Invariant Theory and Reversible-Equivariant Vector Fields

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Abstract

In this paper we obtain results for the systematic study of reversible-equivariant vector fields – namely, in the simultaneous presence of symmetries and reversing symmetries – by employing algebraic techniques from invariant theory for compact Lie groups. We introduce the Hilbert-Poincaré series and their associated Molien formula and we prove the character formulas for the computation of dimensions of spaces of homogeneous anti-invariant polynomial functions and reversible-equivariant polynomial mappings. Two symbolic algorithms are also obtained,

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one for the computation of generators for the module of anti-invariant polynomial functions and the other for the module of reversible-equivariant polynomial mappings both over the ring of invariant polynomials. We show that this computation can be obtained directly from a well-known situation, namely from generators of the ring of invariants and the module of the equivariants.

1 Introduction

Many natural phenomena possess symmetry properties. In particular, symmetries and reversing symmetries can occur simultaneously in many dynamical systems physically motivated. When this occurrence is taken into account in the mathematical formulation, it can simplify significantly the interpretation of such phenomena. The conventional notion of presence of symmetries (equivariances) and reversing symmetries in a system of differential equations consists of phase space transformations, including time transformations for reversing symmetries, that leave the equations of motion invariant. The formulation of this situation is given as follows: consider the system

$$\dot{x} = G(x) \tag{1.1}$$

defined on a finite-dimensional vector space V of state variables, where $G : V \to V$ is a smooth vector field. We assume that V carries an action of a compact Lie group Γ together with a distinguished normal subgroup Γ_+ of index 2 such that system (1.1) transforms under the action of Γ on $\mathbf{R} \times V$ by $(t, x) \mapsto (t, \gamma x)$ if $\gamma \in \Gamma_+$ and by $(t, x) \mapsto (-t, \gamma x)$ if $\gamma \in \Gamma \setminus \Gamma_+$. In other words, $\gamma \in \Gamma_+$ maps trajectories onto trajectories of (1.1) with the direction of time being preserved, while $\gamma \in \Gamma \setminus \Gamma_+$ maps trajectories onto trajectories of (1.1) with the direction of time being reversed. Dynamical systems with such property are called *reversible-equivariant systems* and Γ is called the *reversing symmetry group* of the ordinary differential equation (1.1). The elements of the subgroup Γ_+ act as *spatial symmetries* or simply *symmetries* and the elements of the subset $\Gamma_- = \Gamma \setminus \Gamma_+$ act as *time-reversing symmetries* or simply *reversing symmetries*.

The effect of symmetries in local and global dynamics has been investigated for decades by a great number of authors in many papers. In particular, if a system of type (1.1) is provided with an external bifurcation parameter, then it is well known that symmetry properties of such a system affect the genericity of the occurrence of local bifurcations. The first systematic treatment of symmetries in bifurcation theory has been given by Golubitsky, Stewart and Schaeffer [14], where singularity theory in combination with group representation theory has provided a powerful approach to a wide range of situations. Since then, the subject has become an important area of research in dynamical systems and its applications. The importance of reversing symmetries in dynamics was first recognized in the context of Hamiltonian mechanics, going back to the works of Birkhoff [5] on the 3-body problem and DeVogelaere [9] on periodic motions in conservative systems. More recently, Devaney [8], Arnol'd [2] and Sevryuk [22] have renewed the interest on the concept of reversibility in dynamical systems. See Lamb and Roberts [18] for a historical survey and comprehensive bibliography. The interest on reversible-equivariant bifurcations is much more recent. As it turns out, the first impulse to the subject came from the classification of the reversible-equivariant linear systems in terms of group representation theory (see Lamb and Roberts [19]). An approach based on singularity theory, in the same lines as Golubitsky et al. [14], has been developed by Baptistelli and Manoel [4] for the analysis of the so called "self-dual" case. In another direction, Buono et al. [7] have employed equivariant transversality in order to tackle the case of "separable bifurcations".

The starting point for local or global analysis of equivariant systems is to find the general form of the vector field $G : V \to V$ in (1.1) that satisfies the requirements of equivariance. The Theorems by Schwarz and Poénaru (see Golubitsky et al. [14]) reduce this task to a purely algebraic problem in Invariant Theory, which, in turn, can be solved by symbolic computation. See Sturmfels [26] for example, or Gatermann [13], where the tools of computational invariant theory are developed with the view towards their applications to equivariant bifurcation theory.

