On the adjugate of a matrix

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Let \(|\lambda I - A| = \lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda + c_0\) be the characteristic polynomial of an \(n\times n\) matrix \(A\) over a given field \(K\). The elegant proof of the Cayley-Hamilton theorem of [1, p. 50] can be easily modified to prove that (see [2, p. 40]):

\begin{equation}
(-1)^{n-1} A^{\text{adj}} = A^{n-1} + c_{n-1} A^{n-2} + \cdots + c_1 I
\end{equation}

where \(A^{\text{adj}}\) stands for the adjugate of \(A\) (or classical adjoint — the transpose of the cofactor matrix of \(A\)) and \(I\) for the identity matrix of order \(n\). More generally, it can be easily modified to prove that (see [3, p. 38]):

\begin{equation}
(\lambda I - A)^{\text{adj}} = A^{n-1} + (\lambda + c_{n-1}) A^{n-2} + \cdots + (\lambda^n + c_{n-1} \lambda^{n-2} + c_{n-2} \lambda^{n-3} + \cdots + c_1) I
\end{equation}

Proof. We start from the basic fact that:

\begin{equation}
(\lambda I - A) \cdot (\lambda I - A)^{\text{adj}} = (\lambda^n + c_{n-1} \lambda^{n-1} + \cdots + c_1 \lambda + c_0) I
\end{equation}

and by noting that, by definition of adjugate, \((\lambda I - A)^{\text{adj}}\) is a polynomial in \(\lambda\) of degree \(n-1\) with coefficients in the space of the \(n\times n\) matrices over \(K\), say,

\begin{equation}
(\lambda I - A)^{\text{adj}} = D_{n-1} \lambda^{n-1} + D_{n-2} \lambda^{n-2} + \cdots + D_1 \lambda + D_0
\end{equation}

By equating the terms in \(\lambda\) of the same order in both sides of equation (3), we obtain:

\begin{align*}
D_{n-1} &= I \\
-A \cdot D_{n-1} + D_{n-2} &= c_{n-1} I \\
-A \cdot D_{n-2} + D_{n-3} &= c_{n-2} I \\
&\quad \vdots \quad \vdots \\
-A \cdot D_1 + D_0 &= c_1 I \\
-A \cdot D_0 &= c_0 I
\end{align*}

Finally, by multiplying the first equation by \(A^n\), the second by \(A^{n-1}\), and so on, up to the last equation, and by adding the new equations up, we obtain the Cayley-Hamilton theorem. This is the proof in [1, p. 50]. If, instead, we multiply the first equation by \(A^{n-1}\), the second by \(A^{n-2}\), etc., and stop precisely before the last equation, by summing up we obtain at the left-hand side \(D_0\), which is \((-1)^{n-1} A^{\text{adj}}\), say, by putting \(\lambda = 0\). Hence, we get (1). (2) can be proven by obtaining the coefficients of \((\lambda I - A)^{\text{adj}}\) step by step through the same procedure.

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References

