Integral Calculus On Quantum Exterior Algebras

Serkan Karaçuha

MATHEMATICS DEPARTMENT, UNIVERSITY OF PORTO

November 9, 2013

Contents

Differential Graded Algebra

DGA over A

- $\Omega(A) := \bigoplus_{n \geq 0} \Omega^n(A)$ such that $\Omega^0(A) = A$,

Differential Graded Algebra

DGA over A

- $\Omega(A) := \bigoplus_{n\geq 0} \Omega^n(A)$ such that $\Omega^0(A) = A$,
- $d : \Omega^k(A) \to \Omega^{k+1}(A)$ linear map of degree one, which satisfies

Differential Graded Algebra

DGA over A

- $\Omega(A) := \bigoplus_{n\geq 0} \Omega^n(A)$ such that $\Omega^0(A) = A$,
- $d : \Omega^k(A) \to \Omega^{k+1}(A)$ linear map of degree one, which satisfies
  (i) $d^2 = 0$,

Differential Graded Algebra

DGA over A

- $\Omega(A) := \bigoplus_{n\geq 0} \Omega^n(A)$ such that $\Omega^0(A) = A$,
- $d : \Omega^k(A) \to \Omega^{k+1}(A)$ linear map of degree one, which satisfies
  - (i) $d^2 = 0$,
  - (ii) $d(\omega\nu) = d(\omega)\nu + (-1)^k \omega d(\nu)$, $\forall \omega \in \Omega^k(A)$, $\nu \in \Omega(A)$.

FODC

The pair $(\Omega^1(A), d)$ is referred to as a *first order differential calculus* on A.

Non-commutative Connection

Connection

Given an *FODC* $(\Omega^1(A), d)$ over A and a right A-module M, a linear map $\nabla^0 : M \to M \otimes_A \Omega^1(A)$ satisfying

$$\nabla^0(ma) = \nabla^0(m)a + m \otimes_A d(a)$$

is called a *connection* in M.

Non-commutative Hom-connection

Hom-connection(T.Brzezinski)

A right *hom-connection* w.r.t. a *dga* $(\Omega(A), d)$ over A, is a pair (M, ∇_0), where M is a right A-module and

$$\nabla_0 : Hom_A(\Omega^1(A), M) \to M$$

is a linear mapping s.t.

$$\nabla_0(fa) = \nabla_0(f)a + f(d(a)) \qquad \forall a \in A, f \in Hom_A(\Omega^1(A), M)$$

Non-commutative Hom-connection

Hom-connection

Any hom-connection (M, ∇_0) can be extended to maps $\nabla_m : Hom_A(\Omega^{m+1}(A), M) \to Hom_A(\Omega^m(A), M)$ by

$$\nabla_m(f)(\omega) = \nabla_0(f\omega) + (-1)^{m+1} f(d\omega),$$

$\forall f \in Hom_A(\Omega^{m+1}(A), M)$, $\omega \in \Omega^m(A)$.
The vector space $\bigoplus_{m\geq 0} Hom_A(\Omega^m(A), M)$ is a right $\Omega(A)$-module by the action

$$f\omega(\nu) := f(\omega\nu)$$

where $\omega \in \Omega^m(A)$, $f \in Hom_A(\Omega^{m+n}(A), M)$, $\nu \in \Omega^n(A)$.

Non-commutative Hom-connection

Curvature

- The right A-module homomorphism $F := \nabla_0 \circ \nabla_1$ is called the *curvature* of the hom-connection (M, ∇_0)

Non-commutative Hom-connection

Curvature

- The right A-module homomorphism $F := \nabla_0 \circ \nabla_1$ is called the *curvature* of the hom-connection (M, ∇_0)
- (M, ∇_0) is said to be *flat* provided that $F = 0$. We can associate a chain complex $(\bigoplus_{m \geq 0} Hom_A(\Omega^m(A), M), \nabla)$ to a flat hom-connection (M, ∇_0).