Recall that the space $\vec{\mathcal{P}}_V(\Gamma)$ of all equivariant polynomial mappings on V is a finitely generated module over the ring $\mathcal{P}_V(\Gamma)$ of invariant polynomial functions on V – which is also finitely generated according to the Hilbert-Weyl Theorem (Theorem 2.4). Therefore, the problem of finding the general form of the equivariant vector fields is reduced to the computation of generating sets for $\mathcal{P}_V(\Gamma)$ as a ring and for $\vec{\mathcal{P}}_V(\Gamma)$ as a module. What makes it feasible through computational methods is the simple observation that $\mathcal{P}_V(\Gamma)$ and $\vec{\mathcal{P}}_V(\Gamma)$ are graded algebras by the polynomial degree, that is,

$$\mathcal{P}_V(\Gamma) \;=\; \bigoplus_{d=0}^{\infty} \mathcal{P}_V^d(\Gamma) \qquad \text{and} \qquad \vec{\mathcal{P}}_V(\Gamma) \;=\; \bigoplus_{d=0}^{\infty} \vec{\mathcal{P}}_V^d(\Gamma)\;,$$

where $\mathcal{P}_V^d(\Gamma)$ is the space of homogeneous polynomial invariants of degree d and $\vec{\mathcal{P}}_V^d(\Gamma)$ is the space of equivariant mappings with homogeneous polynomial components of degree d. Since these are finite dimensional vector spaces, generating sets may be found by Linear Algebra together with the Hilbert-Poincaré series and their associated Molien formulas. In fact, the symbolic computation packages GAP [11] and SINGULAR [16] have all these tools implemented in their libraries. In the reversible-equivariant case we face a similar situation. The requirement of G being reversible-equivariant with respect to Γ can be written as

$$G(\gamma x) \;=\; \sigma(\gamma)\gamma\, G(x) \ ,$$

where $\sigma : \Gamma \to \mathbf{Z}_2 = \{\pm 1\}$ is a Lie group homomorphism which is 1 on Γ_+ and -1 on Γ_-. The space $\vec{\mathcal{Q}}_V(\Gamma)$ of reversible-equivariant polynomial mappings is a finitely generated graded module over the ring $\mathcal{P}_V(\Gamma)$ and so the same methods described before could be adapted to work in this context.

In this paper we follow a different approach, based on a link existent between the invariant theory for Γ and for its normal subgroup Γ_+. In order to provide this link, we observe that $\mathcal{P}_V(\Gamma)$ is a subring of $\mathcal{P}_V(\Gamma_+)$ and so this may be regarded as a module over $\mathcal{P}_V(\Gamma)$. Next we introduce the space $\mathcal{Q}_V(\Gamma)$ of *anti-invariant* polynomial functions: a polynomial function $\tilde{f} : V \to \mathbf{R}$ is called anti-invariant if

$$\tilde{f}(\gamma x) \;=\; \sigma(\gamma)\tilde{f}(x) \ ,$$

for all $\gamma \in \Gamma$ and $x \in V$. It follows then that $\mathcal{Q}_V(\Gamma)$ is a finitely generated graded module over $\mathcal{P}_V(\Gamma)$. Now, our first main result states that there are decompositions

$$\mathcal{P}_V(\Gamma_+) \;=\; \mathcal{P}_V(\Gamma) \oplus \mathcal{Q}_V(\Gamma)$$

and

$$\vec{\mathcal{P}}_V(\Gamma_+) \;=\; \vec{\mathcal{P}}_V(\Gamma) \oplus \vec{\mathcal{Q}}_V(\Gamma)$$

as direct sum of modules over $\mathcal{P}_V(\Gamma)$. The principal tools here are the *relative Reynolds operators*

$$R^{\Gamma}_{\Gamma_+} : \mathcal{P}_V(\Gamma_+) \to \mathcal{P}_V(\Gamma_+) \qquad \text{and} \qquad \vec{R}^{\Gamma}_{\Gamma_+} : \vec{\mathcal{P}}_V(\Gamma_+) \to \vec{\mathcal{P}}_V(\Gamma_+)$$

(onto $\mathcal{P}_V(\Gamma)$ and $\vec{\mathcal{P}}_V(\Gamma)$, respectively) and their "twisted counterparts", namely the *relative Reynolds σ-operators*,

$$S^{\Gamma}_{\Gamma_+} : \mathcal{P}_V(\Gamma_+) \to \mathcal{P}_V(\Gamma_+) \qquad \text{and} \qquad \vec{S}^{\Gamma}_{\Gamma_+} : \vec{\mathcal{P}}_V(\Gamma_+) \to \vec{\mathcal{P}}_V(\Gamma_+)$$

(onto $\mathcal{Q}_V(\Gamma)$ and $\vec{\mathcal{Q}}_V(\Gamma)$, respectively), which are homomorphisms of $\mathcal{P}_V(\Gamma)$-modules (see Section 3).

