ALMOST SYMMETRIC NUMERICAL SEMIGROUPS

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In this talk are mentioned notions and results due to Herzog, Kunz, Jäger, Heinzer, Huckaba, Papick, Fröberg, B., ....

We fix for all the section the following notation. $S$ is a numerical semigroup, i.e. a subsemigroup of $\mathbb{N}$, with zero and with finite complement in $\mathbb{N}$. $M = S \setminus \{0\}$ is the maximal ideal of $S$, $e$ is the multiplicity of $S$, that is the smallest positive integer of $S$, $g$ is the Frobenius number of $S$, that is the greatest integer which does not belong to $S$. Moreover we set $n = \text{Card}(\{s \in S \mid s < g\})$.

A relative ideal of $S$ is a nonempty subset $I$ of $\mathbb{Z}$ (which is the quotient group of $S$) such that $I + S \subseteq I$ and $I + s \subseteq S$, for some $s \in S$. A relative ideal which is contained in $S$ is an integral ideal of $S$. 
If $I$, $J$ are relative ideals of $S$, then the following are relative ideals too:

$$I + J = \{ i + j; \ i \in I, j \in J \}$$

$$I - J = \{ z \in \mathbb{Z} \mid z + J \subseteq I \}$$

Moreover the ideal generated by $z_1, \ldots, z_h$ is

$$(z_1 + S) \cup \cdots \cup (z_h + S)$$
We have also

\[ I \subseteq S - (S - I) = \bigcap_{I \subseteq z + S} (z + S) \]

and if the equality holds \( I \) is \textit{bidual}. More generally

\[ I \subseteq J - (J - I) = \bigcap_{I \subseteq z + J} (z + J) \]

If \( I \) is a relative ideal of \( S \), then \( \text{Ap}_e(I) = \text{Ap}(I) = I \setminus (e + I) \)
is the set of the \( e \) smallest elements in \( I \) in the \( e \) congruence classes mod \( e \) and is called the \textit{Apery set} of \( I \) (with respect to \( e \)). In particular \( \text{Ap}(S) \) is the Apery set of \( S \). Since \( g \) is the greatest gap of \( S \), \( g + e \) is the largest element in \( \text{Ap}(S) \).
The following Lemma, corresponding to Nakayama’s Lemma for local rings, is very easy to prove for numerical semigroups:

**Lemma 1** If \( I \) is a relative ideal of \( S \), then the unique minimal set of generators of \( I \) is \( I \setminus (M + I) \).

Since \( e + I \subseteq M + I \), then \( I \setminus (M + I) \subseteq I \setminus (e + I) = \text{Ap}_e(I) \) and by Lemma 1 each relative ideal \( I \) of \( S \) needs at most \( e \) generators.

In particular \( M \setminus 2M \) is the minimal set of generators of \( M \) and its cardinality is the **embedding dimension** \( \nu \) of \( S \). We have \( \nu \leq e \) and if equality holds the semigroup \( S \) is called of **maximal embedding dimension**.
Problem The following problem was asked by Wilf and is still unsolved

\[ g + 1 \leq n \nu \]
A particular relative ideal of $S$ plays a special role. It is the canonical ideal $\Omega = \{ g - x \mid x \in \mathbb{Z} \setminus S \}$. So

$$x \notin S \Rightarrow g - x \in \Omega$$

$$x \in S \Rightarrow g - x \notin \Omega$$

Of course it is $S \subseteq \Omega \subseteq \mathbb{N}$ (if $s \in S$ then $x = g - s \notin S$, so $s = g - (g - s) = g - x \in \Omega$).
Proposition 2 (I. Serban) If $A_{p_e}(S) = \{p_0, \ldots, p_{e-1}\}$, then $A_{p_e}(\Omega) = \{p_{e-1} - p_{e-1}, p_{e-1} - p_{e-2} \ldots, p_{e-1} - p_0\}$.

Proof. Recall that $g + e = p_{e-1}$.

We have $p_i \in A_{p_e}(S) = S \setminus (e + S)$. Thus

$$p_i \in S \Rightarrow g - p_i \notin \Omega \Rightarrow g + e - p_i = p_{e-1} - p_i \notin \Omega + e$$

On the other hand

$$p_i \notin S + e \Rightarrow p_i - e \notin S \Rightarrow g - (p_i + e) = g + e - p_i \in \Omega$$
Proposition 3 1) $\Omega - (\Omega - I) = I$, for each ideal $I$ of $S$.