Non-commutative Hom-connection

Curvature

- The right A-module homomorphism $F := \nabla_0 \circ \nabla_1$ is called the *curvature* of the hom-connection (M, ∇_0)
- (M, ∇_0) is said to be *flat* provided that $F = 0$. We can associate a chain complex $(\bigoplus_{m \geq 0} Hom_A(\Omega^m(A), M), \nabla)$ to a flat hom-connection (M, ∇_0).
- We set $M = A$ and $\Omega_m^* = Hom_A(\Omega^m(A), A)$ to get the following *complex of integral forms* on A

$$\cdots \xrightarrow{\nabla_3} \Omega_3^* \xrightarrow{\nabla_2} \Omega_2^* \xrightarrow{\nabla_1} \Omega_1^* \xrightarrow{\nabla_0} A$$

Twisted Multi-Derivations and Hom-Connections

Twisted Multi-Derivation

Right Twisted Multi-Derivation

- By a *right twisted multi-derivation* in an algebra A we mean a pair (∂, σ), where $\sigma : A \to M_n(A)$ is an algebra homomorphism and $\partial : A \to A^n$ is a k-linear map such that, for all $a, b \in A$,

$$\partial(ab) = \partial(a)\sigma(b) + a\partial(b).$$

- A^n is understood as an $(A\text{-}M_n(A))$-bimodule. If we write $\sigma(a) = (\sigma_{ij}(a))_{i,j=1}^n$ and $\partial(a) = (\partial_i(a))_{i=1}^n$ for an element $a \in A$, then we obtain the following n equations

$$\partial_i(ab) = \sum_j \partial_j(a)\sigma_{ji}(b) + a\partial_i(b), i = 1, 2, \ldots, n.$$

Twisted Multi-Derivation

Right Twisted Multi-Derivation

Given a right twisted multi-derivation (∂, σ) on A we construct a FODC on the free left A-module

$$\Omega^1 = A^n = \bigoplus_{i=1}^{n} A\omega_i$$

with basis $\omega_1, \ldots, \omega_n$ which becomes an A-bimodule by $\omega_i a = \sum_{j=1}^{n} \sigma_{ij}(a)\omega_j$ for all $1 \leq i \leq n$. The map

$$d : A \to \Omega^1, \qquad a \mapsto \sum_{i=1}^{n} \partial_i(a)\omega_i$$

is a derivation and makes (Ω^1, d) a FODC on A.

Twisted Multi-Derivation

Free Right Twisted Multi-Derivation

- A map $\sigma : A \to M_n(A)$ can be equivalently understood as an element of $M_n(End_k(A))$. We write $\bullet$ for the product in $M_n(End_k(A))$, $\mathbb{I}$ for the unit in $M_n(End_k(A))$ and σ^T for the transpose of σ.

Twisted Multi-Derivation

Free Right Twisted Multi-Derivation

- A map $\sigma : A \to M_n(A)$ can be equivalently understood as an element of $M_n(End_k(A))$. We write $\bullet$ for the product in $M_n(End_k(A))$, $\mathbb{I}$ for the unit in $M_n(End_k(A))$ and σ^T for the transpose of σ.
- We call a right twisted multi-derivation (∂, σ) *free*, provided there exist algebra maps $\bar{\sigma} : A \to M_n(A)$ and $\hat{\sigma} : A \to M_n(A)$ such that

$$\bar{\sigma} \bullet \sigma^T = \mathbb{I}, \qquad \sigma^T \bullet \bar{\sigma} = \mathbb{I},$$
$$\hat{\sigma} \bullet \bar{\sigma}^T = \mathbb{I}, \qquad \bar{\sigma}^T \bullet \hat{\sigma} = \mathbb{I}.$$

  We denote it by $(\partial, \sigma; \bar{\sigma}, \hat{\sigma})$.