For our second main result, we shall fix a Hilbert basis $\{u_1, \ldots, u_s\}$ of $\mathcal{P}_V(\Gamma_+)$ and a generating set $\{H_0, \ldots, H_r\}$ of the module $\vec{\mathcal{P}}_V(\Gamma_+)$ over the ring $\mathcal{P}_V(\Gamma_+)$. Note that these two sets can be readily obtained by the standard methods of computational invariant theory applied to Γ_+. By applying the operator $S^{\Gamma}_{\Gamma_+}$ to u_i, we obtain a generating set $\{\tilde{u}_1, \ldots, \tilde{u}_s\}$ of $\mathcal{Q}_V(\Gamma)$ as a module over the ring $\mathcal{P}_V(\Gamma)$. Now, by adding the polynomial function $\tilde{u}_0 \equiv 1$ to the set $\{\tilde{u}_1, \ldots, \tilde{u}_s\}$ we obtain a generating set of $\mathcal{P}_V(\Gamma_+)$ as a module over the ring $\mathcal{P}_V(\Gamma)$. Finally, we multiply the elements of $\{\tilde{u}_0, \ldots, \tilde{u}_s\}$ by the elements of $\{H_0, \ldots, H_r\}$ and apply the operator $\vec{S}^{\Gamma}_{\Gamma_+}$ to that to get the desired generating set of $\vec{\mathcal{Q}}_V(\Gamma)$ as a module over the ring $\mathcal{P}_V(\Gamma)$ (see Section 4). It should be pointed out that it is not evident, a priori, that by applying $S^{\Gamma}_{\Gamma_+}$ to a Hilbert basis of $\mathcal{P}_V(\Gamma_+)$ one produces a generating set of $\mathcal{Q}_V(\Gamma)$ as module over the ring $\mathcal{P}_V(\Gamma)$. In fact, it is quite remarkable that such a simple prescription gives a non-trivial procedure to obtain a generating set of $\mathcal{Q}_V(\Gamma)$ from a Hilbert basis of $\mathcal{P}_V(\Gamma_+)$. We have also included here a brief presentation of the Hilbert-Poincaré series and their associated Molien formulas. In addition, we prove character formulas for the anti-invariants and reversible-equivariants. Although these tools are not used in the proofs of the main results, we felt that, on one hand, it is instructive to show that the standard concepts of invariant theory can be adapted to the reversible-equivariant context; on the other hand, it is quite convenient for future references to have all this material collected in one place.

Structure of the Paper The paper is organized as follows. In Section 2 we recall some concepts and results from representation theory and invariant theory of compact Lie groups. In Section 3 we present the reversible-equivariant framework. In our first main result, we analyze the structure of the space of anti-invariant polynomial functions (Corollary 3.7) and the space of reversible-equivariant polynomial mappings (Corollary 3.9). We end the section obtaining the Hilbert-Poincaré series, their associated Molien formulas and character formulas for the dimensions of the spaces of homogeneous anti-invariant polynomial functions and homogeneous reversible-equivariant polynomial mappings of fixed (but arbitrary) degree. In Section 4 we present the second main result of the paper, namely the procedures for the computation of generating sets for anti-invariant polynomial functions and reversible-equivariant polynomial mappings as modules over the ring of invariant polynomial functions.

2 Preliminaries

In this section we present the definitions and some basic results concerning representations of compact Lie groups and their associated characters. We also recall some elements of the invariant theory of compact Lie groups including the Hilbert-Poincaré series, their associated Molien formulas and the character formulas for the dimensions of spaces of homogeneous invariant polynomial functions and equivariant polynomial mappings.

2.1 Representations and Characters

We start by recalling some notions of representation theory. For a more detailed account, see for example Bröcker and tom Dieck [6], Sattinger [21] in the case of compact Lie groups and Fulton and Harris [10], James and Liebeck [17] in the case of finite groups.

