\}$.

Proposition 6.8 *Let (E,H) be a simple hereditary collection and let $(L,E),(L',E)\in \mathrm{BR}(E,H)$. Then the following conditions are equivalent:*

(i) $(L,E)\rho$ covers $(L',E)\rho$ in $\mathrm{BR}_0(E,H)$;

(ii) there exists an MPS $\varphi: L\to L'$ fixing the elements of E;

(iii) $\mathrm{Fl}\,(L',E)=\mathrm{Fl}\,(L,E)\setminus\{Z_\ell\}$ for some smi $l\in\mathring{L}$.

Proof. (i) $\Rightarrow$ (ii). If $(L,E)\rho$ covers $(L',E)\rho$ in $\mathrm{BR}_0(E,H)$, then the (onto) $\vee$-map $\varphi: L\to L'$ cannot be factorized as the composition of two proper (onto) $\vee$-maps, and so φ is an MPS.

(ii) $\Rightarrow$ (iii). By Proposition 10.7, $\mathrm{Ker}\,\varphi=\rho_{k,\ell}$ for some $k,\ell\in L$ such that k covers ℓ and ℓ is smi, hence $\ell\neq T$. Therefore we may assume that $L'=L/\rho_{k,\ell}$. Clearly, $Z_{k\vee\ell}=Z_k$ and so (10) yields $\mathrm{Fl}\,(L',E)=\mathrm{Fl}\,(L,E)\setminus\{Z_\ell\}$. Note that we are assuming that $(L',E)\in \mathrm{BR}(E,H)\subseteq \mathrm{FLg}$, hence $E\subseteq L'\setminus\{B\}$ and so $\ell\neq B$ (since B is covered only by elements of E). Therefore (iii) holds. as desired.

(iii) $\Rightarrow$ (i). It follows easily from (10) that $\mathrm{Ker}\,\varphi$ has one class with two elements and all the others are singular, hence $|L'|=|L|-1$ and so $(L,E)\rho$ covers $(L',E)\rho$ in $\mathrm{BR}_0(E,H)$. $\square$

This will help us to characterize minimal and sji boolean representations of (E,H) in terms of their flats.

Proposition 6.9 *Let (E,H) be a simple hereditary collection and let $(L,E)\in \mathrm{BR}(E,H)$. Then the following conditions are equivalent:*

(i) (L,E) is minimal;

(ii) for every MPS $\varphi: L\to L'$ fixing the elements of E, $(L',E)\notin \mathrm{BR}(E,H)$;

(iii) for every smi $l\in\mathring{L}$, the matrix obtained by removing the row ℓ from $M(L,E)$ is not a matrix boolean representation of (E,H);

(iv) for every smi $\ell\in\mathring{L}$, $\mathrm{Fl}\,(L,E)\setminus\{Z_\ell\}\notin \mathrm{Im}\,\theta$.

Proof. (i) $\Leftrightarrow$ (iv). By Proposition 6.8.

(i) $\Rightarrow$ (iii). Let $l\in\mathring{L}$ be an smi, and let k be the unique element of L covering ℓ. By Proposition 6.8, $L'=L/\rho_{k,\ell}$ is a lattice and $M(L',E)$ is precisely the matrix obtained by removing the row ℓ from $M(L,E)$. If $M(L',E)$ is a boolean representation of (E,H), then $(L,E)\rho$ covers $(L',E)\rho$ in $\mathrm{BR}_0(E,H)$ and so (L,E) is not minimal.

(iii) $\Rightarrow$ (iv). Suppose that $\mathrm{Fl}\,(L,E)\setminus\{Z_\ell\}\in \mathrm{Im}\,\theta$ for some smi $\ell\in\mathring{L}$. Then $\mathrm{Fl}\,(L,E)\setminus\{Z_\ell\}=\mathrm{Fl}\,(L',E)$ for some $(L',E)\in \mathrm{BR}(E,H)$. It is straightforward to check that $M(L',E)$ is the matrix obtained by removing the row ℓ from $M(L,E)$. Thus (iii) fails. $\square$

Proposition 6.10 *Let (E,H) be a simple hereditary collection and let $(L,E) \in \mathrm{BR}(E,H)$. Then the following conditions are equivalent:*

(i) (L,E) *is sji;*

(ii) up to isomorphism, there is at most one MPS $\varphi : L \to L'$ fixing the elements of E and such that $(L',E) \in \mathrm{BR}(E,H)$;

(iii) there exists at most one smi $l \in \mathring{L}$ such that the matrix obtained by removing the row ℓ from $M(L,E)$ is still a matrix boolean representation of (E,H);

(iv) there exists at most one smi $\ell \in \mathring{L}$ such that $\mathrm{Fl}\,(L,E) \setminus \{Z_\ell\} \in \mathrm{Im}\,\theta$.

Proof. Clearly, (L,E) is sji if and only if $(L,E)\rho$ covers exactly one element in $\mathrm{BR}_0(E,H)$. Now we apply Proposition 6.8, proceeding analogously to the proof of Proposition 6.9. $\square$

We call a reduced matrix boolean representation M of (E,H) *rowmin* if any matrix obtained by removing a row of M is no longer a boolean representation of (E,H).

Proposition 6.11 *Let (E,H) be a simple hereditary collection and let $(L,E) \in \mathrm{BR}(E,H)$ be minimal. Then $\widehat{M}(L,E)$ is rowmin.*

Proof. By Proposition 6.9, we cannot remove from $\widehat{M}(L,E)$ a row corresponding to an smi element of $\mathring{L}$. Suppose now that B is smi. Then B is covered in L by a unique element e, necessarily in E since L is $\vee$-generated by E and so the unique 1 in the column of $M(L,E)$ determined by e occurs at the row of B. Since $\{e\}$ is independent due to (E,H) being simple, it follows that the row of B cannot be removed either. $\square$

The next example shows that the converse of Proposition 6.11 does not hold.

Example 6.12 *Let (E,H) be the matroid defined by $E = \hat{4}$ and $H = P_3(E) \setminus \{123\}$. Then*

$$M = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a boolean representation of (E,H) of minimum degree (therefore rowmin) but $M \neq \widehat{M}(L,E)$ for every minimal $(L,E) \in \mathrm{BR}(E,H)$.

Indeed, it follows from the analysis developed later in Example 8.1 that $M = \widehat{M}(L,E)$ for some sji $(L,E) \in \mathrm{BR}(E,H)$ (with $(L,E)\theta = \{E, 123, 14, 1, 2, \emptyset\}$), being therefore a boolean representation of (E,H). It has minimum degree since it has only 3 rows and there exist independent 3-subsets of E. However, the description of the minimal cases in Example 8.1 shows that M does not arise from any of them.

7 Closure operators

In this section we relate the closure operator induced by a hereditary collection with the closure operator induced by a representation. In the follow-up, we denote by Cl the closure operator on 2^E induced by (E,H). Given $(L,E) \in \mathrm{BR}(E,H)$, we denote by Cl_L the closure operator on 2^E as defined in Section 4.

Lemma 7.1 *Let (E, H) be a simple hereditary collection and let $(L, E) \in \mathrm{BR}(E, H)$. Let $X \subseteq E$. Then:*

(i) $\mathrm{Cl}\, X \subseteq \mathrm{Cl}_L X$*;*

(ii) $\mathrm{Cl}\, X = \mathrm{Cl}_L X$ *if* $L = \mathrm{Fl}\,(E, H)$*.*

Proof. (i) We have $\mathrm{Cl}\, X = \cap\{Z \in \mathrm{Fl}\,(E, H) \mid X \subseteq Z\}$ and in view of (6) and Proposition 5.5 also

$$\mathrm{Cl}_L X = Z_{\vee X} \in \mathrm{Fl}\,(E, H).$$

Since $X \subseteq \mathrm{Cl}_L X$, we get $\mathrm{Cl}\, X \subseteq \mathrm{Cl}_L X$.

(ii) Assume that $L = \mathrm{Fl}\,(E, H)$. Let $Z \in \mathrm{Fl}\,(E, H)$ be such that $X \subseteq Z$. It suffices to show that $\mathrm{Cl}_L X \subseteq Z$, i.e. $Z_{\vee X} \subseteq Z$. Now in $\mathrm{Fl}\,(E, H)$ we have $\vee X = \mathrm{Cl}\, X$ and $Z_{\mathrm{Cl}\, X} = \{e \in E \mid e \in \mathrm{Cl}\, X\} = \mathrm{Cl}\, X$, hence we must show that $\mathrm{Cl}\, X \subseteq Z$. This follows from $X \subseteq Z$ and Z being closed, therefore we are done. $\square$

Proposition 7.2 *Let (E, H) be a simple hereditary collection admitting a boolean representation $(L, E) \in \mathrm{BR}(E, H)$ and let $X \subseteq E$. Then the following conditions are equivalent:*

(i) $X \in H$*;*

(ii) X *admits an enumeration* $x_1, \ldots, x_k$ *such that*

$$\mathrm{Cl}_L(x_1, \ldots, x_k) \supset \mathrm{Cl}_L(x_2, \ldots, x_k) \supset \ldots \supset \mathrm{Cl}_L(x_k); \tag{14}$$

(iii) X *admits an enumeration* $x_1, \ldots, x_k$ *such that*

$$\mathrm{Cl}(x_1, \ldots, x_k) \supset \mathrm{Cl}(x_2, \ldots, x_k) \supset \ldots \supset \mathrm{Cl}(x_k). \tag{15}$$

Proof. (i) $\Rightarrow$ (ii). Assume $X \in H$. Since $(L, E) \in \mathrm{BR}(E, H)$, it follows that X is c-independent. By Proposition 4.2, this is equivalent to saying that X admits an enumeration $x_1, \ldots, x_k$ such that (14) holds.

(ii) $\Rightarrow$ (iii). Suppose that $x_i \in \mathrm{Cl}\,(x_{i+1}, \ldots, x_k)$ for some i. By Lemma 7.1(i), we get $x_i \in \mathrm{Cl}_L(x_{i+1}, \ldots, x_k)$ and so $\mathrm{Cl}_L(x_i, \ldots, x_k) = \mathrm{Cl}_L(x_{i+1}, \ldots, x_k)$, a contradiction. Hence $x_i \notin \mathrm{Cl}\,(x_{i+1}, \ldots, x_k)$ for every i and so (15) holds.

(iii) $\Rightarrow$ (i). Consider the representation of (E, H) by $(L', E) = (\mathrm{Fl}\,(E, H), E)$. By Lemma 7.1(ii), we have

$$\mathrm{Cl}_{L'}(x_1, \ldots, x_k) \supset \mathrm{Cl}_{L'}(x_2, \ldots, x_k) \supset \ldots \supset \mathrm{Cl}_{L'}(x_k).$$

It follows from Proposition 4.2 that X is c-independent with respect to $M(\mathrm{Fl}\,(E, H), E)$, and so $X \in H$ by Lemma 5.6. $\square$

We can now prove another characterization of boolean representability:

Theorem 7.3 *Let (E, H) be a simple hereditary collection. Then the following conditions are equivalent:*

(i) (E, H) *admits a boolean representation;*

(ii) every $X \in H$ admits an enumeration $x_1, \ldots, x_k$ satisfying (15).

Proof. (i) $\Rightarrow$ (ii). By Proposition 7.2.

(ii) $\Rightarrow$ (i). Let $X \in H$. In view of (15), we can use the flats $\mathrm{Cl}\,(x_i, \ldots, x_k)$ as a witness for X, hence X is c-independent with respect to $M(\mathrm{Fl}\,(E, H), E)$. Lemma 5.6 yields the reciprocal implication and so $M(\mathrm{Fl}\,(E, H), E)$ is a reduced boolean representation of (E, H). $\square$

We can also characterize which lattices provide boolean representations:

Proposition 7.4 *Let (E, H) be a boolean representable simple hereditary collection and let $\varphi : (\mathrm{Fl}(E, H), E) \to (L, E)$ be a $\vee$-map fixing the elements of E. Then the following conditions are equivalent:*

(i) $(L, E) \in \mathrm{BR}(E, H)$;

(ii) every $X \in H$ admits an enumeration $x_1, \ldots, x_k$ satisfying (14).

Proof. (i) $\Rightarrow$ (ii). By Proposition 7.2.

(ii) $\Rightarrow$ (i). Let $X \subseteq E$. We show that $X \in H$ if and only if X is c-independent (with respect to $M(L, E)$).

Assume that $X \in H$. Since $\mathrm{Cl}_L Y = (\vee Y) \downarrow \cap E = Z_{\vee Y}$ for every $Y \subseteq E$ by (6), it follows from (ii) that the rows $\mathrm{Cl}_L(x_i, \ldots, x_k)$ act as a witness for X in $M(L, E)$ and so X is c-independent.

Conversely, assume that X is c-independent. By Proposition 6.1, $M(L, E)$ is a submatrix of $M(\mathrm{Fl}\,(E, H), E)$ and so X is c-independent with respect to $M(\mathrm{Fl}\,(E, H), E)$. Hence $X \in H$ by Lemma 5.6. $\square$

Corollary 7.5 *Let (E, H) be a boolean representable simple hereditary collection and let $F \in \mathrm{FISFl}(E, H)$. For every $X \subseteq E$, let $\mathrm{Cl}_F X = \cap\{Z \in F \mid Z \supseteq X\}$. Then the following conditions are equivalent:*

(i) $F = \mathrm{Fl}\,(L, E)$ for some $(L, E) \in \mathrm{BR}(E, H)$;

(ii) every $X \in H$ admits an enumeration $x_1, \ldots, x_k$ satisfying

$$\mathrm{Cl}_F(x_1, \ldots, x_k) \supset \mathrm{Cl}_F(x_2, \ldots, x_k) \supset \ldots \supset \mathrm{Cl}_F(x_k); \tag{16}$$

Proof. As noted before, since F is a $\cap$-subsemilattice of $\mathrm{Fl}\,(E, H)$, it constitutes a lattice of its own with the determined join

$$(X \vee Y) = \mathrm{Cl}_F(X \cup Y) \quad (X, Y \in F).$$

Identifying E with $\{\mathrm{Cl}_F\{e\} \mid e \in E\}$, we can view (F, E) as an element of FLg, isomorphic to (L, E) in view of Proposition 3.3. Now we apply Proposition 7.4. $\square$

The important subcase of boolean representations of matroids was studied in [9, 10] and the following fundamental result was proved. For the sake of completeness, we include here a short alternative proof:

Theorem 7.6 [10, Theorem 4.1] *Let (E, H) be a simple matroid. Then $M(\mathrm{Fl}\,(E, H), E)$ is a boolean representation of (E, H).*

Proof. In view of Theorems 5.8 and 7.3, it suffices to show that every $X \in H$ admits an enumeration $x_1, \ldots, x_k$ satisfying (15). Thus we only have to show that

$$x_i \notin \mathrm{Cl}\,(x_1, \ldots, x_{i-1}) \tag{17}$$

for $i = 2, \ldots, k$. Indeed, suppose that $x_i \in \mathrm{Cl}\,(x_1, \ldots, x_{i-1})$. By Proposition 5.4, we have $x_i \in C \subseteq \{x_1, \ldots, x_i\}$ for some circuit C, hence $C \subseteq X \in H$, a contradiction. Thus (17) holds as required. $\square$

Example 7.7 *Let (E, H) be a simple hereditary collection with $|E| = 4$.*

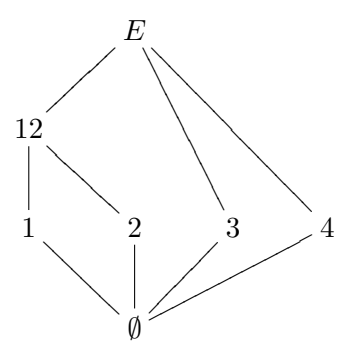
(i) If H has 0, 3 or 4 independent 3-subsets, then (E, H) is a matroid and therefore boolean representable.

(ii) If H has 1 independent 3-subset, then (E, H) does not satisfy (PR) and so is not boolean representable.

(iii) If H has 2 independent 3-subsets, then (E, H) is not a matroid but it is boolean representable.