Twisted Multi-Derivation

Proposition (Brzezinski,El Kaoutit, Lomp)

An upper-triangular right twisted multi-derivation (∂, σ) is free if and only if $\sigma_{11}, \ldots, \sigma_{nn}$ are automorphisms of A.

Theorem (Brzezinski,El Kaoutit, Lomp)

For any free right twisted multi-derivation $(\partial, \sigma; \bar{\sigma}, \hat{\sigma})$ on A with the induced FODC $(\Omega^1(A), d)$ with generators ω_i, the map

$$\nabla : Hom_A(\Omega^1(A), A) \to A, \qquad f \mapsto \sum_i \partial_i^\sigma \left(f\left(\omega_i\right)\right)$$

is a hom-connection, where $\partial_i^\sigma := \sum_{j,k} \bar{\sigma}_{kj} \circ \partial_j \circ \hat{\sigma}_{ki}$, for each $i = 1, 2, \ldots, n$.

Differential Calculi on Quantum Exterior Algebras

DC On Quantum Exterior Algebras

Quantum Exterior Algebras

- We call an $n \times n$-matrix $Q = (q_{ij})$ over K a *multiplicatively antisymmetric matrix* if $q_{ij}q_{ji} = q_{ii} = 1$ for all i, j.

DC On Quantum Exterior Algebras

Quantum Exterior Algebras

- We call an $n \times n$-matrix $Q = (q_{ij})$ over K a *multiplicatively antisymmetric matrix* if $q_{ij}q_{ji} = q_{ii} = 1$ for all i, j.
- Let M be an A-bimodule which is free as left and right A-module with basis $\{\omega_1, \ldots, \omega_n\}$. The *quantum exterior algebra* of M over A w.r.t. a multiplicatively antisymmetric matrix Q is defined as

$$\bigwedge^Q(M) := T_A(M)/\langle \omega_i \otimes \omega_j + q_{ij}\omega_j \otimes \omega_i, \omega_i \otimes \omega_i \mid i, j = 1, \ldots, n\rangle.$$

DC On Quantum Exterior Algebras

Quantum Exterior Algebras

- The quantum exterior algebra is a free left and right A-module of rank 2^n with basis

$$\{1\} \cup \{\omega_{i_1} \wedge \omega_{i_2} \cdots \wedge \omega_{i_k} \mid i_1 < i_2 < \cdots < i_k,\ 1 \leq k \leq n\}.$$

Question

When a bimodule derivation $d : A \to M$ can be extended to an exterior derivation $d : \bigwedge^Q(M) \to \bigwedge^Q(M)$ of the quantum exterior algebra?

DC On Quantum Exterior Algebras

Proposition

Let (∂, σ) be a right twisted multi-derivation of rank n on a k-algebra A with associated FODC $(\Omega^1(A), d)$. Let Q be an $n \times n$ multiplicatively antisymmetric matrix over k. Then $d : A \to \Omega^1(A)$ can be extended to make $\Omega = \bigwedge^Q(\Omega^1(A))$ an n-dimensional differential calculus on A with $d(\omega_i) = 0$ for all $i = 1, \ldots, n$ if and only if

$$\partial_i \partial_j = q_{ji} \partial_j \partial_i \text{ and } \partial_i \sigma_{kj} - q_{ji} \partial_j \sigma_{ki} = q_{ji} \sigma_{kj} \partial_i - \sigma_{ki} \partial_j, \ \forall i < j, \ \forall k.$$

DC On Quantum Exterior Algebras

Theorem(Karaçuha, Lomp)

Let (∂, σ) be a free upper triangular twisted multi-derivation on A with associated FODC (Ω^1, d). Suppose that $d : A \to \Omega^1$ can be extended to an n-dimensional differential calculus (Ω, d) where $\Omega = \bigwedge^Q(\Omega^1)$ is the quantum exterior algebra of Ω^1 for some matrix Q. Then the following hold:

