NON-ASSOCIATIVE UNITARY BANACH ALGEBRAS

MARÍA BURGOS
UNIVERSIDAD DE ALMERÍA

By a normed algebra we mean a real or complex (possibly nonassociative) algebra A endowed with a norm $\|\cdot\|$ satisfying $\|xy\| \leq \|x\|\|y\|$ for all $x, y \in A$. A complete normed associative algebra will be called a Banach algebra. A normed algebra is called norm-unital if it has a unit $\mathbf{1}$ such that $\|\mathbf{1}\| = 1$. Unitary elements of a norm-unital normed associative algebra A are defined as those invertible elements u of A satisfying $\|u\| = \|u^{-1}\| = 1$. By a unitary normed associative algebra we mean a norm-unital associative normed algebra A such that the convex hull of the set of its unitary elements is norm-dense in the closed unit ball of A. Relevant examples of unitary Banach algebras are all unital C^*-algebras and the discrete group algebras $\ell_1(G)$ for every group G.

The study of unitary Banach algebras is quite recent (see [1, 2, 4, 5, 6, 8]). Both [4] and [6] were mainly concerned with the achievement of characterizations of unital C^*-algebras among unitary Banach algebras. In [1], unitary Banach algebras are considered by themselves, showing that all unitary Banach algebras are quotients of discrete group algebras. We continue this line in this talk, thus devoting the most part of it to the development of a general theory of unitary Banach algebras. We revisit the concepts of maximality and unique maximality (which are closely related to that of unitarity). By the sake of usefulness, we introduce the notions of strong maximality and strong unique maximality, and clarify how all these notions are related among them, as well as with that of unitarity. To this end, we also introduce the concept of minimality of the equivalent norm (a weakening of the classical notion of minimality of the norm [3]). We realize that, in the commutative case, unital C^*-algebras are nothing other than strongly uniquely maximal complex Banach algebras.

For a real (respectively, complex) norm unital Banach algebra A, consider Property $(\mathcal{S})$ which follows:

$(\mathcal{S})$ There exists a linear (respectively, conjugate-linear) algebra involution on A mapping each unitary element to its inverse.

It is known that Property $(\mathcal{S})$ is fulfilled in the case that A is a unital C^*-algebra, a discrete group algebra, or a finite-dimensional unitary Banach algebra. However, in general, unitary Banach algebras need not satisfy Property $(\mathcal{S})$, even if they are complex and commutative [1]. We show that unitary semisimple commutative complex Banach algebras satisfy Property $(\mathcal{S})$, and that, endowed with the involution given by such a property, they become hermitian $*$-algebras. From this theorem we deduce that, in the commutative case, unital C^*-algebras are nothing other than strongly maximal unitary semisimple complex Banach algebras, and we show that a uniquely maximal complex Banach algebra "close enough to be commutative" is isometrically isomorphic to a commutative C^*-algebra. Later, we study Property

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$(\mathcal{S})$ in the noncommutative case. To this end, we introduce "good" groups as those groups G such that every primitive ideal of the complex Banach $*$-algebra $\ell_1(G)$ is $*$-invariant, and prove that, if A is a real or complex unitary semisimple Banach algebra such that the group U_A is good, then A satisfies Property $(\mathcal{S})$. It seems to be an open problem whether or not every group is good. Anyway, we show that this problem has an affirmative answer if (and only if) every primitive unitary complex Banach algebra satisfies Property $(\mathcal{S})$, if (and only if) every primitive unitary real Banach algebra satisfies Property $(\mathcal{S})$.

If X is a complex Hilbert space, then the algebra $\mathcal{L}(X)$ (of all bounded linear operators on X) is a (complex) C^*-algebra, and hence it is unitary. It seems to be an open problem whether or not all complex Banach spaces X such that $\mathcal{L}(X)$ is unitary are in fact Hilbert spaces. Some partial affirmative answers to this problem have been given in [2]. We obtain some new partial affirmative answers to this problem . We prove that a complex Banach space X is a Hilbert space if (and only if) $\mathcal{L}(X)$ is unitary and satisfies Property $(\mathcal{S})$. Therefore, if every group is good, then all complex Banach spaces X such that $\mathcal{L}(X)$ is unitary are in fact Hilbert spaces.