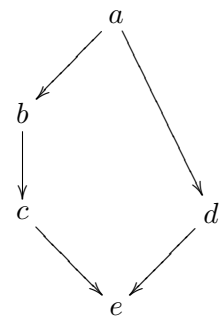
Indeed, (i) and (ii) are straightforward (in view of [9, Theorem 5.3]). In (iii), we may assume that 123 and 124 denote the independent 3-subsets. Since 34 is also a basis, then (E, H) is not a matroid. However, since $\mathrm{Fl}\,(E, H) = \mathcal{P}_1(E) \cup \{12, E\}$, it follows easily from Theorem 7.3 that (E, H) is boolean representable.

In fact, in this case the lattice of flats can be depicted as



and so there exist maximal chains $\emptyset \subset 1 \subset 12 \subset E$ and $\emptyset \subset 4 \subset E$ of different length. Hence $\mathrm{Fl}\,(E, H)$ does not satisfy the *Jordan-Dedekind* condition and so is not semimodular by [6, Theorem 374]. We recall that a lattice L is said to be *semimodular* if there is no sublattice

of the form



with d covering e.

A lattice is called *atomic* if every element is a join of atoms (B being the join of the empty set). It is said to be *geometric* if it is both semimodular and atomic. It is well known that a lattice is geometric if and only if it is isometric to the lattice of flats of some matroid [13, Theorem 1.7.5].

Hence the above example shows that properties such as semimodularity or the Jordan-Dedekind condition, which hold in the lattice of flats of a matroid, may fail in the lattice of flats of a boolean representable hereditary collection, even though it is simple and paving (see Subsection 9.2).

8 Examples

We present now some examples where we succeed on identifying all the minimal and sji boolean representations.

Example 8.1 *Let (E, H) be the matroid defined by $E = \hat{4}$ and $H = P_3(E) \setminus \{123\}$. We compute all the minimal and sji representations of (E, H).*

It is routine to compute $\mathrm{Fl}\,(E, H) = P_1(E) \cup \{14, 24, 34, 123, 1234\}$. Which $F \in \mathrm{FISFl}\,(E, H)$ correspond to lattice representations (i.e. $F \in \mathrm{Im}\,\theta$)? We claim that $F \in \mathrm{Im}\,\theta$ if and only if one of the following conditions is satisfied:

$$123 \in F \text{ and } |\{1, 2, 3\} \cap F| \geq 2, \tag{18}$$

$$|\{14, 24, 34\} \cap F| \geq 2. \tag{19}$$

In view of Corollary 7.5, it is easy to see that any of the conditions implies $F \in \mathrm{Im}\,\theta$ (note that $4 = 14 \cap 24 \in F$ in the case (19)).

Conversely, assume that $F \in \mathrm{Im}\,\theta$ and suppose that $|\{14, 24, 34\} \cap F| \leq 1$. Without loss of generality, we may assume that $24, 34 \notin F$. Since $234 \in H$, it follows from Corollary 7.5 that there exists an enumeration x, y, z of 234 such that

$$\mathrm{Cl}_F(x) \subset \mathrm{Cl}_F(xy) \subset \mathrm{Cl}_F(xyz).$$

The only possibility for $\mathrm{Cl}_F(xy)$ in F is now 123. Hence $\mathrm{Cl}_F(x) \in \{2, 3\}$. Out of symmetry, we may assume that $2 \in F$. On the other hand, since $13 \in H$, there exists an enumeration a, b of 13 such that 2

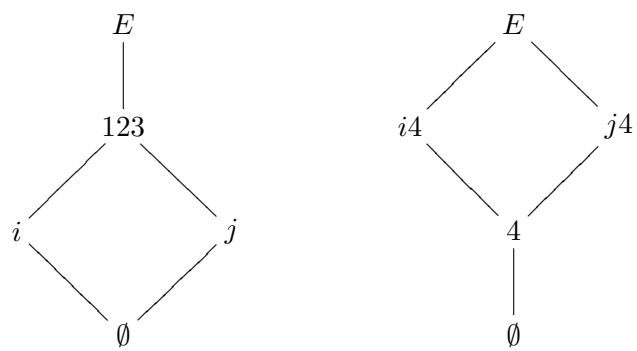
$$\mathrm{Cl}_F(a) \subset \mathrm{Cl}_F(ab) = 123.$$

The only possibilities for $\mathrm{Cl}_F(a)$ in F are now 1 or 3, hence (18) holds.

We consider now the minimal case. Assume that $F \in \operatorname{Im}\theta$. Since we can view (F, E) as a lattice projecting onto F through θ, and by Proposition 6.9, the key lies with the smi elements of F (with respect to intersection). More precisely, $(F, E) \in \mathrm{BR}(E, H)$ is minimal if and only if removal of an smi element of $F \setminus \{E, \emptyset\}$ takes us outside $\operatorname{Im}\theta$. It follows easily from our characterization of $\operatorname{Im}\theta$ that this corresponds to having

$$F = \{E, 123, i, j, \emptyset\} \quad \text{or} \quad F = \{E, i4, j4, 4, \emptyset\}$$

for some distinct $i, j \in \hat{3}$, leading to the lattices



Note that, if we wish to identify the generators E in these lattices, we only have to look for $\mathrm{Cl}\, e$ for each $e \in E$. For instance, in the first lattice, the top element corresponds to the generator 4.

Following Remark 6.7, we can count the number of minimal lattice representations

- up to identity in the $\vee$-generating set E $(3 + 3 = 6)$;
- up to some bijection of E inducing a $\vee$-bijection on the lattices $(1 + 1 = 2)$.

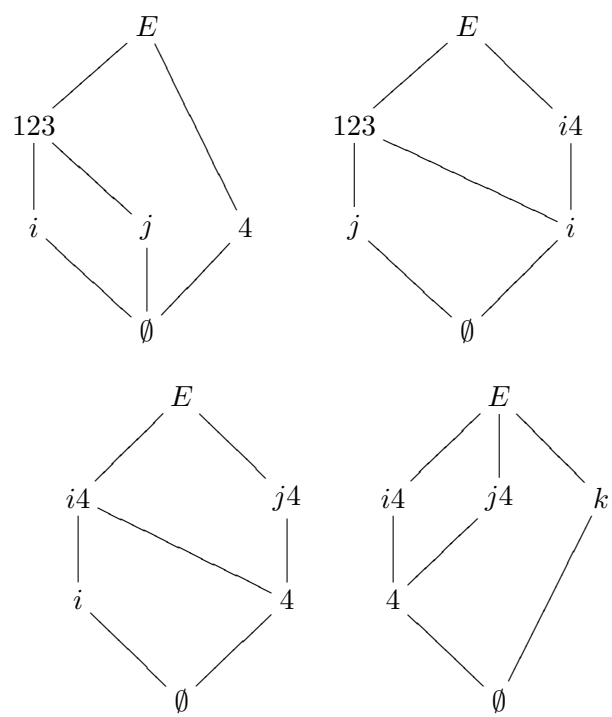
With respect to the sji representations, it follows from Proposition 6.10 that $(F, E) \in \mathrm{BR}(E, H)$ is sji if and only if there is at most one smi element of $F \setminus \{E, \emptyset\}$ whose removal keeps us inside $\operatorname{Im}\theta$. We claim that this corresponds to either the minimal case or having

$$F = \{E, 123, i, j, 4, \emptyset\} \quad \text{or} \quad F = \{E, 123, i4, i, j, \emptyset\} \quad \text{or} \quad F = \{E, i4, j4, k, 4, \emptyset\} \tag{20}$$

for some $i, j, k \in \hat{3}$ with $i \neq j$. It is immediate that the cases in (20) lead to (F, E) sji, the only possible removals being respectively 4, $i4$ and k. Conversely, assume that F corresponds to an sji non minimal case. Suppose first that F satisfies (18). If none of the pairs $k4$ is in F, then F must contain precisely three singletons to avoid the minimal case, and one of them must be 4 to avoid having a mutiple choice in the occasion of removing one of them. Hence we may assume that $i4 \in F$ and so also $i = 123 \cap i4$. If $j4 \in F$ for another $j \in \hat{3}$, then also $j \in F$ and we would have the option of removing either $i4$ or $j4$. Hence F contains $E, 123, i4, i, \emptyset$, and possibly any other singletons. In fact, it must contain at least one in view of (18) but obviously not both. Thus F is of the first form in (20) in this case.

Now assume that F satisfies (19) but not (18). Assume that $i4, j4 \in F$ for some distinct $i, j, k \in \hat{3}$. Then $123 \notin F$, otherwise $i, j \in F$ and we can remove either $i4$ or $j4$. Clearly, a

third pair $k4$ is forbidden, otherwise we could remove any one of the three pairs. Thus F contains $E, i4, j4, 4, \emptyset$, and possibly any other singletons. In fact, it must contain at least one to avoid the minimal case but obviously not two, since any of them could be removed. Thus F is of the second form in (20) and we have identified all the sji cases. In the second case of (20), we must separate the subcases $k = i$ and $k \notin \{i, j\}$. Thus the sji non minimal cases lead to the lattices



Following Remark 6.7 as in the minimal case, the number of sji lattice representations in both counts (which includes the minimal ones) is respectively $6+3+6+6+3 = 24$ and $1+1+1+1+1 = 5$.

It is easy to see that

$$\mathrm{Fl}\,(E, H) = \{E, 123, 34, 2, 3, \emptyset\} \cup \{E, 14, 24, 1, 4, \emptyset\}$$

provides a decomposition of the top boolean representation $\mathrm{Fl}\,(E, H)$ as the join of two sji's. In matrix form, and in view of Proposition 6.5, this corresponds to express the matrix

$$M(\mathrm{Fl}\,(E, H), E) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

as the stacking of the matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Note that the maximal decomposition of $\mathrm{Fl}\,(E, H)$ as join of sji's would include 24 factors.

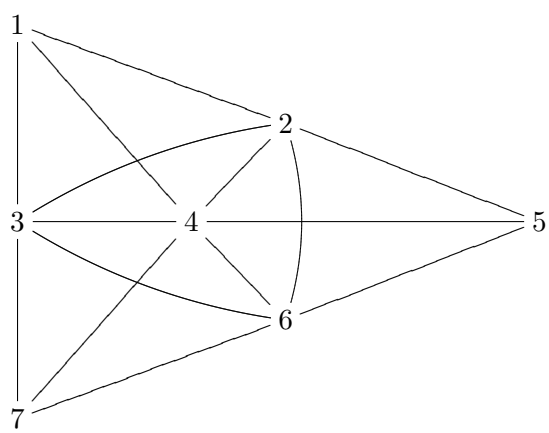
It is also easy to see, in view of Proposition 5.7(ii) (which allows us to discard those rows corresponding to non smi elements) that $\mathrm{mindeg}\,(E, H) = 3$: we take the minimal representation defined by $F = \{E, 14, 24, 4, \emptyset\}$ and discard the row corrresponding to $4 = 14 \cap 24$. We can also discard the useless row of zeroes corresponding to $\emptyset$ to get the matrix

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

We cannot do better than this since there are independent 3-sets in H. Therefore $\mathrm{mindeg}\,(E, H) = 3$.

Example 8.2 *Let (E, H) be the Fano matroid defined by $E = \hat{7}$ and $H = P_3(E) \setminus \{125, 137, 146, 236, 247, 345, 567\}$. We compute all the minimal and sji representations of (E, H).*

Note that $\mathcal{L} = P_3(E) \setminus H$ is precisely the set of *lines* in the Fano plane [20] (the projective plane of order 2 over the two element field):



We can view the Fano plane as a PEG (see Subsection 10.3 of the Appendix). We list a few of its properties:

(F1) Any two distinct lines intersect at a single point.

(F2) Every point belongs to exactly three lines.

(F3) Any two points belong to some line.

(F4) If K consists of 5 lines, then there exist two points a, b such that K consists of all lines containing either a or b.

We note that (F4) follows from (F1) since the two lines not in K must cover exactly 5 points, and we may take a, b as the two remaining points.

It is easy to check that $\mathrm{Fl}\,(E, H) = P_1(E) \cup \mathcal{L}$: the lines are obviously closed, the 2-subsets are not, and every 4-subset of E contains an independent set and has therefore closure E by Proposition 5.3.

Let $F \in \mathrm{FISFl}\,(E, H)$. We claim that $F \in \operatorname{Im}\theta$ if and only if $|F \cap \mathcal{L}| \geq 5$ or

$$|F \cap \mathcal{L}| = 4 \text{ and no 3 lines of } F \cap \mathcal{L} \text{ intersect at a point.} \tag{21}$$

Assume that $F \in \operatorname{Im}\theta$. Suppose that $|F \cap \mathcal{L}| < 5$ and (21) fails. If $|F \cap \mathcal{L}| \leq 3$, we can extend F to some set F' satisfying $|F' \cap \mathcal{L}| = 4$ and having 3 lines intersecting at a point. Otherwise, let $F' = F$.

Suppose that $F' \cap \mathcal{L} = \{X_1, X_2, X_3, X_4\}$ and X_1, X_2, X_3 intersect at a certain point p. For $i = 1, 2, 3$, take some point $x_i \in X_i \setminus (X_4 \cup \{p\})$. The points are all distinct by (F1). Let $Y = x_1x_2x_3$. Since $X_4 \cap Y = \emptyset$, it follows from (F1) that $Y \in H$. It follows from Corollary 7.5 that there exists an enumeration y_1, y_2, y_3 of Y such that

$$\mathrm{Cl}_{F'}(y_1) \subset \mathrm{Cl}_{F'}(y_1y_2) \subset \mathrm{Cl}_{F'}(y_1y_2y_3).$$

Hence $\mathrm{Cl}_{F'}(y_1y_2) \in F' \cap \mathcal{L}$. Since $Y \cap X_4 = \emptyset$, it follows that $|Y \cap X_i| \geq 2$ for some $i \in \hat{3}$. Hence $x_j \in X_i$ for some $j \neq i$, yielding $|X_i \cap X_j| \geq 2$ and contradicting (F1).

Assume now that (21) holds. Let $X = xyz \in H$. By (F3), we may write $x'yz, xy'z, xyz' \in \mathcal{L}$ for some $x', y', z' \in E$. Since $x' \neq x$ due to $xyz \in H$, $x'yz \in \mathcal{L}$ (and similarly, $y' \neq y$), it follows that the three lines $x'yz, xy'z, xyz'$ are distinct.

Suppose that $x'yz, xy'z, xyz' \notin F$. Since $|x, y, z, x', y', z'| \leq 6$, there exists some point p which occurs in no line among $x'yz, xy'z, xyz'$. Since $|\mathcal{L}| = 7$, it follows that the three lines of $\mathcal{L}$ containing p must be all in $F \cap L = \mathcal{L} \setminus \{x'yz, xy'z, xyz'\}$, contradicting (21). Thus we may assume without loss of generality that $x'yz \in F$.

On the other hand, in view of (F1) and (21), the number of intersections of lines in F is precisely $\binom{4}{2} = 6$, which implies that F contains at least 6 points among the 7 of E. Hence $y \in F$ or $z \in F$ and we may assume without loss of generality that $y \in F$. Thus

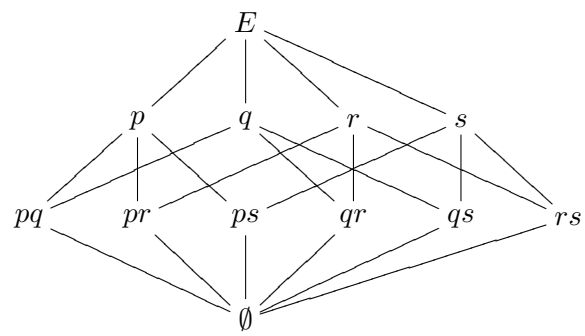
$$\mathrm{Cl}_F(y) = y \subset x'yz = \mathrm{Cl}_F(yz) \subset E = \mathrm{Cl}_F(xyz)$$

and so $F \in \operatorname{Im}\theta$ by Corollary 7.5.

Finally, if $|F \cap \mathcal{L}| \geq 5$, it suffices to show that F contains some $F' \in \mathrm{FISFl}\,(E, H)$ satisfying (21). Let K be any 5-subset of lines in F, and let $a, b \in E$ be given by (F4). Let F' be obtained from F by excluding the lines not in K and the line containing a, b. It is easy to check that F' satisfies (21). This completes the proof for the characterization of $\operatorname{Im}\theta$.