Let Γ be a compact Lie group acting linearly on a finite-dimensional real vector space V. This action corresponds to a representation ρ of the group Γ on V through a linear homomorphism from Γ to the group $\mathbf{GL}(V)$ of invertible linear transformations on V. We denote by (ρ, V) the vector space V together with the linear action of Γ induced by ρ. We often write γx for the action of the linear transformation $\rho(\gamma)$ of an element $\gamma \in \Gamma$ on a vector $x \in V$, whenever there is no danger of confusion. Two representations of a group Γ on the vector spaces V_1 and V_2 are called *equivalent* or Γ-*isomorphic* if there exists a nonsingular linear transformation $T : V_1 \to V_2$ such that $T(\gamma x) = \gamma T(x)$, for all $\gamma \in \Gamma$ and all $x \in V_1$.

The *character* χ_V of a representation (ρ, V) is given by the traces of the linear maps that represent the elements of Γ:

$$\chi_V(\gamma) \;=\; \mathrm{tr}\Big(\rho(\gamma)\Big)\,,$$

for every $\gamma \in \Gamma$. We note that a character is an example of a *class function*, that is,

$$\chi_V(\delta\gamma\delta^{-1}) \;=\; \chi_V(\gamma)\,,$$

for all $\delta, \gamma \in \Gamma$.

2.2 Haar Integral and the Trace Formula

Every compact Lie group Γ admits an (unique up to a constant multiple) *invariant measure*, denoted by $\mathrm{d}\gamma$. We may assume that the measure is normalized so that $\int_\Gamma \mathrm{d}\gamma = 1$. In this case, the measure $\mathrm{d}\gamma$ is called the *Haar measure* of Γ and the integral with respect with this measure is called *normalized Haar integral*. See Bröcker and tom Dieck [6] for a proof of the existence of the Haar measure on a compact Lie group. For finite groups the Haar integral reduces to the "averaging over the group" formula:

$$\int_\Gamma \varphi(\gamma)\,\mathrm{d}\gamma \;=\; \frac{1}{|\Gamma|}\sum_{\gamma\in\Gamma}\varphi(\gamma)\;.$$

Using invariant integration it is possible to construct a Γ-invariant inner product $(\cdot,\cdot)$ on V, that is, $(\gamma u, \gamma v) = (u, v)$ for all $u, v \in V$ and $\gamma \in \Gamma$. See, for example, Golubitsky et al. [14, Proposition XII 1.3]. In that way, we can identify Γ with a closed subgroup of the orthogonal group $\mathbf{O}(n)$, so we assume without loss of generality that Γ acts orthogonally on V. As a consequence, we shall use throughout the following additional property of characters for real representations of compact Lie groups:

$$\chi_V(\gamma^{-1}) \;=\; \chi_V(\gamma)\,,$$

for all $\gamma \in \Gamma$.

Let $\Sigma \subset \Gamma$ be a subgroup. The *fixed point space* of Σ is the subspace of V given by

$$\mathrm{Fix}_V(\Sigma) \;=\; \{x \in V : \gamma x = x\,,\ \forall\, \gamma \in \Sigma\}\ .$$

The next proposition provides an integral formula to compute the dimension of $\mathrm{Fix}_V(\Sigma)$.

Proposition 2.1 (Trace Formula) *Let Γ be a compact Lie group acting on a vector space V and let $\Sigma \subset \Gamma$ be a Lie subgroup. Then*

$$\dim \mathrm{Fix}_V(\Sigma) \;=\; \int_\Sigma \chi_V(\gamma)\,\mathrm{d}\gamma\ ,$$

where χ_V is the character of (ρ, V) and $\int_\Sigma$ denotes the normalized Haar integral.

Proof. See Golubitsky et al. [14, Theorem XIII 2.3]. □

If Σ is finite, then the Trace Formula becomes

$$\dim \mathrm{Fix}_V(\Sigma) = \frac{1}{|\Sigma|} \sum_{\gamma\in\Sigma} \chi_V(\gamma)\ .$$

The following theorem is concerned with the integration over a compact Lie group given by an iteration of integrals. It is used in Section 3, with character theory, to obtain the dimension of vector spaces of homogeneous polynomials.