In particular $\Omega - (\Omega - S) = \Omega - \Omega = S$

2) $\Omega - (I \cap J) = (\Omega - I) \cup (\Omega - J)$

3) $\Omega$ is an irreducible relative ideal, i.e. $\Omega$ is not the intersection of any set of relative ideals properly containing $\Omega$. 
Proof. 2) Since

\[(\Omega - I) \subseteq (\Omega - I) \cup (\Omega - J) = H\]

then

\[(\Omega - H) \subseteq \Omega - (\Omega - I) = I\]

In the same way

\[(\Omega - H) \subseteq J\]

Thus

\[\Omega - H \subseteq I \cap J\]

hence

\[\Omega - (I \cap J) \subseteq \Omega - (\Omega - H) = H\]

The other inclusion is trivial.
3) If $\Omega = I \cap J$, then

$$S = \Omega - \Omega = \Omega - (I \cap J) = (\Omega - I) \cup (\Omega - J)$$

Thus

$$\Omega - I = S \quad \text{or} \quad \Omega - J = S$$

and so

$$I = \Omega - (\Omega - I) = \Omega - S = \Omega \quad \text{or} \quad J = \Omega - (\Omega - J) = \Omega - S = \Omega$$
The properties 1) and 3) characterize $\Omega$ in the following sense:

**Proposition 4** Let $\Omega'$ be a relative ideal of $S$. Then

$$
\Omega' - (\Omega' - I) = I
$$

for each ideal $I$ of $S$ if and only if $\Omega' = z + \Omega$, for some $z \in \mathbb{Z}$.

**Proof.** Since

$$
\Omega' - (\Omega' - \Omega) = \Omega = \bigcap_{\Omega \subseteq z + \Omega'} (z + \Omega')
$$

and $\Omega$ is irreducible, we have $\Omega = z + \Omega'$. The converse is trivial.
Proposition 5 Let \( \Omega' \) be a relative ideal of \( S \). Then \( \Omega' \) is irreducible if and only if \( \Omega' = z + \Omega \), for some \( z \in \mathbb{Z} \).

Proof. Since

\[
\Omega' = \Omega - (\Omega - \Omega') = \bigcap_{\Omega' \subseteq z + \Omega} (z + \Omega)
\]

and \( \Omega' \) is irreducible, we have \( \Omega' = z + \Omega \). The converse is trivial.
The type $t$ of a numerical semigroup $S$ is the minimal number of generators of the canonical ideal, that is $t = \text{Card}(\Omega \setminus (\Omega + M))$.

**Proposition 6** For the type $t$ of $S$ the following holds:

1. $t = \text{Card}((S - M) \setminus S)$

2. $S = (h_1 + \Omega) \cap \cdots \cap (h_t + \Omega)$, for some $h_i \in \mathbb{Z}$, where the intersection is irredundant.

3. Each relative principal ideal of $S$ is an irredundant intersection of $t$ irreducible relative ideals.
4. Each relative ideal is a finite intersection of irreducible relative ideals.

5. \(1 \leq t \leq e - 1\).

Proof. 1. \(t = \text{Card}(\Omega \setminus (\Omega + M))\)

\[= \text{Card}(\Omega - (\Omega + M) \setminus (\Omega - \Omega))\]

\[= \text{Card}((\Omega - \Omega) - M \setminus S) = \text{Card}((S - M) \setminus S)\]

2. Suppose that \(\Omega\) is minimally generated by \(z_1, \ldots, z_t\). Then

\[\Omega - S = \Omega = \bigcup_{z_i = 1}^{t} (z_i + S) = \bigcup_{z_i = 1}^{t} (z_i + (\Omega - \Omega))\]
\[ = \bigcup_{z_i=1}^{t} (\Omega - (\Omega - z_i)) = \Omega - \bigcap_{z_i=1}^{t} (\Omega - z_i) \]

So that, dualizing again

\[ S = (\Omega - (\Omega - S)) = \bigcap_{z_i=1}^{t} (\Omega - z_i) \]

Moreover the intersection is irredudant.

3. Easy by 2.
**Fact** The type $t$ of a semigroup $S$ is at the same time the number of components of an irredundant intersection of a principal ideal in irreducible relative ideals and in irreducible integral ideals.

The two notions are really different, in fact:

$I$ irreducible as relative id. $\Rightarrow I$ irreducible as integral id.

but the converse does not hold. There are in general many irreducible integral ideals. On the contrary, as we saw, there is essentially only one irreducible relative ideal, the canonical ideal $\Omega$ and all those of the form $x + \Omega$, $x \in \mathbb{Z}$. 
Example Let $S = \langle 3, 8, 10 \rangle \Rightarrow \{0, 3, 6, 8, \rightarrow\}$. The canonical ideal is

$$\Omega = \{0, 2, 3, 5, 6, 8, \rightarrow\}$$

So each irreducible relative ideal of $S$ is of the form $x + \Omega$. On the other hand, consider the partial order on $S$ given by

$$s_1 \leq s_2 \iff s_1 + s_3 = s_2, \text{ for some } s_3 \in S$$

and, if $s \in S$, set $B(s) = \{z \in S \mid z \leq s\}$. It turns out (cf. R., G.S., G.G.) that $I = S \setminus B(s)$ is an irreducible integral semigroup ideal of $S$ (in fact each integral semigroup ideal containing $I$ contains $s$) and is not necessarily of the form $x + \Omega$. For example, for $s = 12$, we get

$$I = \{8, 10, 11, 13, \rightarrow\}$$
If \( t = 1 \) or equivalently \( \Omega = S \), then the numerical semigroup \( S \) is classically called \textit{symmetric}.