Now, similarly to Example 8.1, the lattice (F, E) is a minimal boolean representation if and only if removal of an smi element of $F \setminus \{E, \emptyset\}$ takes us outside $\operatorname{Im}\theta$. It follows easily from our characterization of $\operatorname{Im}\theta$ (and the fact that $|F \cap \mathcal{L}| \geq 5$ implies that F contains some $F' \in \mathrm{FISFl}\,(E, H)$ satisfying (21)) that this corresponds to F satisfying (21) and having

only the 6 points that are necessarily present as the outcome of intersections of lines in F. Writing $F \cap \mathcal{L} = \{p, q, r, s\}$ and denoting by xy the intersection of $x, y \in F \cap \mathcal{L}$, we see that the minimal boolean representations are, up to isomorphism, given by the lattice



Up to permuting rows/columns, the matrix representation $\widehat{M}(F, E)$ (where we keep only the rows of $M(F, E)$ corresponding to the smi elements of $F \setminus \{E\}$) is then of the form

$$\begin{pmatrix} 0\,0\,1\,1\,0\,1\,1 \\ 0\,1\,1\,0\,1\,0\,1 \\ 1\,0\,0\,1\,1\,0\,1 \\ 1\,1\,0\,0\,0\,1\,1 \end{pmatrix} \tag{22}$$

Following Remark 6.7 as in Example 8.1, the number of minimal lattice representations in both counts is respectively 7 and 1.

It follows from Proposition 6.10 that $(F, E) \in \mathrm{BR}(E, H)$ is sji if and only if there is at most one smi element of $F \setminus \{E, \emptyset\}$ whose removal keeps us inside $\operatorname{Im}\theta$. We claim that this corresponds to either the minimal case or one of the following:

(A) F satisfies (21) and contains all the points;

(B) $|F \cap L| = 5$ and F contains only 6 points.

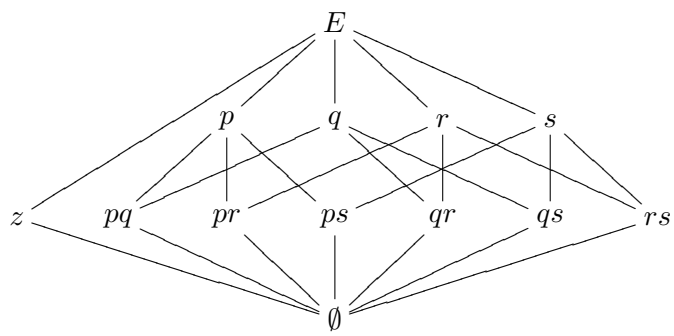
Assume that (A) holds. We have already remarked that the intersections of the 4 lines in F yield 6 distinct points, hence the unique possible removal is the 7th point.

Assume now that (B) holds. Clearly, we cannot remove a point. Let $a, b \in E$ be given by (F4). Then the unique line we can remove is the line containing a and b, in order to satisfy (21). Thus both (A) and (B) correspond to sji (non minimal) cases.

Assume now that F corresponds to an sji non minimal case. It is imediate that F must contain 4 or 5 lines. Suppose that $|F \cap L| = 4$. Then F satisfies (21) and must contain all the points to avoid the minimal case. Hence (A) holds. Finally, we suppose that $|F \cap L| = 5$. Let $a, b \in E$ be given by (F4). Since we have $\binom{5}{2} = 10$ pairs of lines and 6 of these pairs intersect in either a or b, F must contain only 6 points. Therefore (B) holds.

Which lattice corresponds to (A)? Let us use the same notation as in the minimal case and denote the seventh point by z. Since every line in F must intersect each of the other three, and always at different points, it follows that z does not belong to any line in F.

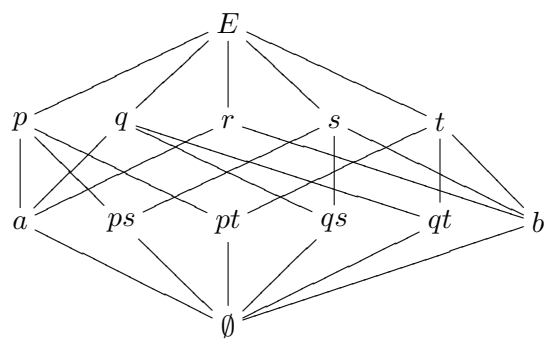
Therefore we obtain the lattice



Up to permuting rows/columns, the matrix representation $\widehat{M}(F,E)$ is then of the form

$$\begin{pmatrix} 0\,0\,1\,1\,0\,1\,1 \\ 0\,1\,1\,0\,1\,0\,1 \\ 1\,0\,0\,1\,1\,0\,1 \\ 1\,1\,0\,0\,0\,1\,1 \\ 1\,1\,1\,1\,1\,1\,0 \end{pmatrix}$$

In case (B), let r denote the line containing a and b, and let p, q (respectively s, t) be the two other lines containing a (respectively b). Using the same notation for intersection of lines, we get the lattice



Up to permuting rows/columns, the matrix representation $\widehat{M}(F,E)$ is then of the form

$$\begin{pmatrix} 0\,0\,1\,1\,0\,1\,1 \\ 0\,1\,1\,0\,1\,0\,1 \\ 1\,0\,0\,1\,1\,0\,1 \\ 1\,1\,0\,0\,0\,1\,1 \\ 1\,1\,1\,1\,0\,0\,0 \end{pmatrix}$$

Following Remark 6.7 as in the minimal case, the number of sji lattice representations in both counts is respectively $7 + 7 + 21 = 35$ and $1 + 1 + 1 = 3$.

It is easy to see that

$$\begin{aligned} \mathrm{Fl}\,(E,H) = \{&E, 125, 146, 236, 345, 567, 1, 2, 3, 4, 5, 6, \emptyset\} \\ \cup\, \{&E, 137, 146, 236, 247, 567, 1, 2, 3, 4, 6, 7\emptyset\} \end{aligned}$$

provides a decomposition of the top boolean representation $\mathrm{Fl}\,(E,H)$ as the join of two sji's. In matrix form, and in view of Proposition 6.5, this corresponds to the stacking of matrices

$$\begin{pmatrix} 0\,0\,1\,1\,0\,1\,1 \\ 0\,1\,0\,1\,1\,1\,0 \\ 0\,1\,1\,0\,1\,0\,1 \\ 1\,0\,0\,1\,1\,0\,1 \\ 1\,0\,1\,0\,1\,1\,0 \\ 1\,1\,0\,0\,0\,1\,1 \\ 1\,1\,1\,1\,0\,0\,0 \end{pmatrix} = \begin{pmatrix} 0\,0\,1\,1\,0\,1\,1 \\ 0\,1\,1\,0\,1\,0\,1 \\ 1\,0\,0\,1\,1\,0\,1 \\ 1\,1\,0\,0\,0\,1\,1 \\ 1\,1\,1\,1\,0\,0\,0 \end{pmatrix} \oplus_b \begin{pmatrix} 0\,1\,0\,1\,1\,1\,0 \\ 0\,1\,1\,0\,1\,0\,1 \\ 1\,0\,0\,1\,1\,0\,1 \\ 1\,0\,1\,0\,1\,1\,0 \\ 1\,1\,1\,1\,0\,0\,0 \end{pmatrix}$$

where we depict only the rows corresponding to the smi elements of the lattices (minus the top).

Note that the maximal decomposition of $\mathrm{Fl}\,(E,H)$ as join of sji's would include 35 factors.

We claim that $\mathrm{mindeg}\,(E,H) = 4$ taking the matrix representation (22) given for the minimal case. Indeed, suppose that M is a 3-row matrix representation of (E,H). Since all the 2-subsets are c-independent, all columns must be different and nonzero in view of Proposition 2.2. Since each of our 7 columns has 3 entries, there exist 4 columns (corresponding to some distinct $a,b,c,d \in E$) having at most one zero. By Proposition 2.2, all the 3-subsets of $\{a,b,c,d\}$ must be dependent, thus lines. In particular, two lines may have two points in common, contradicting (F1). Therefore $\mathrm{mindeg}\,(E,H) = 4$.

Further information on the Fano plane can be found in [16].

Before presenting the next example, we need to recall two standard concepts from graph theory.

Given a finite graph G, the *girth* of G, denoted by $\mathrm{gth}\,G$, is the length of the shortest cycle in G (assumed to be ∞ is G is acyclic). Note that $\mathrm{gth}\,G \geq 3$ for any finite graph. We denote the maximum degree of a vertex in G by $\mathrm{maxdeg}\,G$.

If $G = (V,E)$ is a connected graph, we can define a metric d on V by

$$d(v,w) = \text{ length of the shortest path connecting } v \text{ and } w \text{ (counting edges).}$$

The *diameter* of G, denoted by $\mathrm{diam}\,G$, is the maximum value in the image of d. If G is not connected, we define $\mathrm{diam}\,G = \infty$.

Finally, let $K_{m,n}$ denote the complete bipartite graph on $m+n$ vertices.

Example 8.3 *Let $b \geq 5$ and $(E,H) = U_{3,b}$. We compute all the minimal and sji representations of (E,H).*

It is immediate that $\mathrm{Fl}\,(E,H) = P_2(E) \cup \{E\}$. Given $F \in \mathrm{FISFl}\,(E,H)$, we define a finite undirected graph $F\gamma$ with vertex set $E = \hat{b}$ and edges p — q whenever p,q are distinct and $pq \notin F$. We claim that

$$F \in \mathrm{Im}\,\theta \;\Leftrightarrow\; (\mathrm{gth}\,F\gamma > 3 \text{ and } |E \setminus F| \leq 1). \tag{23}$$

Indeed, assume that $F \in \mathrm{Im}\,\theta$. Suppose that $\mathrm{gth}\,F\gamma = 3$. Then there exist distinct $p,q,r \in E$ such that $pq, pr, qr \notin F$. Hence $\mathrm{Cl}_F(xy) = E$ for all distinct $x,y \in \{p,q,r\}$. Since $pqr \in H$, this contradicts $F \notin \mathrm{Im}\,\theta$ in view of Corollary 7.5. Thus $\mathrm{gth}\,F\gamma > 3$.

Suppose next that $x, y \in E \setminus F$ are distinct. Let $z, t, w \in E \setminus \{x, y\}$ be distinct. By Corollary 7.5, xyz admits an enumeration x_1, x_2, x_3 satisfying

$$\mathrm{Cl}_F(x_1) \subset \mathrm{Cl}_F(x_1x_2) \subset \mathrm{Cl}_F(x_1x_2x_3). \tag{24}$$

Since $F \subseteq P_2(E) \cup \{E\}$, we get $x_1, x_1x_2 \in F$ and so $x_1 = z$ and $i_z z \in F$ for some $i_z \in \{x, y\}$. Similarly, $i_t t, i_w w \in F$ for some $i_t, i_w \in \{x, y\}$. We may thus assume that $i_z = i_t = x$, hence $x = i_z z \cap i_t t \in F$, a contradiction. Therefore $|E \setminus F| \leq 1$.

Conversely, assume that $\operatorname{gth} F\gamma > 3$ and $|E \setminus F| \leq 1$. Let $x, y, z \in E$ be distinct. By Corollary 7.5, it suffices to show that xyz admits an enumeration x_1, x_2, x_3 satisfying (24). Since $\operatorname{gth} F\gamma > 3$, we have $\{xy, xz, yz\} \cap F \neq \emptyset$. We may assume that $xy \in F$. Since $|E \setminus F| \leq 1$, we have either $x \in F$ or $y \in F$. In any case, (24) is satisfied by some enumeration of x, y, z and so (23) holds.

The minimal cases are once more characterized by the following property: removal of an smi element of $F \setminus \{E, \emptyset\}$ must make (23) fail. It is easy to see that the smi elements of $F \setminus \{E, \emptyset\}$ are precisely the 2-sets and the points which are not intersections of 2-sets in F, i.e. vertices of degree $\geq |E| - 2$ in $F\gamma$. We claim that (F, E) is minimal if and only

(M) $\operatorname{gth} F\gamma > 3$, $\operatorname{diam} F\gamma = 2$ and

$$\operatorname{maxdeg} F\gamma \geq |E| - 2 \Rightarrow |E \setminus F| = 1. \tag{25}$$

Assume that (F, E) is minimal. Then $\operatorname{gth} F\gamma > 3$ and $|E \setminus F| = 1$ by (23). Suppose that $\operatorname{maxdeg} F\gamma \geq |E| - 2$. Then some points of E are smi elements of $F\gamma$. If $E \subseteq F$, we could remove one of these smi points and still satisfy (23). Hence $|E \setminus F| = 1$.

Finally, since $|E| > 2$ and $\operatorname{gth} F\gamma > 3$, we have $\operatorname{diam} F\gamma \geq 2$. Suppose that $x, y \in E$ lie at distance > 2 in $F\gamma$. Then we could add an edge x — y and still satisfy (23). Since adding an edge corresponds to removal of the smi xy from F, this contradicts (L, E) being minimal.

Conversely, assume that $\operatorname{gth} F\gamma > 3$, $\operatorname{diam} F\gamma = 2$ and (25) holds. Since $\operatorname{diam} F\gamma = 2$, it is clear that we cannot add any extra edge and keep $\operatorname{gth} F\gamma > 3$, hence removal of 2-sets from F is forbidden. On the other hand, in view of (23), we can only remove a point from F if $E \subseteq F$, and by (25) this can only happen if $\operatorname{maxdeg} F\gamma < |E| - 2$. However, as remarked before, this implies that no point is an smi element of F. Therefore (L, E) is minimal as claimed.

Next we show that (F, E) is sji if and only if $\operatorname{gth} F\gamma > 3$ and one of the following cases holds:

(A) There exists a unique 2-subset $\{u, v\}$ of E such that $d(u, v) > 2$ in $F\gamma$, and (25) holds.

(B) $\operatorname{diam} F\gamma = 2$ and

$$F\gamma \cong K_{2,n} \Rightarrow |E \setminus F| = 1. \tag{26}$$

Indeed, assume that (A) holds. Clearly, the unique edge that can be added to the graph and keep its girth above 3 is u — v. On the other hand, since (25) holds, the possibility of removal of an smi point is excluded. Thus (F, E) is sji in this case.

Assume now that (B) holds. We cannot remove a 2-set from F, since adding an edge to a graph of diameter 2 brings along girth 3. On the other hand, having an option on

removing an smi point would imply the existence of two points of degree $\geq |E|-2$, which implies $F\gamma \cong K_{2,n}$. But in view of (26), only one of these points can be present on F. Thus (L,E) is sji also in this case.

Conversely, assume that (F,E) is sji. Suppose first that $\operatorname{diam} F\gamma = 2$. As remarked before, we cannot remove a 2-set from F, and smi points correspond to degree $\geq |E|-2$. Therefore there is at most one such vertex in F. Since $K_{2,n}$ has two, then (26) holds and we fall into case (B).

Finally, assume that $\operatorname{diam} F\gamma > 2$. Then there exist $u, v \in E$ at distance 3 in $F\gamma$, and adding an edge u — v does not spoil (23). Since (L,E) is sji, then the pair u, v is unique. Similarly to the characterization of the minimal case, (25) must hold to prevent removal of an smi point. Therefore (A) holds.