1. $\overline{\omega} a = \det(\sigma)\overline{\omega}$,for all $a \in A$, where $\det \sigma = \sigma_{11} \circ \cdots \circ \sigma_{nn}$.
2. The maps $\Theta_m : \Omega^m \to Hom_A(\Omega^{n-m}(A), A)$ given by $\Theta_m(v) = (-1)^{m(n-1)}\beta v$ for all $v \in \Omega^m$ are isomorphisms of right A-modules.
3. Moreover if

DC On Quantum Exterior Algebras

Theorem(Karaçuha, Lomp)

$$\partial_i^\sigma = \left(\prod_j q_{ij}\right) \det(\sigma)^{-1} \partial_i \det(\sigma) \qquad \forall i = 1, \ldots, n$$

holds, then $\Theta = (\Theta_m)_{m=0}^n$ is a chain map, that is, the following diagram commutes:

$$\begin{array}{ccccccccc}
A & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{n-1} & \xrightarrow{d} & \Omega^n \\
\Theta_0 \downarrow & & \Theta_1 \downarrow & & & & \Theta_{n-1} \downarrow & & \Theta_n \downarrow \\
\Omega_n^* & \xrightarrow[\nabla_{n-1}]{} & \Omega_{n-1}^* & \xrightarrow[\nabla_{n-2}]{} & \cdots & \xrightarrow[\nabla_1]{} & \Omega_1^* & \xrightarrow[\nabla_0]{} & A
\end{array}$$

Multivariate Quantum Polynomials

Multivariate Quantum Polynomials

Skew Derivations

- We have a diagonal bimodule structure on $\Omega^1 = A^n$ if $\sigma_{ij} = \delta_{ij}\sigma_i$ for all i, j where $\sigma_1, \dots, \sigma_n$ are endomorphisms of A. Moreover if σ is diagonal and (∂, σ) is a right twisted multi-derivation on A, then the maps ∂_i, for all $a, b \in A$ and i, satisfy

$$\partial_i(ab) = \partial_i(a)\sigma_i(b) + a\partial_i(b)$$

which are then called right σ_i-*skew derivations.*

Multivariate Quantum Polynomials

Skew Derivations

- Conversely, given any right σ_i-derivations ∂_i on A, for $i = 1, \ldots, n$ one can form a corresponding diagonal twisted multi-derivation (∂, σ) on A. Such *diagonal* twisted multi-derivation (∂, σ) is free if and only if the maps $\sigma_1, \ldots, \sigma_n$ are automorphisms. The associated A-bimodule structure on $\Omega^1 = A^n$ with left A-basis $\omega_1, \ldots, \omega_n$ is given by

$$\omega_i a = \sigma_i(a)\omega_i$$

for all i and $a \in A$.

Multivariate Quantum Polynomials

Corollary

Let A be an algebra over a field K, σ_i automorphisms and ∂_i right σ_i-skew derivations on A, for $i = 1, \ldots, n$ and let (Ω^1, d) be the associated FODC on A.

1. The derivation $d : A \to \Omega^1$ extends to an n-dimensional differential calculus (Ω, d) where $\Omega = \bigwedge^Q(\Omega^1)$ is the quantum exterior algebra with respect to some Q such that $d(\omega_i) = 0$ for all $i = 1, \ldots, n$ if and only if

$$\partial_i\sigma_j = q_{ji}\sigma_j\partial_i \qquad \text{and} \qquad \partial_i\partial_j = q_{ji}\partial_j\partial_i \qquad \forall i < j.$$

2. If $\partial_i\sigma_j = q_{ji}\sigma_j\partial_i$ for all i, j and $\partial_i\partial_j = q_{ji}\partial_j\partial_i$ for all $i < j$, then the de Rham and the integral complexes on A are isomorphic relative to (Ω, d).

