Proportionally modular Diophantine inequalities and proportionally modular numerical semigroups

A talk based on a joint work with J.C. Rosales

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Proportionally modular Diophantine inequalities

\[ ax \mod b \leq cx \]

\(a, b, c\) positive integers.

- \(a\): factor of the inequality
- \(b\): modulus of the inequality
- \(c\): proportion of the inequality

Set of solutions: \(\{x \in \mathbb{Z} : ax \mod b \leq cx\}\).

(Non-negative solutions)

Equivalence: Two proportionally modular Diophantine inequalities are equivalent if they have the same set of solutions.
System of proportionally modular Diophantine inequalities

\[
\begin{align*}
a_1 x \mod b_1 & \leq c_1 x \\
\vdots \\
 a_m x \mod b_m & \leq c_m x
\end{align*}
\]

\(a_i, b_i, c_i (1 \leq i \leq n)\) positive integers.

Set of solutions  \(\left\{\right\}\)  Analogous definitions

Equivalence
**Problem:** To get an equivalent system in which the value for all \( b_i \) is the same one, which in addition is a prime integer.

**Answer:** Yes

**Tool:** Theory of proportionally modular numerical semigroups
- Intervals (Submonoids generated by closed intervals)

**Gift:** New characterization of numerical semigroups with a Toms’ decomposition.

**Tool:** Theory of proportionally modular numerical semigroups
- Quotients of numerical semigroups
Proportionally modular numerical semigroup

\[ S(a, b, c) = \{ x \in \mathbb{Z} : ax \mod b \leq cx \} \]

\( S(a, b, c) \) is a numerical semigroup.

Lemma.

1. \( S(a, b, c) = S(a \mod b, b, c) \).
2. If \( c \geq a \) then \( S(a, b, c) = \mathbb{N} \).

We reduce our study to the case \( 1 \leq c < a < b \).
Submonoids generated by closed intervals

Let $T$ be the submonoid of $(\mathbb{R}^+, +)$ generated by the closed interval $[\alpha, \beta]$, where $\alpha, \beta \in \mathbb{R}$ and $0 \leq \alpha < \beta$.

$$S([\alpha, \beta]) = T \cap \mathbb{N}$$

$S([\alpha, \beta])$ is a numerical semigroup.

**Lemma (R-GS-GG-UB'03).** Let $\alpha, \beta \in \mathbb{R}$ be such that $0 < \alpha < \beta$, and let $x$ be a positive integer. Then $x \in S([\alpha, \beta])$ if and only if there exists a positive integer $k$ such that $\frac{x}{k} \in [\alpha, \beta]$. Therefore, $x \notin S([\alpha, \beta])$ if and only if there exists an integer $n$ for which $\frac{x}{n+1} < \alpha < \beta < \frac{x}{n}$. 
Lemma (R-GS-GG-UB’03). Let $a, b$ and $c$ be positive integers such that $c < a < b$. Then $S(a, b, c) = S\left(\left[ \frac{b}{a}, \frac{b}{a-c} \right]\right)$.

Conversely, if $a_1, a_2, b_1$ and $b_2$ are positive integers such that $\frac{b_1}{a_1} < \frac{b_2}{a_2}$, then $S\left(\left[ \frac{b_1}{a_1}, \frac{b_2}{a_2} \right]\right) = S(a_1 b_2, b_1 b_2, a_1 b_2 - a_2 b_1)$.

**Problem:** To change the interval $I = \left[ \frac{b}{a}, \frac{b}{a-c} \right]$ without changing the semigroup.

**Example.** Let $n \in \mathbb{N} \setminus \{0\}$.

- $n \in S([n, n + 1]) = S\left(\left[ \frac{n(n+1)}{n+1}, \frac{n(n+1)}{(n+1)-1} \right]\right)$
- $n \notin S([n + \delta, n + 1 - \delta]), \forall \delta \in (0, \frac{1}{2})$. 
Lemma. Let $I = [\alpha, \beta]$ be given with $\alpha, \beta \in \mathbb{R}^+$. There exists $\varepsilon > 0$ such that $S([\alpha, \beta]) = S([\alpha - \delta, \beta + \delta])$ for every $\delta \in [0, \varepsilon)$.

Proof.

$(\subseteq)$ $I \subseteq J \Rightarrow S(I) \subseteq S(J)$.

$(\supseteq)$
- $\exists x \in \mathbb{N} \setminus \{0\}$ s.t. $x \in S([\alpha - \frac{\varepsilon}{n+2}, \beta + \frac{\varepsilon}{n+2}]) \forall n \in \mathbb{N}$.
- $\forall n \in \mathbb{N} \exists k_n \in \mathbb{N} \setminus \{0\}$ s.t. $\alpha - \frac{\varepsilon}{n+2} < \frac{x}{k_n} < \beta + \frac{\varepsilon}{n+2}$.
  - $\alpha - \frac{\varepsilon}{n+2} \geq \frac{\alpha}{2} \Rightarrow k_n \leq \lceil \frac{2x}{\alpha} \rceil$.
  - $k_n = k \forall n$ (by passing to a subsequence).
- $\alpha - \frac{\varepsilon}{n+2} \leq \frac{x}{k} \leq \beta + \frac{\varepsilon}{n+2}, \forall n \in \mathbb{N}$.
- $x \in S([\alpha, \beta])$.