We prove next that

$$\operatorname{mindeg} U_{3,2b} = b(b-1) \tag{27}$$

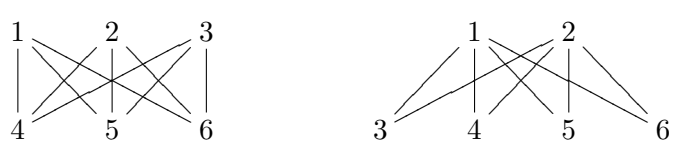
holds for every $b \geq 3$. Indeed, assume that M is an $R \times E$ boolean representation of $U_{3,2b}$ with minimum degree. By Proposition 5.7(ii), we can add all the boolean sums of rows in M and have still a boolean representation of $U_{3,2b}$, and we can even add a row of zeroes (we are in fact building the matrix $M^\nu \in \mathcal{M}$ from Section 3). Now by Proposition 3.4 we have $M^\nu = M(L,E)$ for some $(L,E) \in \operatorname{BR} U_{3,2b}$, and so $F = (L,E)\theta$ satisfies $\operatorname{gth} F\gamma > 3$ by (23). By Turán's Theorem [4, Theorem 7.1.1], the maximum number of edges in a triangle-free graph with $2b$ vertices is reached by the complete bipartite graph $K_{b,b}$ which has b^2 edges. Therefore $F\gamma$ has at most b^2 edges. Since 2^E has $\binom{2b}{2} = b(2b-1)$ 2-sets, it follows that F has at least $b(2b-1) - b^2 = b(b-1)$ 2-sets. Since the 2-sets repesent necessarily smi elements of M^ν, it follows that $M = \widehat{M}(L,E)$ has at least $b(b-1)$ elements and so $\operatorname{mindeg} U_{3,2b} \geq b(b-1)$. Equality is now realized through $F\gamma = K_{b,b}$. Note that in this case no vertex has degree $\geq |E|-2$, hence all the points are meets of closed 2-sets and the smi rows of the matrix are precisely the $b(b-1)$ rows defined by the complement graph of $K_{b,b}$. Therefore $\operatorname{mindeg} U_{3,2b} = b(b-1)$.

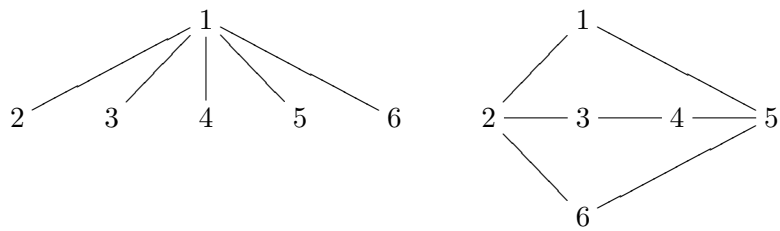
With respect to the odd case, we show that

$$\operatorname{mindeg} U_{3,2b+1} = b^2 \tag{28}$$

holds for every $b \geq 3$. The argument is similar to the the proof of (27). By Turán's Theorem [4, Theorem 7.1.1], the maximum number of edges in a triangle-free graph with $2b+1$ vertices is reached by the complete bipartite graph $K_{b,b+1}$ which has $(b+1)b$ edges. Therefore $F\gamma$ has at most $(b+1)b$ edges. Since 2^E has $\binom{2b+1}{2} = (2b+1)b$ 2-sets, it follows that F has at least $(2b+1)b - (b+1)b = b^2$ 2-sets. Therefore (28) holds.

It is now a simple exercise, for instance, to check that the minimal representations of $U_{3,6}$ correspond (up to permutation of vertices) to the graphs





and to $F_1, F_2, F_3 \in \mathrm{FISFl}\,(E, H)$ given respectively by

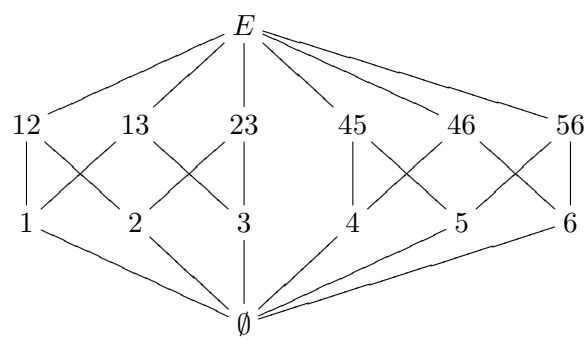
$$F_1 = \{E, 12, 13, 23, 45, 46, 56, 1, 2, 3, 4, 5, 6, \emptyset\};$$

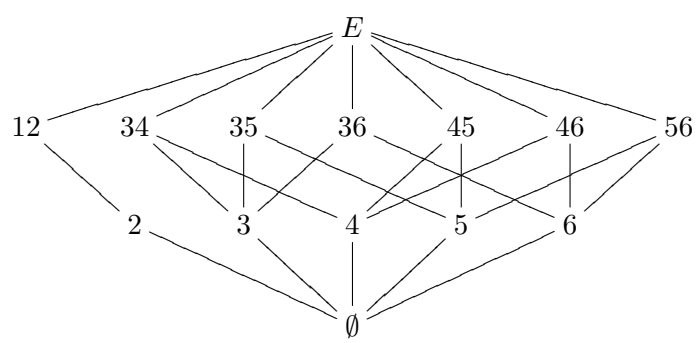
$$F_2 = \{E, 12, 34, 35, 36, 45, 46, 56, 1, 3, 4, 5, 6, \emptyset\};$$

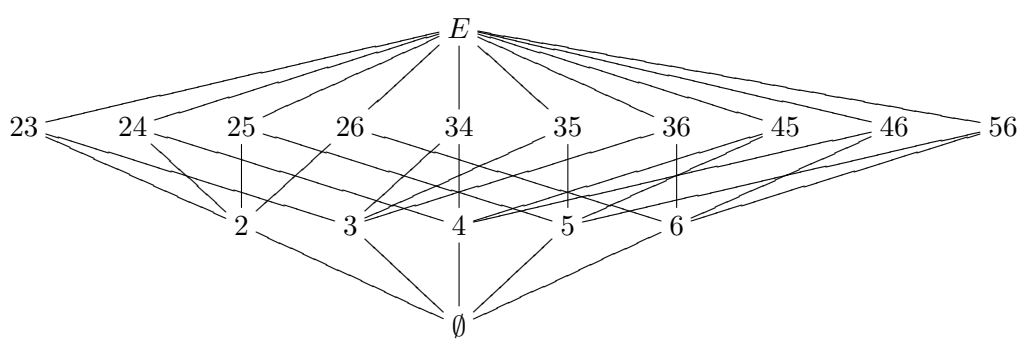
$$F_3 = \{E, 23, 24, 25, 26, 34, 35, 36, 45, 46, 56, 2, 3, 4, 5, 6, \emptyset\};$$

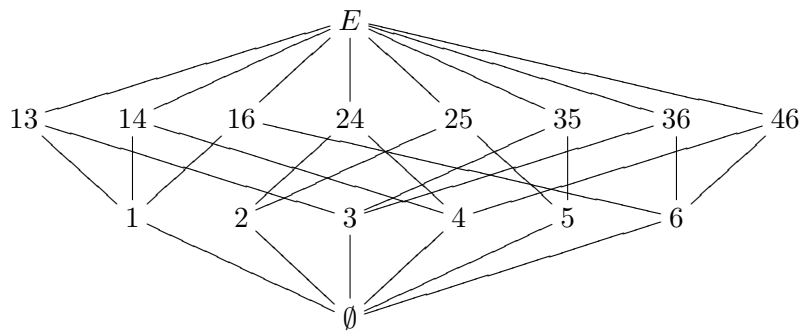
$$F_4 = \{E, 13, 14, 16, 24, 25, 35, 36, 46, 1, 2, 3, 4, 5, 6, \emptyset\}.$$

The corresponding lattices are now





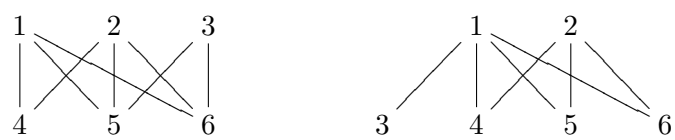




The non minimal sji representations of $U_{3,6}$ can be easily computed. In fact, it is easy to see that if (A) holds, then by adding an edge u — v to the graph $F\gamma$ we get a graph of diameter 2 and still girth > 3. The converse is not true, but a brief analysis of all the possible removals of one edge from a minimal case graph to reach (A) gives us all such sji representations.

Those of type (B) are obtained by adding the seventh point to the minimal representation given by $K_{1,5}$ (the other types already have the seven points or are excluded by (26).

Therefore the graphs corresponding to the sji representations of type (A) are



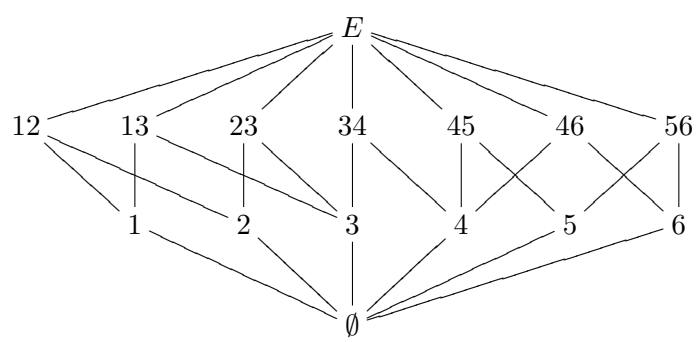
obtained by removing an edge from $K_{3,3}$ and $K_{2,4}$, respectively. Adding the case (B) representation, we obtain types

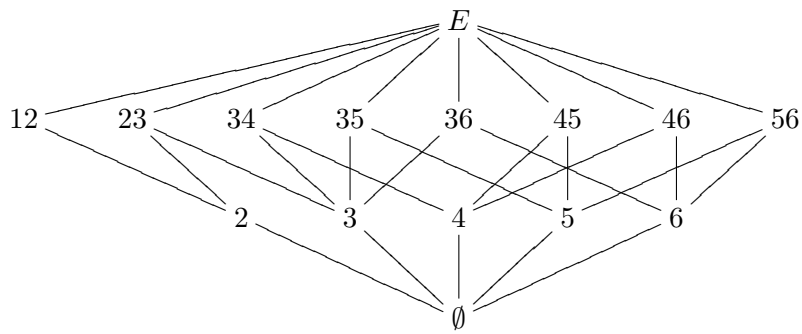
$$F_5 = \{E, 12, 13, 23, 34, 45, 46, 56, 1, 2, 3, 4, 5, 6, \emptyset\};$$

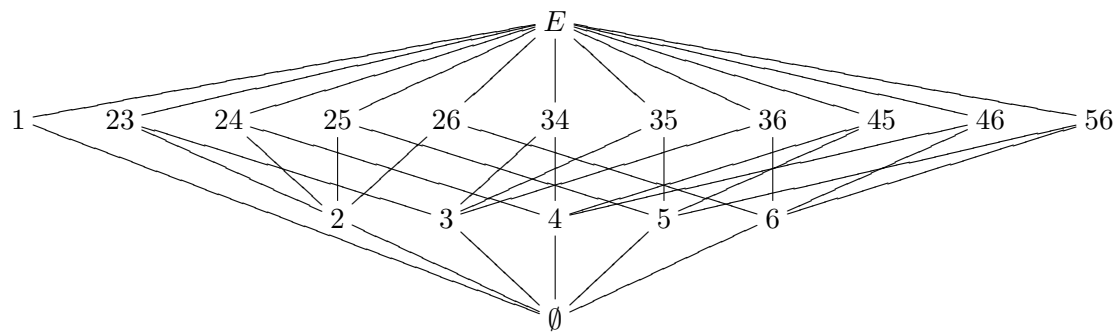
$$F_6 = \{E, 12, 23, 34, 35, 36, 45, 46, 56, 1, 3, 4, 5, 6, \emptyset\};$$

$$F_7 = \{E, 23, 24, 25, 26, 34, 35, 36, 45, 46, 56, 1, 2, 3, 4, 5, 6, \emptyset\}.$$

The corresponding lattices are







It is easy to count $20 + 15 + 6 + 180 = 221$ minimal lattice representations for $U_{3,6}$ only (but they reduce to $1 + 1 + 1 + 1 = 4$ in the alternative counting of Remark 6.7)! The sji's (including the minimal cases) amount to $221 + 180 + 120 + 6 = 527$ and $4 + 1 + 1 + 1 = 7$ in both countings. Note also that $\operatorname{mindeg} U_{3,6} = 6$ by (27).

Note that the lattices in the examples in which $E \subseteq F$, after removal of the top and bottom elements, are essentially the *Levi graphs* of the graphs $F\gamma$. The Levi graph of $F\gamma$ can be obtained by introducing a new vertex at the midpoint of every edge (breaking thus the original edge into two), and the connection to the lattice is established by considering that each of the new vertices lies above its two adjacent neighbours.

Note also that famous graphs of girth > 3 and diameter 2 such as the *Petersen graph* [21] turn out to encode minimal respresentations via the function γ (in $U_{3,10}$, since the Petersen graph has 10 vertices).

9 Additions

This section contains results which are relevant to the theory but were not needed for the main sections.

9.1 Rank functions

Let (E, H) be a hereditary collection. The *rank function* $r_H : 2^E \to \mathbb{N}$ is defined by

$$Xr_H = \max\{|I| : I \in 2^X \cap H\}.$$

The maximum value of r_H is the *rank* of $E, H)$.

Given a function $f : 2^E \to \mathbb{N}$, consider the following axioms for all $X, Y \subseteq E$:

(A1) $X \subseteq Y \quad \Rightarrow \quad Xf \leq Yf$;

(A2) $\exists I \subseteq X : |I| = If = Xf$;

(A3) $(Xf = |X| \ \wedge \ Y \subseteq X) \quad \Rightarrow \quad Yf = |Y|$.

It is easy to see that the three axioms are independent.

Proposition 9.1 *Given a function $f : 2^E \to \mathbb{N}$, the following conditions are equivalent:*

(i) $f = r_H$ for some hereditary collection (E, H);

(ii) f satisfies axioms (A1)–(A3).

Proof. (i) $\Rightarrow$ (ii). It follows immediately from the equivalence

$$Xr_H = |X| \Leftrightarrow X \in H.$$

(ii) $\Rightarrow$ (i). Let $H = \{I \subseteq E : If = |I|\}$. By (A3), H is closed under taking subsets. Taking $X = \emptyset$ in (A2), we get $\emptyset f = 0$, hence $\emptyset \in H$ and so (E, H) is a hereditary collection. Now, for every $X \in E$, we have

$$Xr_H = \max\{|I| : I \in 2^X \cap H\} = \max\{|I| : I \subseteq X, \ If = |I|\}.$$

By (A2), we get $Xr_H \geq Xf$, and $Xr_H \leq Xf$ follows from (A1). Hence $f = r_H$ as required. $\square$

We collect next some elementary properties of rank functions:

Proposition 9.2 *Let (E, H) be a hereditary collection and let $X, Y \subseteq E$. Then:*

(i) $Xr_H \leq |X|$;

(ii) $Xr_H + Yr_H \geq (X \cup Y)r_H$;

(iii) $Xr_H + Yr_H \geq (X \cup Y)r_H + (X \cap Y)r_H$ if some independent subset of maximum size of $X \cap Y$ can be extended to some independent subset of maximum size of $X \cup Y$;

(iv) $Xr_H + Yr_H \geq (X \cup Y)r_H + (X \cap Y)r_H$ if (E, H) is a matroid.

Proof. (i) By (A2).

(ii) Assume that $(X \cup Y)r_H = |I|$ with $I \in 2^{X \cup Y} \cap H$. Then $I \cap X, I \cap Y \in H$ and so $(X \cup Y)r_H = |I| \leq |I \cap X| + |I \cap Y| \leq Xr_H + Yr_H$.

(iii) We may assume that $(X \cup Y)r_H = |I|$ and $(X \cap Y)r_H = |I \cap X \cap Y|$. It follows that $(X \cup Y)r_H + (X \cap Y)r_H = |I| + |I \cap X \cap Y| = |I \cap X| + |I \cap Y| \leq Xr_H + Yr_H$.

(iv) This is well known, but we can include a short deduction from (iii).

Let $K \subseteq L \subseteq E$, and assume that J is an independent subset of maximum size of K. Let J' be a maximal independent subset of L containing J. If (E, H) is a matroid, it follows from the exchange property that $|J'| = Lr_H$. Now we apply part (iii) to $K = X \cap Y$ and $L = X \cup Y$. $\square$

Proposition 9.3 *Let (E, H) be a hereditary collection of rank r.*

(i) If $X, Y \in \mathrm{Fl}(E, H)$ and $Xr_H = Yr_H$, then

$$X \subseteq Y \quad \Leftrightarrow \quad X = Y.$$

(ii) E is the unique flat of rank r.

Proof. (i) Let $X, Y \in \mathrm{Fl}\,(E, H)$. Assume that $X \subseteq Y$ and let $I \subseteq X$ satisfy $I \in H$ and $|I| = Xr_H = Yr_H$. If $p \in Y \setminus X$, then X closed yields $I \cup \{p\} \in H$ and $Yr_H > |I| = Xr_H$, a contradiction. Therefore $X = Y$ and (i) holds.