Multivariate Quantum Polynomials

Quantum Polynomial Algebra

- $Q = (q_{ij})$ is a $n \times n$ multiplicatively antisymmetric matrix over a field k. The multivariate quantum polynomial algebra with respect to Q is defined as:

$$A = \mathcal{O}_Q(k^n) := k\langle x_1, \ldots, x_n\rangle / \langle x_i x_j - q_{ij} x_j x_i \mid 1 \leq i, j \leq n\rangle.$$

- For two generic monomials x^α and x^β with $\alpha, \beta \in \mathbb{N}^n$ one has

$$x^\alpha x^\beta = \left(\prod_{1 \leq j < i \leq n} q_{ij}^{\alpha_i \beta_j} \right) x^{\alpha+\beta} = \mu(\alpha, \beta) x^{\alpha+\beta},$$

where $\mu(\alpha, \beta) = \prod_{1 \leq j < i \leq n} q_{ij}^{\alpha_i \beta_j}$.

Multivariate Quantum Polynomials

Quantum Polynomial Algebra

- We define automorphisms $\sigma_1, \ldots, \sigma_n$ and right σ_i-derivations of A as follows: For a generic monomial x^α with $\alpha \in \mathbb{N}^n$ one sets

$$\sigma_i(x^\alpha) := \lambda_i(\alpha) x^\alpha \qquad \text{and} \qquad \partial_i(x^\alpha) := \alpha_i \delta_i(\alpha) x^{\alpha - \epsilon^i}$$

where $\lambda_i(\alpha) = \prod_{j=1}^n q_{ij}^{\alpha_j}$, $\delta_i(\alpha) = \prod_{i<j} q_{ij}^{\alpha_j}$ and $\epsilon^i \in \mathbb{N}^n$ such that $\epsilon^i_j = \delta_{ij}$.
Then by the previous Corollary we get

Multivariate Quantum Polynomials

Corollary

Let $A = \mathcal{O}_Q(K^n)$ be the multivariate quantum polynomial algebra and let $\Omega = \bigwedge^Q(\Omega^1)$ be the associated quantum exterior algebra. Then the derivation $d : A \to \Omega^1$ with $d(x^\alpha) = \sum_{i=1}^n \partial_i(x^\alpha)\omega_i$ makes Ω into a differential calculus such that the de Rham complex and the integral complex are isomorphic.

Manin's Quantum n-space

Coordinate Ring of Quantum n-space

Manin's Quantum n-space

Let $q \in k \setminus \{0\}$. For the matrix $Q = (q_{ij})$ with $q_{ij} = q$ and $q_{ji} = q^{-1}$ for all $i < j$ and $q_{ii} = 1$, the algebra $\mathcal{O}_Q(k^n)$ is called the *coordinate ring of quantum n-space* or *Manin's quantum n-space* and will be denoted by $A = k_q[x_1, \ldots, x_n]$. We have the following defining relations of the algebra A

$$x_i x_j = q x_j x_i, \qquad i < j.$$

Coordinate Ring of Quantum n-space

Manin's Quantum n-space

For $\alpha \in \mathbb{N}^n$ and $1 \leq i \leq n$ we have:

$$\lambda_i(\alpha) x^\alpha x_i \;=\; x^{\alpha+\epsilon^i} \;=\; \overline{\lambda}_i(\alpha) x_i x^\alpha,$$

where

$$\lambda_i(\alpha) = \prod_{i<j} q^{\alpha_j} \qquad \text{and} \qquad \overline{\lambda}_i(\alpha) = \prod_{j<i} q^{-\alpha_j}.$$

More generally

$$x^{\alpha+\beta} = \left(\prod_{j=1}^{n-1} \lambda_j(\alpha)^{\beta_j} \right) x^\alpha x^\beta = \prod_{1 \leq s < j \leq n} q^{\alpha_s \beta_j} x^\alpha x^\beta$$

Coordinate Ring of Quantum n-space

An FODC On Manin's Quantum n-space

We take the following two-parameter first order differential calculus Ω^1 which is freely generated by $\{\omega_1, \ldots \omega_n\}$ over A subject to the relations

$$\omega_i x_j = q x_j \omega_i + (p-1) x_i \omega_j, \qquad i < j,$$

$$\omega_i x_i = p x_i \omega_i,$$

$$\omega_j x_i = pq^{-1} x_i \omega_j, \qquad i < j.$$

Set $\pi_i(\alpha) = \prod_{s<i} p^{\alpha_s}$, $i = 1, \ldots, n$ for the following lemma.