\( S \) is \textit{almost symmetric} if \( \Omega \subseteq S - M \), equivalently if \( t = g + 2 - 2n \).

\textbf{Example} (see board)

\[ S = \langle 5, 8, 9, 12 \rangle = \{0, 5, 8, 9, 10, 12, \rightarrow\} \]

\[ \Omega = \{0, 4, 5, 7, 8, 9, 10, 12, \rightarrow\} = \langle 0, 4, 7 \rangle \]

\[ S - M = \{0, 4, 5, 7, \rightarrow\} \]
Theorem 7 (B.) The following conditions are equivalent for a numerical semigroup $S$ of maximal ideal $M$:

1. $S$ is almost symmetric.

2. Each ideal of $M - M$ is bidual as ideal of $S$.

3. $M - e$ is the canonical ideal of $M - M$

Proof. If $S = \mathbb{N}$, the three conditions trivially hold. Suppose $S \not\subseteq \mathbb{N}$, so that $M - M = S - M$ is a semigroup which properly contains $S$. 

2. $\Rightarrow$ 3. We have to prove that $M - (M - I) \subseteq I$, for each ideal $I$ of $M - M$. First notice that

$$M - I = S - I$$

If $M - I \subsetneq S - I$, then $z + I \subseteq S$ and $z + I \not\subseteq M$ for some $z \in \mathbb{Z}$ thus $z + I = S$ and $S$ is an ideal of $M - M$, a contradiction. Thus

$$M - (M - I) \subseteq S - (M - I) = S - (S - I) = I$$

3. $\Rightarrow$ 2. Each relative ideal $I$ of $M - M$ is a finite intersection of irreducible relative ideals, i.e. of ideals of the form $x + M$, because $M - e$ is the canonical ideal of $M - M$. Since $M$ is bidual in $S$ (in fact $S - (S - M) \subsetneq S$ is equal to $M$) also $I$, as intersection of bidual ideals, is bidual.
Theorem 8 (M. La Valle) If $S$ is an almost symmetric semigroup, then

$$g + 1 \leq n\nu$$

Proof. Since $S$ is almost symmetric, $t = g + 2 - 2n$, thus $n = \frac{g-t}{2} + 1$. Thus the Wilf inequality becomes

$$g + 1 \leq \left(\frac{g-t}{2} + 1\right)\nu$$

or equivalently

$$2 + \nu t \leq (\nu - 2)g + 2\nu$$

It is enough to prove

$$\nu t \leq (\nu - 2)g$$
equivalently
\[ \nu(g - t) \geq 2g \]
Since \( t < e \), it is enough to prove
\[ \nu(g - e) \geq 2g \] that is \( \nu \geq \frac{2g}{g - e} \)

Case 1. If \( e < g/2 \), then \( g - e > g/2 \) and
\[ \frac{2g}{g - e} < \frac{2g}{g/2} = 4 \]
Since, for \( \nu \leq 3 \), Wilf’s inequality is true, in this case the theorem is proved.

Case 2. if \( e = g + 1 \), it is easy: \( \nu = e \) and \( n = 1 \) thus \( g + 1 = e \leq 1e \)
Case 3. If $g/2 < e < g$, we have to compute the number $\nu$ of (necessary) generators of $S$.

We have at least:

$n - 1$ generators smaller than $g$

$2e - (g + 1)$ generators between $g + 1$ and $2e$

$(e + s) - 2e - 1$ generators between $2e$ and $e + s$, where $S = \{s_0 = 0, s_1 = e, s_2 = s, \ldots (s_i < s_{i+1})\}$. Therefore:

$$\nu \geq (n - 1) + (2e - g - 1) + (s - e - 1) = n + e + s - g - 3$$

and

$$\nu n \geq n^2 + ne + ns - ng - 3n = f(n)$$
We claim that
\[ f(n) \geq g + 1 \]

By induction on \( n \geq 3 \). For \( n = 3 \), it is
\[ f(n) = 3(e + s - g) \]

But for \( n = 3 \) an almost symmetric semigroup is of one of the following forms:
\[ S = \{0, g - 3, g - 1, g + 1, \rightarrow\} \]
\[ S = \{0, g - 2, g - 1, g + 1, \rightarrow\} \]

In the first case
\[ f(3) = 3g - 12 \geq g + 1 \iff g \geq 7 \]
In the second case

\[ f(3) = 3g - 9 \geq g + 1 \iff g \geq 5 \]

Thus, taking care of a finite small number of easy cases (\( g < 7 \) in the first case and \( g < 5 \) in the second case) the inductive hypothesis is verified.

Now the inductive step:

\[ f(n + 1) - f(n) = (2n - 2) + (s + e - g) > 0 \]

So if \( f(n) > g + 1 \), then also \( f(n + 1) > g + 1 \).