(ii) By part (i). □

It follows that the flats of rank $r - 1$ are maximal in $\mathrm{Fl}\,(E, H) \setminus \{E\}$. Such flats are called *hyperplanes*.

The following result relates the rank function with the closure operator Cl induced by a (simple) hereditary collection.

Proposition 9.4 *Let (E, H) be a simple hereditary collection admitting a boolean representation and let $X \subseteq E$. Write $L = \mathrm{Fl}(E, H)$. Then Xr_H is the maximum k such that (15) holds for some $x_1, \ldots, x_k \in X$, and the maximum k such that (14) holds for some $x_1, \ldots, x_k \in X$.*

Proof. The first equality follows from Proposition 7.2 and the definition of r_H. The second follows from Theorem 5.8 and Lemma 7.1(ii). □

9.2 Paving hereditary collections

A hereditary collection of rank $r > 2$ is said to be *paving* if it has no circuits of size $< r$ (or equivalently, of rank less than $r - 1$).

Lemma 9.5 *Let (E, H) be a hereditary collection of rank $r > 2$. Then the following conditions are equivalent:*

(i) (E, H) is paving;

(ii) $P_{r-1}(E) \subseteq H$;

(iii) $P_{r-2}(E) \subseteq \mathrm{Fl}(E, H)$.

Proof. (i) $\Rightarrow$ (ii). Since every dependent subset of (E, H) must contain a circuit.

(ii) $\Rightarrow$ (i). Trivial.

(ii) $\Rightarrow$ (iii). By Proposition 5.2(ii).

(iii) $\Rightarrow$ (i). Suppose that C is a circuit of rank $< r$ and let $x \in C$. Then $|C \setminus \{x\}| \leq r - 2$, hence $C \setminus \{x\}$ is closed and $C \setminus \{x\} \in H$ yields $C \in H$, a contradiction. Thus (E, H) is paving. □

Next we provide a simple characterization of boolean representable paving hereditary collections:

Proposition 9.6 *Let (E, H) be a paving hereditary collection of rank r. Then the following conditions are equivalent:*

(i) (E, H) *is boolean representable;*

(ii) $\forall X \in H \quad |X| = r \Rightarrow \exists x \in X : x \notin \mathrm{Cl}(X \setminus \{x\})$*;*

(iii) $\forall X \in H \quad |X| = r \Rightarrow \exists x \in X : \mathrm{Cl}(X \setminus \{x\}) \neq E$*.*

Proof. (i) $\Rightarrow$ (ii). By Proposition 7.2.

(ii) $\Rightarrow$ (iii). Immediate.

(iii) $\Rightarrow$ (i). By Theorem 7.3, it suffices to show that every $X \in H$ admits an enumeration $x_1, \ldots, x_k$ such that

$$\mathrm{Cl}(x_1, \ldots, x_k) \supset \mathrm{Cl}(x_2, \ldots, x_k) \supset \ldots \supset \mathrm{Cl}(x_k).$$

By condition (iii) in Lemma 9.5, this condition is satisfied if $|X| < r$. Hence we may assume that $|X| = r$ and we only need to show that there exists some enumeration $x_1, \ldots, x_r$ of X such that

$$\mathrm{Cl}\, X \supset \mathrm{Cl}(X \setminus \{x_1\}) \supset \{x_3, \ldots, x_r\} \supset \ldots \supset \{x_r\}.$$

Since $\mathrm{Cl}\, X = E$ by Proposition 5.3, condition (iii) yields the required inclusion. $\square$

In connection with Proposition 9.6, we can mention several equivalent characterizations of matroids among paving hereditary collections:

Proposition 9.7 *Let (E, H) be a paving hereditary collection of rank r. Then the following conditions are equivalent:*

(i) (E, H) *is a matroid;*

(ii) every $(r-1)$-subset of E is contained in a unique hyperplane;

(iii) if X is an $(r-1)$-subset of E, then $\mathrm{Cl}\, X \neq E$*.*

Proof. (i) $\Rightarrow$ (ii). By [13, Proposition 2.1.21].

(ii) $\Rightarrow$ (iii). Immediate.

(iii) $\Rightarrow$ (i). First, note that (E, H) is boolean representable by Proposition 9.6. Let $I, J \in H$ with $|I| = |J| + 1$. We must show that $J \cup \{i\} \in H$ for some $i \in I \setminus J$. Since $P_{r-1}(E) \subseteq H$ by Lemma 9.5, we may assume that $|J| = r-1$. Since $\mathrm{Cl}\, J \neq E$ and $\mathrm{Cl}\, I = E$ by Proposition 5.3, we get $I \not\subseteq \mathrm{Cl}\, J$. Take $i \in I \setminus \mathrm{Cl}\, J$. Since $P_{r-2}(E) \subseteq \mathrm{Fl}\,(E, H)$ by Lemma 9.5 and $\mathrm{Cl}\, J \subset \mathrm{Cl}(J \cup \{i\})$, it follows from Proposition 7.2 that $J \cup \{i\} \in H$. Therefore (E, H) is a matroid. $\square$

Note that (E, H) being boolean representable and all its bases having rank r does *not* imply that (E, H) is a matroid, a counterexample being provided by $E = \hat{6}$ and $H = P_4(E) \setminus \{2456, 3456\}$.

9.3 Boolean operations

Boolean representability behaves badly with respect to intersection and union, as we show next.

First, we recall a well-known fact: every hereditary collection (E, H) is the intersection of matroids on E, namely the intersection of the matroids M_X over all circuits X of (E, H), where M_X is the matroid consisting of all subsets of E not containing X [13]. Since all simple matroids are boolean representable by Theorem 7.6, it follows that all simple hereditary collections are the intersection of boolean representable hereditary collections. Therefore boolean representable hereditary collections are not closed under intersection.

Example 9.8 *Let $E = \hat{6}$ and $J_1 = P_3(E) \setminus \{123, 125, 135, 235, 146, 246, 346, 456\}$, $J_2 = P_2(E) \cup \{123, 124, 125, 126\}$. Then (E, J_1) and (E, J_2) are both boolean representable hereditary collections, but $(E, J_1 \cup J_2)$ is not.*

It is easy to check that $1235 \in \mathrm{Fl}\,(E, J_1)$. Since $|xyz \cap 1235| = 2$ for every $xyz \in J_1$, it follows from Proposition 9.6 that (E, J_1) is boolean representable. Similarly, since $12 \in \mathrm{Fl}\,(E, J_2)$, we show that that (E, J_2) is boolean representable.

Now $J_1 \cup J_2 = P_3(E) \setminus \{135, 235, 146, 246, 346, 456\}$ and it is straightforward to check that $\mathrm{Cl}\,13 = \mathrm{Cl}\,15 = \mathrm{Cl}\,35 = 1235$. By Proposition 9.6, $(E, J_1 \cup J_2)$ is not boolean representable.

However, closure under union can be satisfied in particular circumstances:

Proposition 9.9 *Let $(E, H_1), (E, H_2)$ be simple boolean representable hereditary collections of rank 3. If*

$$X \in \mathrm{Fl}(E, H_i) \setminus \{E\} \Rightarrow |X| \leq 3 \tag{29}$$

holds for $i = 1, 2$, then $(E, H_1 \cup H_2)$ is boolean representable.

Proof. For $i = 1, 2$, write $F_i = \mathrm{Fl}\,(E, H_i)$ and let $L_i = \{X \in F_i : |X| = 3\}$. We define also the set of *potential lines*

$$P_i = \{X \subseteq E : |X| = 3 \text{ and } |X \cap Y| \leq 1 \text{ for every } Y \in F_i \setminus \{E\}\}.$$

It is easy to see that $H_i \cap P_i = \emptyset$. Indeed, suppose that $xyz \in H_i \cap P_i$. Since (E, H_i), we may apply Proposition 9.6 and assume, without loss of generality, that there exists some $Y \in F_i \setminus \{E\}$ containing xy. This contradicts $xyz \in P_i$, hence $H_i \cap P_i = \emptyset$.

Let $F = \mathrm{Fl}\,(E, H_1 \cup H_2)$ and define

$$W = (L_1 \cap L_2) \cup (L_1 \cap P_2) \cup (P_1 \cap L_2);$$

$$W' = \{X \subseteq E : |X| = 2 \text{ and } X \cup \{q\} \notin (L_1 \cup P_1) \cap (L_2 \cup P_2) \text{ for every } q \in E\}.$$

We claim that

$$W \cup W' \subseteq F. \tag{30}$$

Let $xyz \in W$. We may assume that $xyz \in L_1$. Suppose first that $xyz \in L_2$. If $I \subseteq xyz$, $I \in H_1 \cup H_2$ and $p \notin xyz$, then $I \in H_i$ for some i and so $xyz \in L_i$ yields $I \cup \{p\} \in H_i \subseteq H_1 \cup H_2$ as required.

Hence we may assume that $xyz \in P_2$. Let $I \subseteq xyz$ and $p \notin xyz$. If $I \in H_1$, all is similar to the preceding case, hence we assume that $I \in H_2 \setminus H_1$. Since $H_2 \cap P_2 = \emptyset$, we have $|I| \leq 2$ and so $I \in H_1$. If $p \notin xyz$, It follows from $xyz \in F_1$ that $I \cup \{p\} \in H_1 \subseteq H_1 \cup H_2$. Thus $W \subseteq F$.

Assume now that $xy \in W'$. Let $z \notin xy$. It suffices to show that $xyz \in H_1 \cup H_2$. Since $xy \in W'$, we have $xyz \notin L_i \cup P_i$ for some i. Thus $|xyz \cap Y| = 2$ for some $Y \in F_i \setminus \{E\}$, and Y closed yields $xyz \in H_i \subseteq H_1 \cup H_2$ as required. Therefore $xy \in F$ and (30) holds.

Now let $\mathrm{Cl}\, X$ (respectively $\mathrm{Cl}_i X$) denote the closure of $X \subseteq E$ in $(E, H_1 \cup H_2)$ (respectively (E, H_i)). Let $xyz \in H_1 \cup H_2$. By Proposition 9.6, we must show that $\mathrm{Cl}\,(xy) \neq E$ or $\mathrm{Cl}\,(xz) \neq E$ or $\mathrm{Cl}\,(yz) \neq E$. We may assume that $xyz \in H_1$. By Proposition 9.6, we may assume also that $\mathrm{Cl}_1(xy) \neq E$.

Since $xy \in W'$ implies $xy \in F$ by (30), we have that $xyq \in (L_1 \cup P_1) \cap (L_2 \cup P_2)$ for some $q \in E$. If $xyq \in W \subseteq F$, we immediately get $\mathrm{Cl}\,(xy) \neq E$. It remains to consider the case $xyq \in P_1 \cap P_2$. Since $|xyq \cap \mathrm{Cl}_1(xy)| \geq 2$ and $\mathrm{Cl}_1(xy) \neq E$, we reach a contradiction. Thus $\mathrm{Cl}\,(xy) \neq E$ and $(E, H_1 \cup H_2)$ is boolean representable. $\square$

9.4 Truncation

Given a hereditary collection (E, H) and $k \geq 0$, the *k-truncation* of (E, H) is the hereditary collection (E, H_k) defined by $H_k = \{X \subseteq E : |X| \leq k\}$.

Proposition 9.10 *Let (E, H) be a hereditary collection and let $k \geq 0$. Then:*

(i) $\mathrm{Fl}\,(E, H_k) \subseteq \mathrm{Fl}\,(E, H)$*;*

(ii) If $X \subset E$, then $X \in \mathrm{Fl}\,(E, H_k)$ if and only if $X \in \mathrm{Fl}\,(E, H)$ and X does not contain a basis of H_k.

Proof. It suffices to prove (ii). Let $X \subset E$. Assume first that $X \in \mathrm{Fl}\,(E, H_k)$. By Proposition 5.3, X does not contain a basis of (E, H_k). Let $p \in E \setminus X$ and let $I \subseteq X$ be such that $I \in H$. Since I is not a basis of (E, H_k), we have $|I| < k$ and so $I \in H_k$. Now $X \in \mathrm{Fl}\,(E, H_k)$ yields $I \cup \{p\} \in H_k \subseteq H$. Therefore $X \in \mathrm{Fl}\,(E, H)$.

Conversely, assume that $X \in \mathrm{Fl}\,(E, H)$ and X does not contain a basis of (E, H_k). Let $p \in E \setminus X$ and let $I \subseteq X$ be such that $I \in H_k$. Since $H_k \subseteq H$ and $X \in \mathrm{Fl}\,(E, H)$, we get $I \cup \{p\} \in H$. But I is not a basis of (E, H_k), hence $|I| < k$ and so $I \cup \{p\} \in H_k$. Thus $X \in \mathrm{Fl}\,(E, H_k)$ as required. $\square$

The next example shows that boolean representability is not preserved under truncation, even in the simple case.

Example 9.11 *Let $E = \hat{6}$ and let H be the hereditary collection defined by $H = (P_3(E) \setminus \{135, 235, 146, 246, 346, 456\}) \cup \{1234, 1236, 1245, 1256\}$. Then (E, H) is boolean representable, but (E, H_3) is not.*

It is easy to check that $P_1(E) \cup \{12, 1235\} \subseteq \mathrm{Fl}\,(E, H)$. By Theorem 7.3, to show that (E, H) is boolean representable it suffices to show that every $X \in H$ admits an enumeration $x_1, \ldots, x_k$ satisfying (15). We may of course assume that $|X| > 2$. Hence X cannot contain both 4 and 6. Since 1235 is closed, we may assume that $X \subseteq 1235$. Since we may assume that X is a 3-set, we are reduced to the cases $X \in \{123, 125\}$. Now $1 \subset 12 \subset 1235$ yields the desired chain of flats, and so (E, H) is boolean representable.

On the other hand, H_3 is the collection $J_1 \cup J_2$ of Example 9.8, already proved not to be boolean representable.

10 Appendix

We gather here several results which, although not essential for obtaining our main results, can help the interested reader to gain further insight into our approach and methods.

10.1 Categoric alternatives

We note that the category FL is isomorphic to some other categories that bring different viewpoints into our discussions.

Recall that a structure $(S, +, \cdot, 0)$ is a *semiring* if:

- $(S, +, 0)$ is a commutative monoid;
- $(S, \cdot, 0)$ is a semigroup with zero
- $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$ for all $x, y, z \in S$.

The semiring S is *idempotent* if $x+x = x$ for every $x \in S$. It is *null* if $xy = 0$ for all $x, y \in S$. Morphisms and modules over semirings are defined the obvious way (see [17, Chapter 9]).

We introduce the following notation:

FICM: the category of *finite idempotent commutative monoids* together with monoid morphisms;

FINS: the category of *finite idempotent null semirings* together with semiring morphisms;

FBM: the category of *finite unitary right* $\mathbb{B}$*-modules* together with $\mathbb{B}$-module morphisms.

Proposition 10.1 *The categories* FL*,* FICM*,* FINS *and* FBM *are isomorphic.*

Proof. It is well known that the functor FL $\to$ FICM defined by $(L, \leq) \mapsto (L, \vee)$ and identity on arrows defines an isomorphism of categories.

Clearly, the forgetful functor FINS $\to$ FICM is also an isomorphism of categories. The same happens for the forgetful functor FBM $\to$ FICM. Indeed, it is easy to see that each $\mathbb{B}$-module is necessarily idempotent since $x = 1x = (1 + 1)x = 1x + 1x = x + x$ holds for every $x \in S$. On the other hand, each finite idempotent commutative monoid $(M, +, 0)$ determines a unique $\mathbb{B}$-module structure in M since we are forced to have $1x = x$ and $0x = 0$ for every $x \in M$, and the arrows turn out to be the same mappings. $\square$

Now, for each category X $\in$ { FICM, FINS, FBM }, we define another category Xg by taking objects of the form (M, E), where M is an object of X and $E \subseteq M \setminus \{0\}$ a generating set for M. For arrows $\varphi : (M, E) \to (M', E')$, we require also $E\varphi \subseteq E' \cup \{0\}$. With straightforward adaptations, Proposition 10.1 yields

Corollary 10.2 *The categories* FLg*,* FICMg*,* FINSg *and* FBMg *are isomorphic.*

The following result, stated for FL and ideals, which is after all our basic viewpoint in this paper, is inspired by standard concepts in semigroup theory [2], and therefore by the viewpoint FICM. We say that $I \subseteq L$ is an *ideal* (or downset) if $x \leq y \in I$ implies $x \in I$ for all $x, y \in L$. There is an obvious dual version of the Rees quotient when we consider the dual notion of upset.