Coordinate Ring of Quantum n-space

Lemma

For $\alpha \in \mathbb{N}^n$ the entries of the matrix $\sigma(x^\alpha)$ are as follows
$\sigma_{ij}(x^\alpha) = 0$ for $i > j$ and

$$\sigma_{ij}(x^\alpha) = \eta_{ij}(\alpha) x^{\alpha+\epsilon^i-\epsilon^j};$$

$$\eta_{ij}(\alpha) = \begin{cases} \pi_j(\alpha)\overline{\lambda}_i(\alpha)\lambda_j(\alpha)(p^{\alpha_j}-1) & \text{for} \quad i<j, \\ \pi_i(\alpha)\overline{\lambda}_i(\alpha)\lambda_i(\alpha)p^{\alpha_i} & \text{for} \quad i=j. \end{cases}$$

Coordinate Ring of Quantum n-space

The Derivation

We have a derivation $d : K_q[x_1, \ldots, x_n] \to \Omega^1$ such that $d(x_i) = \omega_i$ for all i. For any $\alpha \in \mathbb{N}^n$ we set $d(x^\alpha) = \sum_{i=1}^n \partial_i(x^\alpha)\omega_i$ where

$$\partial_i(x^\alpha) = \delta_i(\alpha)x^{\alpha-\epsilon^i}\,;\, \delta_i(\alpha) = \pi_i(\alpha)\lambda_i(\alpha)\frac{p^{\alpha_i}-1}{p-1}.$$

for all $i = 1, \ldots, n$. Also for i, k we have:

$$\delta_i(\alpha) = q^{\mp 1}\delta_i(\alpha \pm \epsilon^k), \quad \text{if } i < k\,;\, \delta_i(\alpha) = p^{\mp 1}\delta_i(\alpha \pm \epsilon^k), \quad \text{if } i > k.$$

Coordinate Ring of Quantum n-space

Lemma

The pair (∂, σ) is a right twisted multi-derivation of $K_q[x_1, \ldots, x_n]$ satisfying the equations ensuring the extension of the FODC to make $\Omega = \bigwedge^Q(\Omega^1)$ an $n-$dimensional DC with respect to the multiplicatively antisymmetric matrix Q' whose entries are $Q'_{ij} = p^{-1}q$ for $i < j$.

Coordinate Ring of Quantum n-space

Lemma cont.

In particular

$$\partial_i \partial_j = pq^{-1} \partial_j \partial_i, \qquad \forall i < j$$

holds as well as for all i, k, j:

$$\begin{aligned}
\partial_i \sigma_{kj} &= pq^{-1} \sigma_{kj} \partial_i, & i < k \leq j \\
\partial_i \sigma_{kj} &= pq^{-1} \partial_j \sigma_{ki}, & k < i < j \\
\sigma_{ki} \partial_j &= pq^{-1} \sigma_{kj} \partial_i, & k < i < j \\
\partial_i \sigma_{ij} - pq^{-1} \partial_j \sigma_{ii} &= pq^{-1} \sigma_{ij} \partial_i - \sigma_{ii} \partial_j, & i < j
\end{aligned}$$

Coordinate Ring of Quantum n-space

Theorem(Karaçuha,Lomp)

The derivation $d : K_q[x_1, \ldots, x_n] \to \Omega^1$ extends to a differential calculus $\bigwedge^{p^{-1}q}(\Omega^1)$ on $K_q[x_1, \ldots, x_n]$. Furthermore the de Rham and the integral complex associated to the differential calculus $(\bigwedge^{p^{-1}q}(\Omega^1), d)$ are isomorphic.

THANKS!