Historically, unitary Banach algebras have been considered only from an associative point of view, and the main topic of interest has been characterize C^*-algebras among them. In this talk, we leave the associative scope in order to deal by the first time with nonassociative unitary normed algebras. Such a generalization of the theory of unitary normed algebras to the non-associative setting is mainly motivated by the Russo-Dye-type theorem for unital JB^*-algebras, proved by J.D.M. Wright and M.A. Youngson [9]. Due to the fact that the setting of unital non-commutative Jordan algebras becomes the largest non-associative one where a notion of invertible element works reasonably [7], we restrict our attention to norm-unital normed non-commutative Jordan algebras. Unitary elements of such an algebra are defined verbatim as in the associative case, and the the notions of unitarity, maximality, strong maximality, unique maximality, and strong unique maximality are translated literally from the associative setting to the more general one. Since the set of all unitary elements of a norm-unital normed non-commutative Jordan algebra need not be multiplicatively closed, we introduce weakly unitary normed non-commutative Jordan algebras as those norm-unital normed non-commutative Jordan algebras A such that the convex multiplicatively closed hull of unitary elements is dense in the closed unit ball of A. Replacing unitarity with weak unitarity, most results obtained in the associative case remain true in the new setting.

Alternative algebras (respectively, alternative C^*-algebras) are very particular examples of non-commutative Jordan algebras (respectively, non-commutative JB^*-algebras). It is worth mentioning that, as in the particular associative case, for a norm-unital normed alternative algebra A, the set of unitary elements is multiplicatively closed, and hence the concepts of unitarity and weak unitarity are equivalent for A. We prove that every finite-dimensional maximal unitary normed alternative complex algebra is isometrically isomorphic to an alternative C^*-algebra. This generalizes [6, Theorem 6]. We prove that every finite-dimensional maximal unitary normed alternative complex algebra is isometrically isomorphic to an alternative C^*-algebra, and that a semisimple finite-dimensional norm-unital normed complex alternative algebra such that its symmetrized algebra is maximal is (isometrically isomorphic to) an alternative C^*-algebra.

Surprisingly, we obtain that every group is good (that is, every primitive ideal of $\ell_1(G)$ is $*$-invariant) if and only if every unitary semisimple complete normed complex alternative algebra satisfies Property $(\mathcal{S})$, if and only if every unitary semisimple complete normed real alternative algebra satisfies Property $(\mathcal{S})$.

References

[1] J. BECERRA, S. COWELL, A. RODRÍGUEZ, and G. V. WOOD, Unitary Banach algebras. *Studia Math.* **162** (2004), 25-51.
[2] J. BECERRA, A. RODRÍGUEZ, and G. WOOD, Banach spaces whose algebras of operators are unitary: a holomorphic approach. *Bull. London Math. Soc.* **35** (2003), 218-224.
[3] F. F. BONSALL, A minimal property of the norm in some Banach algebras. *J. London Math. Soc.* **29** (1954), 156-164.
[4] E. R. COWIE, *Isometries in Banach algebras.* Ph. D. Thesis, Swansea 1981.
[5] E. R. COWIE, An analytic characterization of groups with no finite conjugacy classes. *Proc. Amer. Math. Soc.* **87** (1983), 7-10.
[6] M. L. HANSEN and R. V. KADISON, Banach algebras with unitary norms. *Pacific J. Math.* **175** (1996), 535-552.
[7] K. McCRIMMON, Noncommutative Jordan rings. *Trans. Amer. Math. Soc.* **158** (1971), 1-33.
[8] G. V. WOOD, Maximal algebra norms. *Contemporary Math.* **321** (2003), 335-345.
[9] J. D. M. WRIGHT and M. A. YOUNGSON, A Russo Dye theorem for Jordan C^*-algebras. In *Functional analysis: surveys and recent results* (Ed. by K. D. Bierstedt and F. Fuchssteiner), 279-282, North Holland Math. Studies **27**, Amsterdam, 1977.