Given $(L, E) \in$ FLg and an ideal I of L, the *Rees quotient* L/I is the quotient of L by the congruence $\sim_I$ defined on L by

$$x \sim_I y \quad \text{if} \quad x = y \text{ or } x, y \in I.$$

The elements of L/I are the equivalence class $B' = I$ (the bottom element) and the singular equivalence classes $\{x\}$ ($x \in L \setminus I$), which we identify with x. The partial ordering of L translates naturally to L/I.

Proposition 10.3 *Let $L \in$ FL and let I be an ideal of $L \setminus \{T\}$. Then $L/I \in$ FLg.*

Proof. Clearly, L/I inherits a naural $\wedge$-semilattice structure, and then becomes a lattice with the determined join. $\square$

It is also possible to import the notion of *quotient submodule* (see [17, Section 9.1]) from FBM to FL. Let S be a $\vee$-subsemilattice of a finite lattice L. We define a $\vee$-congruence $\equiv_S$ on L by $x \equiv_S y$ if $(x \vee s) = (y \vee s')$ for some $s, s' \in S$. Then L/S denotes the quotient $L/\equiv_S$.

Given a finite lattice L, we say that $\xi : L \to L$ is a *closure operator* if the following axioms hold for all $x, y \in L$:

(C1) $x \leq x\xi$;

(C2) $x \leq y \quad \Rightarrow \quad x\xi \leq y\xi$;

(C3) $x\xi = (x\xi)\xi$.

The next proposition summarizes some of the properties of closure operators (see [6, Subsection I.3.12]):

Proposition 10.4 *Let L be a lattice, let $\xi : L \to L$ be a closure operator and let S be a $\wedge$-subsemilattice of L. Then:*

(i) $(x \vee y)\xi = (x\xi \vee y\xi)\xi$ *for all* $x, y \in L$.

(ii) $L\xi$ *is a $\wedge$-subsemilattice of L and constitutes a lattice under the determined join* $(x\xi \vee' y\xi) = (x\xi \vee y\xi)\xi$.

(iii) S *induces a closure operator* $\xi_S : L \to L$ *defined by* $x\xi_S = \wedge\{y \in S \mid y \geq x\}$.

(iv) $\xi_{L\xi} = \xi$ *and* $L\xi_S = S$.

Next we associate closure operators and $\vee$-congruences, making explicit a construction suggested in [17, Theorem 6.3.7].

Proposition 10.5 *Let L be a lattice, let $\xi : L \to L$ be a closure operator and let ρ be a $\vee$-congruence on L. Then:*

(i) $Ker\xi$ is a $\vee$-congruence on L.

(ii) ρ induces a closure operator $\eta_\rho : L \to L$ defined by $x\eta_\rho = \vee(x\rho) = \max(x\rho)$.

(iii) $\eta_{Ker\xi} = \xi$ and $Ker\eta_\rho = \rho$.

Proof. (i) Clearly, $\mathrm{Ker}\,\xi$ is an equivalence relation. Hence we must show that $x\xi = y\xi$ implies $(x\vee z)\xi = (y\vee z)\xi$ for all $x, y, z \in L$. Now $(x\vee z)\xi = (x\xi\vee z\xi)\xi = (y\xi\vee z\xi)\xi = (y\vee z)\xi$ by Proposition 10.4(i).

(ii) Axioms (C1) and (C3) follow immediately from $x \in x\rho$ and $x\eta_\rho\rho = x\rho$.

Assume that $x \leq y$ in L. Then $y = (x \vee y)$, hence

$$y\rho = (x \vee y)\rho = (x\rho \vee y\rho) = (x\eta_\rho\rho \vee y\rho) = (x\eta_\rho \vee y)\rho$$

and so $x\eta_\rho \leq (x\eta_\rho \vee y) \leq y\eta_\rho$ and (C2) holds as well.

(iii) Let $x, y \in L$. Suppose that $y\xi = x\xi$. By (C1), we get $y \leq y\xi = x\xi$ and so $x\eta_{\mathrm{Ker}\,\xi} = \max(x(\mathrm{Ker}\,\xi)) = x\xi$. Thus $\eta_{\mathrm{Ker}\,\xi} = \xi$.

On the other hand,

$$(x, y) \in \mathrm{Ker}\,\eta_\rho \Leftrightarrow x\eta_\rho = y\eta_\rho \Leftrightarrow \max(x\rho) = \max(y\rho) \Leftrightarrow (x, y) \in \rho,$$

therefore $\mathrm{Ker}\,\eta_\rho = \rho$. $\square$

Since $\vee$-congruences are nothing but kernels of $\vee$-maps, Proposition 10.5 establishes a correspondence between kernels of $\vee$-maps and closure operators.

We can combine Propositions 10.4 and 10.5 with the representation of lattices by flats. Given $(L, E) \in \mathrm{FLg}$, let $\mathrm{ISFl}(L, E)$ denote the set of all $\wedge$-subsemilattices of $\mathrm{Fl}(L, E)$, equivalently described as subsets of $\mathrm{Fl}(L, E)$ closed under intersection.

Theorem 10.6 *Let $(L, E) \in \mathrm{FLg}$, let ρ be a $\vee$-congruence on L and let $F \in \mathrm{ISFl}(L, E)$. Then:*

(i) $F_\rho = \{Z_{max(x\rho)} \mid x \in L\} \in \mathrm{ISFl}(L, E)$.

(ii) The relation ρ_F on L defined by $x\rho_F y$ if

$$\cap\{Z \in F \mid Z_x \subseteq Z\} = \cap\{Z \in F \mid Z_y \subseteq Z\}$$

is a $\vee$-congruence.

(iii) $\rho_{F_\rho} = \rho$ and $F_{\rho_F} = F$.

Proof. We combine the correspondences in Propositions 10.4 and 10.5 with the lattice isomorphism $\varphi : (L, \leq) \to (\mathrm{Fl}(L, E), \subseteq) : x \mapsto Z_x$ from Proposition 3.3. Note that a $\wedge$-subsemilattice of L corresponds to a subset of $\mathrm{Fl}(L, E)$ closed under intersection. Thus we only have to check that the three correspondences mentioned above yield the claimed ones.

Starting from ρ, we get η_ρ and $L\eta_\rho$ by Propositions 10.5 and 10.4, respectively, and then $\{Z_m \mid m \in L\eta_\rho\} = F_\rho$ by applying φ.

Starting from F, application of φ^{-1} gives us $F\varphi^{-1}$. Then Propositions 10.4 and 10.5 give us successively $\xi_{F\varphi^{-1}}$ and $\operatorname{Ker}\xi_{F\varphi^{-1}}$. Now, for all $x, y \in L$, we get

$$\begin{aligned} x\xi_{F\varphi^{-1}} = y\xi_{F\varphi^{-1}} &\Leftrightarrow \wedge\{p \in F\varphi^{-1} \mid p \geq x\} = \wedge\{p \in F\varphi^{-1} \mid p \geq y\} \\ &\Leftrightarrow \cap\{Z \in F \mid Z \supseteq Z_x\} = \cap\{Z \in F \mid Z \supseteq Z_y\} \\ &\Leftrightarrow x\rho_F y \end{aligned}$$

and we are done. □

10.2 Decomposing $\vee$-maps

Once again, we import to the context of finite lattices a concept originated in semigroup theory. Following [17, Section 5.2], we call an onto $\vee$-map a $\vee$-surmorphism and say that a $\vee$-surmorphism $\varphi : L \to L'$ is a *maximal proper $\vee$-surmorphism* (MPS) of lattices if $\operatorname{Ker}\varphi$ is a minimal nontrivial $\vee$-congruence on L. This amounts to saying that φ cannot be factorized as the composition of two proper $\vee$-surmorphisms.

Given $a, b \in L$, let $\rho_{a,b}$ denote the equivalence relation on L defined by

$$x\rho_{a,b} = \begin{cases} \{a, b\} & \text{if } x = a \text{ or } x = b \\ \{x\} & \text{otherwise} \end{cases}$$

Proposition 10.7 *Let $\varphi : L \to L'$ be a $\vee$-surmorphism of lattices. Then:*

(i) If φ is not one-to-one, then φ factorizes as a composition of MPSs.

(ii) If a covers b and b is smi, then $\rho_{a,b}$ is a minimal nontrivial $\vee$-congruence on L.

(iii) φ is an MPS if and only if $\operatorname{Ker}\varphi = \rho_{a,b}$ for some $a, b \in L$ such that a covers b and b is smi.

Proof. (i) Since L is finite, there exists a minimal nontrivial $\vee$-congruence $\rho_1 \subseteq \operatorname{Ker}\varphi$ and we can factor φ as a composition $L \to L/\rho_1 \to L'$. Now we apply the same argument to $L/\rho_1 \to L'$ and successively.

(ii) Let $x \in L$. We must prove that $(x \vee a, x \vee b) \in \rho_{a,b}$. Since b is smi, a is the unique element of L covering b. Hence either $(x \vee b) = b$ or $(x \vee b) \geq a$. In the first case, we get $x \leq b$ and so $(x \vee a) = a$; in the latter case, we get $(x \vee b) = (x \vee (x \vee b)) \geq (x \vee a) \geq (x \vee b)$ and so $(x \vee b) = (x \vee a)$. Hence $(x \vee a, x \vee b) \in \rho_{a,b}$ and so $\rho_{a,b}$ is a (nontrivial) $\vee$-congruence on L. Minimality is obvious.

(iii) Assume that φ is an MPS and let $a \in L$ be maximal among the elements of L which belong to a nonsingular $\operatorname{Ker}\varphi$ class. Then there exists some $x \in L \setminus \{a\}$ such that $x\varphi = a\varphi$. It follows that $(x \vee a)\varphi = (x\varphi \vee a\varphi) = a\varphi$ and so by maximality of a we get $(x \vee a) = a$ and so $x < a$. Then there exists some $b \geq x$ such that a covers b. Since every $\vee$-map preserves order, we get $a\varphi = x\varphi \leq b\varphi \leq a\varphi$ and so $a\varphi = b\varphi$.

Suppose that b is not smi. Then b is covered by some other element $c \neq a$, hence $b = (a \wedge c)$ and $a, c < (a \vee c)$. It follows that $(a \vee c)\varphi = (a\varphi \vee c\varphi) = (b\varphi \vee c\varphi) = c\varphi$. Since $c \neq (a \vee c)$ and $a < (a \vee c)$, this contradicts the maximality of a. Thus b is smi. Since $\rho_{a,b} \subseteq \operatorname{Ker}\varphi$, it follows from (ii) that $\operatorname{Ker}\varphi = \rho_{a,b}$.

The converse implication is immediate. □

We prove next the dual of Proposition 10.7 for injective $\vee$-maps. We say that a $\vee$-map $\varphi : L \to L'$ is a *maximal proper injective $\vee$-map* (MPI) of lattices if φ is injective and $L\varphi$ is a maximal proper $\vee$-subsemilattice of L'. This amounts to saying that φ cannot be factorized as the composition of two proper injective $\vee$-maps.

Proposition 10.8 *Let $\varphi : L \to L'$ be an injective $\vee$-map of lattices. Then:*

(i) If φ is not onto, then φ factorizes as a composition of MPIs.

(ii) If $a \in L' \setminus \{B\}$ is sji, then the inclusion $\iota : L' \setminus \{a\} \to L'$ is an MPI of lattices.

(iii) φ is an MPI if and only if $L\varphi = L' \setminus \{a\}$ for some sji $a \in L \setminus \{B\}$.

Proof. (i) Immediate since L' is finite and each proper injective $\vee$-map increases the number of elements.

(ii) Let $x, y \in L' \setminus \{a\}$. Since a is sji, the join of x and y in L' is also the join of x and y in $L' \setminus \{a\}$. Hence $L' \setminus \{a\}$ is a $\vee$-semilattice and therefore a lattice with the determined meet. Since $(x\iota \vee y\iota) = (x \vee y) = (x \vee y)\iota$, then ι is a $\vee$-map. Since $|L' \setminus \operatorname{Im} \iota| = 1$, it must be an MPI.

(iii) Assume that φ is an MPI. Let a be a minimal element of $L' \setminus L\varphi$. We claim that a is an sji in L'. Otherwise, by minimality of a, we would have $a = (x\varphi \vee y\varphi)$ for some $x, y \in L$. Since φ is a $\vee$-map, this would imply $a = (x \vee y)\varphi$, contradicting $a \in L' \setminus L\varphi$.

Thus a is an sji in L' and we can factor $\varphi : L \to L'$ as the composition of $\varphi : L \to L' \setminus \{a\}$ with the inclusion $\iota : L' \setminus \{a\} \to L'$. Since φ is an MPI, then $\varphi : L \to L' \setminus \{a\}$ must be onto as required.

The converse implication is immediate. $\square$

Theorem 10.9 *Let $\varphi : L \to L'$ be a $\vee$-map of lattices. Then φ factorizes as a composition of MPSs followed by a composition of MPIs.*

Proof. In view of Propositions 10.7 and 10.8, it suffices to note that φ can always be factorized as $\varphi = \varphi_1\varphi_2$ with φ_1 a $\vee$-surmorphism and φ_2 an injective $\vee$-map. This can be easily achieved taking $\varphi_1 : L \to L\varphi$ defined like φ, and $\varphi_2 : L\varphi \to L'$ to be the inclusion. $\square$

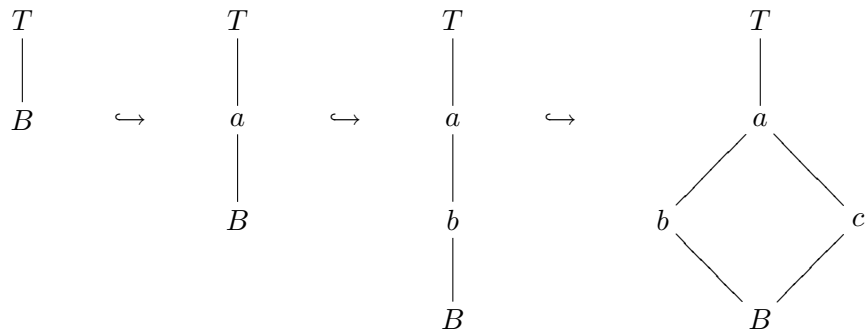
We can produce a partial version of this result for the category FLg:

Corollary 10.10 *Let $\varphi : (L, E) \to (L', E')$ be a $\vee$-surmorphism in* FLg. *Then the decomposition of φ as a composition of MPSs constitutes a composition of maps in* FLg.

Proof. Clearly, the $\vee$-generating set E has a canonical correspondent $E\rho_{a,b}$ in the construction $L/\rho_{a,b}$, and the restriction $\varphi|_E E \to E' \cup \{B\}$ factors adequately if φ is onto. $\square$

The analogous result fails for injective $\vee$-maps. For instance, it is easy to see that the

chain of inclusions



can induce no chain of inclusions between $\vee$-generating sets when we consider $E = \{T\}$ and $E' = \{T, b, c\}$.

10.3 Geometry

Let P be a finite nonempty set and let $\mathcal{L}$ be a nonempty subset of 2^P. We shall always assume that $P \cap 2^P = \emptyset$. We say that $(P, \mathcal{L})$ is a *partial euclidean geometry* (abbreviated to PEG) if the following axioms are satisfied:

(G1) if $L, L' \in \mathcal{L}$ are distinct, then $|L \cap L'| \leq 1$;

(G2) $|L| \geq 2$ for every $L \in \mathcal{L}$.

The elements of P are called *points* and the elements of $\mathcal{L}$ are called *lines*. Given $p \in P$, we denote by $\mathcal{L}(p)$ the set of all lines containing p.