* $\forall x \notin S([\alpha, \beta]) \exists \varepsilon_x > 0$ s.t. $x \notin S([\alpha - \delta, \beta + \delta]) \forall \delta \in [0, \varepsilon_x)$.

* $\varepsilon = \min \{\varepsilon_x \mid x \in \mathbb{N} \setminus S([\alpha, \beta])\}$. 
Lemma. Let $a, b$ and $c$ be positive integers such that $c < a < b$. Then
\[
S(a, b, c) = S\left(\left(\frac{(b-a)b+1}{(b-a)a+1}, \frac{(a-c)b+1}{(a-c)^2}\right)\right).
\]

Corollary. Let $a, b$ and $c$ be positive integers such that $c < a < b$. Let $\alpha, \beta \in \mathbb{R}$ be such that
\[
\frac{(b-a)b+1}{(b-a)a+1} \leq \alpha \leq \frac{b}{a} \leq \frac{b}{a-c} \leq \beta \leq \frac{(a-c)b+1}{(a-c)^2}.
\]
Then $S(a, b, c) = S([\alpha, \beta])$. 
Proposition. Let $S$ be a proportionally modular numerical semigroup. Then there exists $N \in \mathbb{N} \setminus \{0\}$ such that for every $n \in \mathbb{N}$, $n \geq N$, $S$ is the set of integer solutions of a proportionally modular Diophantine inequality with modulus $n$.

Proof.

* $S = \mathbb{N} \Rightarrow S = S(1, n, 1)$, $\forall n \in \mathbb{N}$, $n \geq 1$.
* $S \neq \mathbb{N} \Rightarrow \exists a, b \ (1 < a < b)$ s.t. $S = S([a, b])$.
  \begin{itemize}
  \item $\exists \epsilon > 0$ s.t. $S = S([a - \epsilon, b + \epsilon])$ and $1 < a - \epsilon$.
  \item $\exists N \in \mathbb{N} \setminus \{0\}$ such that 
    $n \geq N \Rightarrow \exists a_1, a_2 \in \mathbb{N}$ s.t. $a - \frac{\epsilon}{2} \leq \frac{n}{a_1} \leq a$, $b \leq \frac{n}{a_2} \leq b + \frac{\epsilon}{2}$.
  \item $S([a, b]) \subseteq S \left( \left[ \frac{n}{a_1}, \frac{n}{a_2} \right] \right) \subseteq S \left( \left[ a - \frac{\epsilon}{2}, b + \frac{\epsilon}{2} \right] \right) = S([a, b])$.
  \item $a_2 < a_1 < n \Rightarrow S = S \left( \left[ \frac{n}{a_1}, \frac{n}{a_2} \right] \right) = S(a_1, n, a_1 - a_2)$.
  \end{itemize}
**Corollary.** Let $S$ be a proportionally modular numerical semigroup. Then there exist $a, b, c \in \mathbb{N} \setminus \{0\}$ such that $c < a < b$, $b$ is a prime number, and $S = S(a, b, c)$. 
**System of proportionally modular Diophantine inequalities**

\[ a_1 x \mod b_1 \leq c_1 x \]
\[ \vdots \]
\[ a_n x \mod b_n \leq c_n x \]

**Remark.** The set of nonnegative integer solutions is the numerical semigroup \( S = \bigcap_{i=1}^{n} S(a_i, b_i, c_i) \).

**Remark.** If \( A = \prod_{i=1}^{n} a_i \), \( B = \prod_{i=1}^{n} b_i \) and \( C = \prod_{i=1}^{n} c_i \), then
\[ S(a_i, b_i, c_i) = S \left( A, \frac{A b_i}{a_i}, \frac{A c_i}{a_i} \right) = S \left( \frac{B a_i}{B_i}, B, \frac{B c_i}{b_i} \right) = S \left( \frac{C a_i}{c_i}, \frac{C b_i}{c_i}, C \right). \]

**Consequence.** There exists an equivalent system with the same factor (modulus or proportion) for all the inequalities in the system.
**Corollary.** Every system of proportionally modular Diophantine inequalities is equivalent to a system of proportionally modular Diophantine inequalities in which all the inequalities have the same modulus, which in addition is a prime integer.
Quotients of numerical semigroups

\[ \frac{S}{d} = \{ x \in \mathbb{N} \mid dx \in S \} \]

\( \frac{S}{d} \) is a numerical semigroup.