The concept of PEG is an abstract combinatorial generalization of the following geometric situation:

Consider a finite set of lines $\mathcal{L}$ in the euclidean space $\mathbb{R}^n$. Consider also a finite subset P of $\cup\mathcal{L} \subset \mathbb{R}^n$ such that:

- if $L, L' \in \mathcal{L}$ and $L \cap L' = \{p\}$, then $p \in P$;
- $|L \cap P| \geq 2$ for every $L \in \mathcal{L}$.

Representing each $L \in \mathcal{L}$ by $L \cap P$, it follows that $(\mathcal{L}, P)$ constitutes a PEG. It is well known that not all PEG's can be represented over an euclidean space (nor any field) (see [3] and [7, Section 2.6]).

In view of Proposition 4.1, if $\operatorname{ht} L = 2$, the subsets of $L \setminus \{B\}$ with at most two elements are the only c-independent subsets of L. What about the case $\operatorname{ht} L = 3$? This is the starting point for a digression into some interesting connections between c-independence and geometry.

Given a lattice L and $\ell \in \mathring{L} = L \setminus \{T, B\}$, we define

$$\ell\iota = \ell{\downarrow} \cap \mathring{L}.$$

If $(L, E) \in \mathrm{FLg}$, we define

$$\operatorname{Lin}(L, E) = \{\ell\iota \cap E : \ell \in \mathring{L},\ |\ell\iota \cap E| \geq 2\},$$

$$\operatorname{Pt}(L, E) = E, \quad \operatorname{Geo}(L, E) = (\operatorname{Pt}(L, E), \operatorname{Lin}(L, E)).$$

Theorem 10.11 *Let* $(L, E) \in \mathrm{FLg}$ *with* $\mathrm{ht}\, L = 3$. *Then* $\mathrm{Geo}\,(L, E)$ *is a PEG.*

Proof. Axiom (G2) holds trivially, it remains (G1) to be checked. Let $k, \ell \in \mathring{L}$ be such that $|k\iota \cap \ell\iota \cap E| \geq 2$. We must show that $k = \ell$.

Suppose that $k \neq \ell$. Without loss of generality, we may assume that $k > (k \wedge \ell)$. On the other hand, if e_1, e_2 are distinct elements of $k\iota \cap \ell\iota \cap E$, then we have $e_1, e_2 \leq (k \wedge \ell)$ and we may assume that $e_1 < (k \wedge \ell)$. Thus we obtain a chain

$$B < e_1 < (k \wedge \ell) < k < T$$

in $\mathcal{L}$, contradicting $\mathrm{ht}\, L = 3$. Therefore $k = \ell$ and we are done. $\square$

Next we associate a matroid to every lattice L of heigth 3: we define $\mathrm{Mat}_0 L$ to contain:

- all the i-subsets of $L \setminus \{B\}$ for $i \leq 2$;
- all the 3-subsets X of $L \setminus \{B\}$ such that $\vee X = T$.

Note that the latter condition is equivalent to saying that $X \not\subseteq \ell\iota$ for every $\ell \in \mathring{L}$. Finally, write $\mathrm{Mat}\, L = (L \setminus \{B\}, \mathrm{Mat}_0 L)$.

Theorem 10.12 *Let* L *be a lattice of height 3. Then* $\mathrm{Mat}\, L$ *is a matroid.*

Proof. It is immediate that $\mathrm{Mat}_0 L$ is a hereditary collection. Let $\{x, y\}$ be a 2-subset of $L \setminus \{B\}$ and let $\{\ell_1, \ell_2, \ell_3\}$ be a 3-subset of $L \setminus \{B\}$ satisfying $(\ell_1 \vee \ell_2 \vee \ell_3) = T$. We must show that $(x \vee y \vee \ell_i) = T$ for some $i \in \hat{3}$. Suppose not. Then $(x \vee y \vee \ell_i) = k_i < T$ for every $i \in \hat{3}$.

Suppose first that $k_1 = k_2 = k_3$. Then $\ell_i \leq k_1 < T$ for $i = 1, 2, 3$, contradicting $(\ell_1 \vee \ell_2 \vee \ell_3) = T$. Hence we have $(k_1 \wedge k_2 \wedge k_3) < k_j$ for some $j \in \hat{3}$. Since $x, y \leq k_i$ for $i = 1, 2, 3$, we get $x, y \leq (k_1 \wedge k_2 \wedge k_3)$. Since x and y are distinct, we may assume that $x < (k_1 \wedge k_2 \wedge k_3)$ and so we get a chain in L of the form

$$B < x < (k_1 \wedge k_2 \wedge k_3) < k_j < T,$$

contradicting $\mathrm{ht}\, L = 3$. Hence $(x \vee y \vee \ell_i) = T$ for some $i \in \hat{3}$ and so $\{x, y, \ell_i\} \in \mathrm{Mat}\, L$. It follows that $\mathrm{Mat}_0 L$ is a matroid. $\square$

Next we use $\mathrm{Mat}\, L$ to characterize the c-independent subsets of L. To do so, we introduce one more concept: a 3-subset $X \subseteq L \setminus \{B\}$ is called a *potential line* (of $\mathrm{Geo}\, L$) if $|X \cap \ell\iota| \leq 1$ for every $\ell \in \mathring{L}$. This is equivalent to saying that $(x \vee y) = T$ for any distinct $x, y \in X$.

Theorem 10.13 *Let* L *be a lattice of height 3 and let* $X \subseteq L \setminus \{B\}$. *Then the following conditions are equivalent:*

(i) X *is c-independent;*

(ii) $X \in \mathrm{Mat}_0 L$ *and is not a potential line;*

(iii) $|X| \leq 2$ *or* $(|X| = 3$, $\vee X = T$ *and* $(x \vee y) < T$ *for some distinct* $x, y \in X)$.

Proof. (i) $\Rightarrow$ (ii) Suppose that X is c-independent. We may assume that $|X| = 3$. By Proposition 4.2, we may write $X = \{x, y, z\}$ to get a chain $(x \vee y \vee z) > (x \vee y) > x > 0$. Since $\operatorname{ht} L = 3$, it follows that $\vee X = T$ and so $X \in \mathrm{Mat}_0 L$. Since $(x \vee y) < T$, X is not a potential line.

(ii) $\Rightarrow$ (iii). Immediate.

(iii) $\Rightarrow$ (i). The case $|X| = 2$ following from Proposition 4.3, assume that $|X| = 3$, $\vee X = T$ and $x \vee y < T$ for some distinct $x, y \in X$. Since $x \neq y$, we may assume that $(x \vee y) > x$ and so we get a chain $\vee X > (y \vee x) > x$. By Proposition 4.2, X is c-independent. $\square$

Next we associate a $\vee$-generated lattice of height 3 to every PEG $\mathcal{G} = (P, \mathcal{L})$ with at least two lines: let $\operatorname{Lat} \mathcal{G} = P \cup \mathcal{L} \cup \{B, T\}$, where $x \leq y$ if and only if

$$x = B \quad \text{or} \quad y = T \quad \text{or} \quad (x \in P \text{ and } y \in \mathcal{L} \text{ and } x \in y).$$

It is immediate that $\operatorname{Lat} \mathcal{G}$ is a lattice of height 3. Moreover, if $W = \{p_1, \ldots, p_k\} \in \mathcal{L}$, then $W = (p_1 \vee \ldots \vee p_k)$, and we can also get the top T as the join of two lines. Thus $(\operatorname{Lat} \mathcal{G}, P) \in \mathrm{FLg}$.

Proposition 10.14 *Let $\mathcal{G} = (P, \mathcal{L})$ with $|\mathcal{L}| \geq 2$. Then* $\operatorname{Geo}(\operatorname{Lat} \mathcal{G}, P) = \mathcal{G}$.

Proof. It follows from the definitions that ¡$\operatorname{Geo}(\operatorname{Lat} \mathcal{G}, P)$ is of the form $(P, \mathcal{L}')$. If $p \in P$, then $p\iota = \{p\}$ in $\operatorname{Lat} G$. If $W = \{p_1, \ldots, p_k\} \in \mathcal{L}$, then $W\iota = \{W, p_1, \ldots, p_k\}$. Thus, by definition of the construction Geo, the elements of $\mathcal{L}'$ are of the form $W\iota \cap P$ for $W \in \mathcal{L}$. Since $W\iota \cap P = W$, we get $\mathcal{L}' = \mathcal{L}$ and so $\operatorname{Geo}(\operatorname{Lat} \mathcal{G}, P) = \mathcal{G}$. $\square$

We say that $h \in \operatorname{Fl}(L, E)$ is a *hyperplane* of (L, E) if h is maximal in $\operatorname{Fl}(L, E) \setminus \{E\}$. For height 4, we can prove the following result:

Proposition 10.15 *Let $(L, E) \in \mathrm{FLg}$ have height 4 and let X be a 4-subset of E. Then the following conditions are equivalent:*

(i) X is c-independent;

(ii) every 3-subset of X is c-independent and $|X \cap h| = 3$ for some hyperplane h of L.

Proof. (i) $\Rightarrow$ (ii). The first claim follows from c-independent sets being closed under inclusion. On the other hand, by Proposition 4.2, X admits an enumeration x_1, x_2, x_3, x_4 such that

$$\mathrm{Cl}_L X \supset \mathrm{Cl}_L(x_2, x_3, x_4) \supset \mathrm{Cl}_L(x_3, x_4) \supset \mathrm{Cl}_L(x_4) \supset \emptyset.$$

Let $h = \mathrm{Cl}_L(x_2, x_3, x_4)$. Since $\operatorname{ht} \operatorname{Fl}(L, E) = \operatorname{ht} L = 4$ by Proposition 3.3, it follows that h is a hyperplane of (L, E). Clearly, $x_1 \notin \mathrm{Cl}_L(x_2, x_3, x_4)$ and so $|X \cap h| = 3$.

(ii) $\Rightarrow$ (i). Write $X \setminus h = \{x_1\}$. Since $X \cap h$ is c-independent, it follows from Proposition 4.4 that $X \cap h$ admits an enumeration x_2, x_3, x_4 such that $(x_2 \vee x_3 \vee x_4) > (x_3 \vee x_4) > x_4$. Since $\{x_2, x_3, x_4\} \subseteq h$ implies $\mathrm{Cl}_L(x_2, x_3, x_4) \subseteq \mathrm{Cl}_L(h) = h$, we get $x_1 \notin \mathrm{Cl}_L(x_2, x_3, x_4) = Z_{x_2 \vee x_3 \vee x_4}$ (by (6)) and so $(x_1 \vee x_2 \vee x_3 \vee x_4) > (x_2 \vee x_3 \vee x_4)$. Thus X is c-independent by Proposition 4.2. $\square$

We can generalize to higher dimensions the concept of PEG to get generalizations of some results obtained in the height 3 case, namely Proposition 10.14. For technical reasons, we include the full space of points as the highest dimension subspace, but it could as well be omitted.

For $m \geq 3$, we say that $(P_1, P_2, \ldots, P_m)$ is an m-PEG over a finite set E if:

(J1) $P_1, \ldots, P_m$ are mutually disjoint subsets of 2^E and $P_m = \{E\}$;

(J2) $\forall p \in P_1\ |p| = 1$;

(J3) $\forall i \in \{2, \ldots, m\}\ \cup P_i \subseteq \cup P_1$;

(J4) $\forall i \in \{2, \ldots, m\}\ \forall p \in P_i\ \exists q \in P_{i-1} : q \subset p$;

(J5) $\forall i, j \in \{2, \ldots, m\}\ \forall p \in P_i\ \forall q \in P_j$, one of the following five conditions holds:

- (J5a) $p \cap q = \emptyset$;
- (J5b) $p \cap q \in P_r$ for some $r < i, j$;
- (J5c) $i < j$ and $p \subset q$;
- (J5d) $i > j$ and $p \supset q$;
- (J5e) $p = q$.

The 3-PEG case corresponds to our original concept of PEG, replacing each point p by $\{p\}$ and adding the full subspace P. A natural example for the general case is given by a (finite) collection of affine subspaces of various ranks in an euclidean space of arbitrary dimension, where the subspaces are defined through collections of points from a finite set E.

Two m-PEGs $(P_1, P_2, \ldots, P_m)$ (over E) and $(P'_1, P'_2, \ldots, P'_m)$ (over E') are said to be isomorphic if there exists a bijection $\varphi : E \to E'$ such that

$$\{e_1, \ldots, e_k\} \in P_i \Leftrightarrow \{e'_1, \ldots, e'_k\} \in P_i$$

holds for all $i \in \hat{m}$ and $e_1, \ldots, e_k \in E$. A particularly important case arises with the canonical bijections $X \to \widetilde{X}$, where $\widetilde{X} = \{\{x\} \mid x \in X\}$.

Given a lattice L and $\ell \in \mathcal{L}$, we define $\mathrm{ht}(\ell)$ to be the maximum length n of a chain $\ell = \ell_0 > \ell_1 > \ldots > \ell_n$ in $\mathcal{L}$. Obviously, $\mathrm{ht}\, L = \mathrm{ht}(1)$.

We recall now the notions of atom and atomic lattice. If we denote by A_L the set of atoms of L, then L is atomic if and only if $(L, A_L) \in \mathrm{FLg}$.

Given an atomic lattice L of height m, we define $\mathrm{Geo}\,(L, A_L) = (P_1, \ldots, P_m)$ by

$$P_i = \{\ell\iota \cap A_L \mid \ell \in L,\ \mathrm{ht}(\ell) = i\} \qquad (i = 1, \ldots, m).$$

We claim that $\mathrm{Geo}\,(L, A_L)$ is an m-PEG over A_L. Axiom (J1) follows from (3) and $T\iota = \mathring{L}$, and (J2) is immediate. Since the elements of P_1 are of the form $\{a\}$ for $a \in \mathcal{A}_L$, (J3) holds. Since every element of L of height i covers some element of height $i-1$, (J4) follows. Finally, let $k, \ell \in L$ have height i and j, respectively. Since $(k\iota \cap A_L) \cap (\ell\iota \cap A_L) = (k \wedge \ell)\iota \cap A_L$, then (J5a) or (J5b) hold if $k \wedge \ell < k, \ell$. Hence we are left with the cases $k < \ell$, $k > \ell$ and

$k = \ell$ which give us respectively (J5c), (J5d) and (J5e) in view of (3). Thus $\mathrm{Geo}\,(L, A_L)$ is an m-PEG over A_L.

Conversely, given an m-PEG $\mathcal{G} = (P_1, \ldots, P_m)$ over E, we define the poset $\mathrm{Lat}_0\mathcal{G} = \{\emptyset\} \cup P_1 \cup \ldots \cup P_m$, ordered by inclusion. By (J5), $\mathrm{Lat}_0\mathcal{G}$ is closed under intersection and constitutes then a $\wedge$-semilattice with bottom element $\emptyset$ and top element E. Hence $\mathrm{Lat}_0\mathcal{G}$ is a lattice with $p \vee q = \cap\{r \in \mathrm{Lat}\,\mathcal{G} \mid p \cup q \subseteq r\}$ (note that $E \in \mathrm{Lat}_0\mathcal{G}$). We claim that P_1 is the set of atoms of $\mathrm{Lat}_0 G$.

Indeed, it follows from (J4) that any atom is contained in P_1, and the converse is a consequence of (J2). Now, by (J2) and (J3), $\mathrm{Lat}_0\mathcal{G}$ is atomic and we can define $\mathrm{Lat}\,\mathcal{G} = (\mathrm{Lat}_0\mathcal{G}, P_1) \in \mathrm{FLg}$.

Theorem 10.16 *Let $\mathcal{G}$ be an m-PEG and let L be an atomic lattice. Then*

(i) $\mathrm{Geo}\,\mathrm{Lat}\,\mathcal{G} \cong \mathcal{G}$*;*

(ii) $\mathrm{Lat}\,\mathrm{Geo}\,(L, A_L) \cong (L, A_L)$*.*

Proof. (i) Let $\mathcal{G} = (P_1, \ldots, P_m)$ be an m-PEG over E. We claim that

$$p \in P_i \quad \Rightarrow \quad \mathrm{ht}(p) = i \text{ in } \mathrm{Lat}\,\mathcal{G}. \tag{31}$$

We use induction on i. The case $i = 1$ follows from (J2), hence we assume that $p \in P_i$ with $i > 1$ and (31) holds for $i - 1$. By (J4) and the induction hypothesis, we have $\mathrm{ht}(p) \geq i$. Suppose that $\mathrm{ht}(p) > i$. This would imply that there would exist distinct $q, r \in P_j$ for some j such that $q \subset r$, contradicting (J5). Hence $\mathrm{ht}(p) = i$ and (31) holds.