**Lemma (R-UB'06).** Let \( n_1, n_2, d \in \mathbb{N} \setminus \{0\} \) such that \( n_1, n_2 \) are relatively primes. Then \( \frac{\langle n_1, n_2 \rangle}{d} \) is a proportionally modular numerical semigroup. Conversely, every proportionally modular numerical semigroup can be represented in this form.

**Lemma (R-GS'08).** Let \( a_1, a_2, b_1, b_2 \in \mathbb{N} \setminus \{0\} \) be such that \( 1 < \frac{b_1}{a_1} < \frac{b_2}{a_2} \). If \( \gcd\{b_1, b_2\} = 1 \), then

\[
S \left( \left[ \begin{array}{c} b_1 \\ a_1 \\ b_2 \\ a_2 \end{array} \right] \right) = \frac{\langle b_1, b_2 \rangle}{a_1 b_2 - a_2 b_1}.
\]
Proposition. Let $S$ be a proportionally modular numerical semigroup. Then there exists $N \in \mathbb{N} \setminus \{0\}$ such that for every $n \in \mathbb{N}$, $n \geq N$, we can represent $S$ as a quotient of $\langle n, n+1 \rangle$ by a positive integer.

Proof.

* $S = \mathbb{N} \Rightarrow S = \frac{\langle n, n+1 \rangle}{n}$, $\forall n \in \mathbb{N}$, $n \geq 1$.

* $S \neq \mathbb{N} \Rightarrow \exists a, b$ $(1 < a < b)$ s.t. $S = S([a, b])$.
  
  • $\exists \varepsilon > 0$ s.t. $S = S([a - \varepsilon, b + \varepsilon])$ and $1 < a - \varepsilon$.
  
  • $\exists N \in \mathbb{N} \setminus \{0\}$ such that
    
    $n \geq N \Rightarrow \exists a_1, a_2 \in \mathbb{N}$ s.t. $a - \frac{\varepsilon}{2} \leq \frac{n}{a_1} \leq a$, $b \leq \frac{n+1}{a_2} \leq b + \frac{\varepsilon}{2}$.
  
  • $S([a, b]) \subseteq S\left(\left[\frac{n}{a_1}, \frac{n+1}{a_2}\right]\right) \subseteq S\left(\left[a - \frac{\varepsilon}{2}, b + \frac{\varepsilon}{2}\right]\right) = S([a, b])$.
  
  • $1 < \frac{n}{a_1} \Rightarrow S = S\left(\left[\frac{n}{a_1}, \frac{n+1}{a_2}\right]\right) = \frac{\langle n, n+1 \rangle}{(n+1)a_1 - na_2}$.
Corollary. Let $a, d \in \mathbb{N} \setminus \{0\}$. Then $\langle \frac{a}{d}, \frac{a+1}{d} \rangle$ is a proportionally modular numerical semigroup. Conversely, every proportionally modular numerical semigroup can be represented in this form.
**Toms’ decompositions**

**Definition** *(Toms’03).* A numerical semigroup $S$ has a Toms’ decomposition if there exist $q_1, \ldots, q_n, m_1, \ldots, m_n, L \in \mathbb{N} \setminus \{0\}$ fulfilling that

1. $\gcd\{q_i, m_i\} = \gcd\{L, q_i\} = \gcd\{L, m_i\} = 1$, for all $i \in \{1, \ldots, n\},$

2. $S = \frac{\langle q_1, m_1 \rangle}{L} \cap \cdots \cap \frac{\langle q_n, m_n \rangle}{L}.$

**Theorem** *(R-GS’08).* Every system proportionally modular numerical semigroup admits a Toms’ decomposition.

**Lemma** *(R-GS-GG-UB’03).* Let $n_1, n_2, u, v \in \mathbb{N} \setminus \{0\}$ such that $un_2 - vn_1 = 1$. Then

$$\langle n_1, n_2 \rangle = \{x \in \mathbb{N} \mid un_2 x \mod n_1 n_2 \leq x\}.$$
Corollary. Let \( a, d_1, \ldots, d_n \in \mathbb{N} \setminus \{0\} \). Then

\[
S = \{ x \in \mathbb{N} \mid \{d_1 x, \ldots, d_n x\} \subseteq \langle a, a + 1 \rangle \}
\]

is a numerical semigroup having a Toms’ decomposition. Conversely, every numerical semigroup having a Toms’ decomposition can be represented in this form.

Corollary. Let \( a, d_1, \ldots, d_n \in \mathbb{N} \setminus \{0\} \). Then

\[
S = \left\{ x \in \mathbb{N} \mid \begin{array}{c}
(a + 1)d_1 x \mod a(a + 1) \leq d_1 x, \\
\vdots \\
(a + 1)d_n x \mod a(a + 1) \leq d_n x.
\end{array} \right\}
\]

is a numerical semigroup having a Toms’ decomposition. Conversely, every numerical semigroup having a Toms’ decomposition can be represented in this form.