Hence $\mathrm{Lat}\,\mathcal{G}$ is a lattice of height m with set of atoms P_1 and we may write $\mathrm{Geo}\,\mathrm{Lat}\,\mathcal{G} = (P'_1, \ldots, P'_m)$, an m-PEG over $P_1 \subseteq \widetilde{E}$. In view of (31), the elements of P'_i are of the form $p\iota \cap P_1$ for $p \in P_i$.

For all $p \in P_i$ and $e \in E$, we have $\{e\} \in p\iota$ in $\mathrm{Lat}\,\mathcal{G}$ if and only if $\{e\} \subseteq p$ in $\mathcal{G}$ if and only if $e \in p$, hence $p\iota \cap P_1 = \widetilde{p}$ and so $e \mapsto \widetilde{e}$ $(e \in E)$ induces an isomorphism between $\mathcal{G}$ and $\mathrm{Geo}\,\mathrm{Lat}\,\mathcal{G}$.

(ii) Let L be an atomic lattice. Then the elements of $\mathrm{Lat}_0\mathrm{Geo}\,(L, A_L)$ are of the form $\ell\iota \cap A_L$ (note that $\emptyset = B\iota \cap A_L$), ordered by inclusion. By (3), $\varphi : L \to \mathrm{Lat}_0\mathrm{Geo}\,(L, A_L)$ defined by $\ell\varphi = \ell\iota \cap A_L$ is an isomorphism of posets and therefore of lattices. Since the set of atoms of a lattice is uniquely determined, φ induces an isomorphism from (L, A_L) onto $\mathrm{Lat}\,\mathrm{Geo}\,(L, A_L)$. $\square$

10.4 Strong maps

The concept of strong map can be defined for lattices and for hereditary collections. We start discussing the lattice case.

We say that a mapping $\varphi : (L, E) \to (L', E')$ of lattices is a *strong map* if

$$\forall Z \in \mathrm{Fl}\,(L', E') \quad (Z \cup \{B\})\varphi^{-1} \cap E \in \mathrm{Fl}\,(L, E).$$

Now assume that φ is a $\vee$-map in FLg. In particular, $E\varphi \subseteq E' \cup \{0\}$. Given $X \subseteq E$, write $\overline{X} = \mathrm{Cl}_L X$. Then φ induces a map $\overline{\varphi} : \mathrm{Fl}\,(L, E) \to \mathrm{Fl}\,(L', E')$ by $Z\overline{\varphi} = \overline{Z\varphi \setminus \{B\}}$.

Proposition 10.17 *Let $\varphi : (L, E) \to (L', E')$ be a $\vee$-map in* FLg*. Then:*

(i) φ is a strong map;

(ii) $\overline{\varphi}$ is a $\vee$-map and $\overline{e}\,\overline{\varphi} = \overline{e\varphi \setminus \{B\}}$ for every $e \in E$.

Proof. (i) Let $Z' \in \mathrm{Fl}\,(L', E')$. Then $Z' = Z_{\ell'} = \ell' \downarrow \cap E'$ for some $\ell' \in L'$. Assume that $(Z' \cup \{B\})\varphi^{-1} = \{x_1, \ldots, x_k\}$, and write $\ell = (x_1 \vee \ldots \vee x_k)$. We claim that

$$(Z' \cup \{B\})\varphi^{-1} \cap E = Z_\ell \in \mathrm{Fl}\,(L, E). \tag{32}$$

Indeed, let $e \in (Z' \cup \{B\})\varphi^{-1} \cap E$. Then $e = x_i$ for some $i \in \hat{k}$ and so $e \leq \ell$. Thus $e \in Z_\ell$. Conversely, assume that $e \in Z_\ell$. Since $Z_\ell \subseteq E$, it suffices to show that $e\varphi \in Z' \cup \{B\}$. Since $E\varphi \subseteq E' \cup \{B\}$, all we need is to prove that $e\varphi \leq \ell'$. Now $e \leq \ell = (x_1 \vee \ldots \vee x_k)$ and φ being a $\vee$-map yields $e\varphi \leq (x_1 \vee \ldots \vee x_k)\varphi = (x_1\varphi \vee \ldots \vee x_k\varphi)$. Since $x_i\varphi \in Z' \cup \{B\}$ and therefore $x_i\varphi \leq \ell'$ for every i, we get $e\varphi \leq \ell'$ as required. Thus (32) holds and so φ is a strong map.

(ii) Let $x, y \in L$. To show that $\overline{\varphi}$ is a $\vee$-map, we need to show that $(Z_x \vee Z_y)\overline{\varphi} = (Z_x\overline{\varphi} \vee Z_y\overline{\varphi})$, i.e.

$$(\overline{Z_x \cup Z_y})\overline{\varphi} = \overline{Z_x\overline{\varphi} \cup Z_y\overline{\varphi}}.$$

Since $\overline{\varphi}$ is order-preserving, we get $Z_x\overline{\varphi} \cup Z_y\overline{\varphi} \subseteq (\overline{Z_x \cup Z_y})\overline{\varphi}$ and so $\overline{Z_x\overline{\varphi} \cup Z_y\overline{\varphi}} \subseteq (\overline{Z_x \cup Z_y})\overline{\varphi}$ since $(\overline{Z_x \cup Z_y})\overline{\varphi}$ is closed. The inclusion

$$(\overline{Z_x \cup Z_y})\overline{\varphi} \subseteq \overline{Z_x\overline{\varphi} \cup Z_y\overline{\varphi}}. \tag{33}$$

remains to be proved. We have

$$(\overline{Z_x \cup Z_y})\overline{\varphi} = \overline{(\overline{Z_x \cup Z_y})\varphi \setminus \{B\}}. \tag{34}$$

Let $Z = \overline{Z_x\overline{\varphi} \cup Z_y\overline{\varphi}} \in \mathrm{Fl}\,(L', E')$. Then $Z_x \cup Z_y \subseteq (Z \cup \{0\})\varphi^{-1}$. By (i), we have $(Z \cup \{B\})\varphi^{-1} \cap E \in \mathrm{Fl}\,(L, E)$, hence $\overline{Z_x \cup Z_y} \subseteq (Z \cup \{B\})\varphi^{-1}$ and so $(\overline{Z_x \cup Z_y})\varphi \setminus \{B\} \subseteq Z$. Since Z is closed, it follows that

$$(\overline{Z_x \cup Z_y})\overline{\varphi} \subseteq \overline{(\overline{Z_x \cup Z_y})\varphi \setminus \{B\}} \subseteq Z,$$

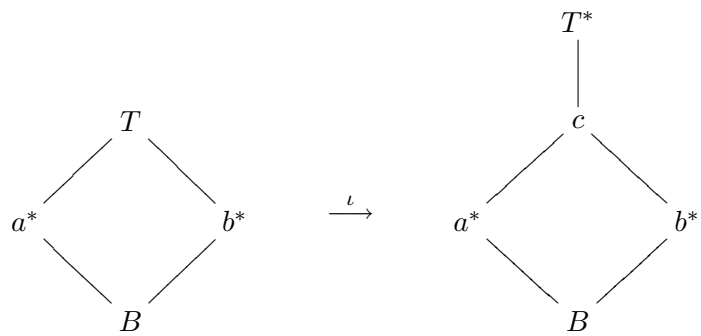
hence (33) holds and so $\overline{\varphi}$ is a $\vee$-map.

Let $e \in E$. We must show that $\overline{\overline{e}\varphi \setminus \{B\}} = \overline{e\varphi \setminus \{B\}}$. The opposite inclusion being immediate, we set $Z = \overline{e\varphi \setminus \{B\}} \in \mathrm{Fl}\,(L', E')$ and show that $\overline{\overline{e}\varphi \setminus \{B\}} \subseteq Z$.

Clearly, $e \in (Z \cup \{B\})\varphi^{-1}$. By (i), $(Z \cup \{B\})\varphi^{-1}$ is closed and so $\overline{e} \subseteq (Z \cup \{B\})\varphi^{-1}$. Hence $\overline{e}\varphi \subseteq Z \cup \{B\}$ and so $\overline{e}\varphi \setminus \{B\} \subseteq Z$. Since Z is closed, we get $\overline{\overline{e}\varphi \setminus \{B\}} \subseteq Z$ as required. $\square$

The following examples show that a strong map is not necessarily a $\vee$-map, even if we assume injectivity or surjectivity:

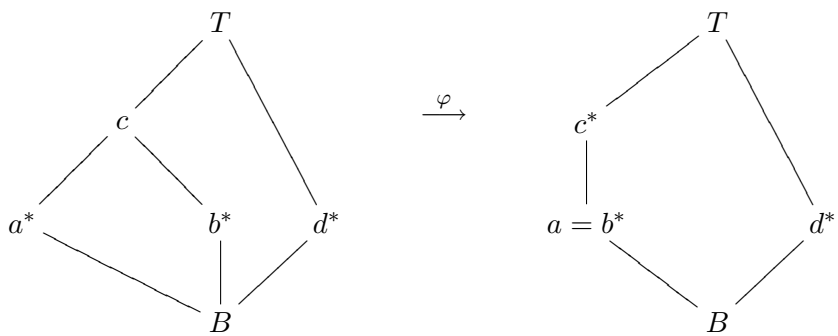
Example 10.18 *Consider the inclusion $\iota : (L, E) \to (L', E')$ for the following lattices, where the elements of the $\vee$-generating sets are marked with an asterisk:*



Then ι is strong but not a $\vee$-map.

Indeed, since $\mathrm{Fl}\,(L, E) = 2^E$, ι is strong. Since $(a \vee b)\iota = T \neq c = (a\iota \vee b\iota)$, ι is not a $\vee$-map.

Example 10.19 *Consider the onto mapping $\varphi : (L, E) \to (L', E')$ defined by*



where the elements of the $\vee$-generating sets are marked with an asterisk. Then φ is strong but not a $\vee$-map.

Indeed, we have $\mathrm{Fl}\,(L, E) = 2^E \setminus \{ad, bd\}$, but the inverse image of a flat from (L', E') contains a if and only if it contains b. Thus φ is strong. Since $(a \vee b)\varphi = c \neq (a\varphi \vee b\varphi)$, then φ is not a $\vee$-map.

We discuss now the concepts of strong and weak maps for hereditary collections. Let $(E, H), (E', H')$ be hereditary collections and let $\varphi : E \to E'$ be a mapping. We say that φ is a *weak map* (with respect to $(E, H), (E', H')$) if

$$\varphi|_X \text{ injective and } X\varphi \in H' \quad \Rightarrow \quad X \in H$$

holds for every $X \subseteq E$. Assume now that $(E, H), (E', H')$ are boolean representable. We say that φ is a *strong map* (with respect to $(E, H), (E', H')$) if

$$X \in \mathrm{Fl}\,(E', H') \Rightarrow X\varphi^{-1} \in \mathrm{Fl}\,(E, H)$$

holds for every $X \subseteq E'$.

Proposition 10.20 *Let $(E, H), (E', H')$ be boolean representable simple hereditary collections and let $\varphi : E \to E'$ be a strong map. Then φ is a weak map.*

Proof. Let $X \subseteq E$ and assume that $\varphi|_X$ is injective and $X\varphi \in H'$. By Proposition 7.2, and since $\varphi|_X$ is injective, X admits an enumeration $x_1, \ldots, x_k$ such that

$$\mathrm{Cl}(x_1\varphi, \ldots, x_k\varphi) \supset \mathrm{Cl}(x_2\varphi, \ldots, x_k\varphi) \supset \ldots \supset \mathrm{Cl}(x_k\varphi).$$

We claim that

$$\mathrm{Cl}(x_1, \ldots, x_k) \supset \mathrm{Cl}(x_2, \ldots, x_k) \supset \ldots \supset \mathrm{Cl}(x_k). \tag{35}$$

Indeed, suppose that $\mathrm{Cl}(x_i, \ldots, x_k) = \mathrm{Cl}(x_{i+1}, \ldots, x_k)$ for some $i \in \{1, \ldots, k-1\}$. Then $x_i \in \mathrm{Cl}(x_{i+1}, \ldots, x_k)$. Since $\{x_{i+1}, \ldots, x_k\} \subseteq (\mathrm{Cl}(x_{i+1}\varphi, \ldots, x_k\varphi))\varphi^{-1}$ and the latter is closed since φ is a strong map, we get $x_i \in \mathrm{Cl}(x_{i+1}, \ldots, x_k) \subseteq (\mathrm{Cl}(x_{i+1}\varphi, \ldots, x_k\varphi))\varphi^{-1}$ and so $x_i\varphi \in \mathrm{Cl}(x_{i+1}\varphi, \ldots, x_k\varphi)$, contradicting $\mathrm{Cl}(x_i\varphi, \ldots, x_k\varphi) \supset \mathrm{Cl}(x_{i+1}\varphi, \ldots, x_k\varphi)$. Hence $\mathrm{Cl}(x_i, \ldots, x_k) \supset \mathrm{Cl}(x_{i+1}, \ldots, x_k)$ for every i and so (35) holds. Now Proposition 7.2 yields $X \in H$ and so φ is a weak map. $\square$

Acknowledgments

The second author acknowledges support from the European Regional Development Fund through the programme COMPETE and by the Portuguese Government through FCT (Fundação para a Ciência e a Tecnologia) under the project PEst-C/MAT/UI0144/2011.

References

[1] A. V. Borovik, I. M. Gelfand and N. White, Representations of matroids in semimodular lattices, *Eur. J. Comb.* 22(6) (2001), 789–799.

[2] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Volume 1, Mathematical Surveys of the American Mathematical Society, No. 7, Providence, R.I., 1961.

[3] H. S. M. Coxeter, Self-dual configurations and regular graphs, *Bull. Amer. Math. Soc.* 56 (1950) 413–455.

[4] R. Diestel, *Graph Theory*, Springer-Verlag, 2000.

[5] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove and D. S. Scott, *Continuous Lattices and Domains*, volume 93 of *Encyclopedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 2003.

[6] G. Grätzer, *Lattice Theory: Foundation*, Springer Basel AG, 2011.

[7] B. Grünbaum, *Configurations of Points and Lines*, Graduate Studies in Mathematics, vol. 103, American Mathematical Society, 2009.

[8] Z. Izhakian, The tropical rank of a tropical matrix, preprint, arXiv:math.AC/060420, 2006.

[9] Z. Izhakian and J. Rhodes, New representations of matroids and generalizations, preprint, arXiv:1103.0503, 2011.

[10] Z. Izhakian and J. Rhodes, Boolean representations of matroids and lattices, preprint, arXiv:1108.1473, 2011.

[11] Z. Izhakian and J. Rhodes, C-independence and c-rank of posets and lattices, preprint, arXiv:1110.3553, 2011.

[12] A. Lascoux and M.-P. Schützenberger, Treillis et bases des groupes de Coxeter, *Electron. J. Combin.* 3 (1996), #R27.

[13] J. G. Oxley, *Matroid Theory*, Oxford Science Publications, 1992.

[14] J. G. Oxley, What is a matroid, In: *LSU Mathematics Electronic Preprint Series*, pages 179–218, 2003.

[15] N. Reading, Order dimension, strong Bruhat order and lattice properties for posets, *Order* 19(1) (2002), 73–100.

[16] J. Rhodes and P. V. Silva, A new notion of vertex independence and rank for finite graphs, preprint, arXiv:1201.3984, 2012.

[17] J. Rhodes and B. Steinberg, *The q-theory of Finite Semigroups*, Springer Monographs in Mathematics, 2009.

[18] M. Wachs, Poset topology: tools and applications, *Geometric Combinatorics* 13 (2007), 497–615.

[19] H. Whitney, On the abstract properties of linear dependence, *American Journal of Mathematics* (The Johns Hopkins University Press), 57(3) (1935), 509–533 (Reprinted in Kung (1986), pp. 55–79).

[20] Wikipedia, http://en.wikipedia.org/wiki/Fano_plane.

[21] Wikipedia, http://en.wikipedia.org/wiki/Petersen_graph.