Theorem 2.2 ("Fubini") *Let Γ be a compact Lie group, Δ a closed subgroup. Let $\mathrm{d}(\gamma\Delta)$ denote the left-invariant normalized Haar measure on Γ/Δ. For any continuous real-valued function φ on Γ,*

$$\int_\Gamma \varphi(\gamma)\,\mathrm{d}\gamma \;=\; \int_{\Gamma/\Delta} \left(\int_\Delta \varphi(\gamma\delta)\,\mathrm{d}\delta\right)\,\mathrm{d}(\gamma\Delta)\ .$$

Proof. See Bröcker and tom Dieck [6, Proposition I 5.16]. □

Remark 2.3 In the theorem above, if $\Delta \subset \Gamma$ is a normal subgroup of finite index $[\Gamma : \Delta] = n$, then Γ/Δ is a finite group and so the integral over Γ/Δ

becomes a finite sum, that is,

$$\int_\Gamma \varphi(\gamma)\,\mathrm{d}\gamma \;=\; \frac{1}{n}\sum_{\gamma\Delta}\Big(\int_\Delta \varphi(\gamma\delta)\,\mathrm{d}\delta\Big)\,.$$

Here the summation runs over the (finite) set of left-cosets of Γ/Δ and γ is a representative of the left-coset $\gamma\Delta$. $\diamond$

In what follows, we shall use $\int_\Gamma \varphi(\gamma)$ for $\int_\Gamma \varphi(\gamma)\,\mathrm{d}\gamma$ to simplify notation.

2.3 Invariant Theory

Let Γ be a compact Lie group acting linearly on V. A real-valued polynomial function $f : V \to \mathbf{R}$ is called Γ-*invariant*, or simply *invariant*, if

$$f(\rho(\gamma)x) \;=\; f(x)\,,$$

for all $\gamma \in \Gamma$ and $x \in V$. We denote by $\mathcal{P}_V(\Gamma)$ the ring of Γ-invariant polynomial functions.

We say that a finite subset $\{u_1, \ldots, u_s\}$ of invariant polynomial functions generates $\mathcal{P}_V(\Gamma)$ (as a ring) if every invariant polynomial function f can be written as a polynomial function of $\{u_1, \ldots, u_s\}$, that is,

$$f(x) \;=\; \sum_{i_1,\ldots,i_s} a_{i_1\ldots i_s}\, u_1^{i_1}(x)\ldots u_s^{i_s}(x)\;, \tag{2.1}$$

where $a_{i_1\ldots i_s} \in \mathbf{R}$. Such a finite set of generators (which is not unique) is called a *Hilbert basis* for $\mathcal{P}_V(\Gamma)$.

Theorem 2.4 (Hilbert-Weyl) *Let Γ be a compact Lie group acting linearly on V. Then there exists a Hilbert basis for $\mathcal{P}_V(\Gamma)$.*

Proof. See Golubitsky et al. [14, Theorem XII 4.2]. $\square$

Now let (ρ, V) and (η, W) be two representations of Γ. A polynomial mapping $G : V \to W$ is called Γ-*equivariant*, or simply *equivariant*, if

$$G(\rho(\gamma)x) \;=\; \eta(\gamma)G(x)\,,$$

for all $\gamma \in \Gamma$ and $x \in V$. We denote by $\vec{\mathcal{P}}_{V,W}(\Gamma)$ the $\mathcal{P}_V(\Gamma)$-module of equivariant polynomial mappings between V and W. When $(\eta, W) = (\rho, V)$, we write $\vec{\mathcal{P}}_V(\Gamma)$ instead of $\vec{\mathcal{P}}_{V,W}(\Gamma)$.

We say that a finite subset $\{H_1, \ldots, H_r\}$ of equivariant polynomial mappings generates $\vec{\mathcal{P}}_{V,W}(\Gamma)$ (as a module over the ring $\mathcal{P}_V(\Gamma)$) if every equivariant mapping $G \in \vec{\mathcal{P}}_{V,W}(\Gamma)$ can be written as

$$G = p_1\, H_1 + \ldots + p_r\, H_r \ ,$$

where $p_1, \ldots, p_r \in \mathcal{P}_V(\Gamma)$.

Theorem 2.5 (Poénaru) *Let Γ be a compact Lie group acting linearly on V. Then there exists a finite set of Γ-equivariant polynomial mappings that generate $\vec{\mathcal{P}}_{V,W}(\Gamma)$ as a module over the ring $\mathcal{P}_V(\Gamma)$.*

Proof. See Golubitsky et al. [14, Theorem XII 6.8]. $\square$

The "Reynolds operators" (see Smith [24]) constitute a basic tool in invariant theory. Let Σ be a closed subgroup of Γ and (ρ, V) a representation of Γ. The *relative Reynolds operators from Σ to Γ* on $\mathcal{P}_V(\Sigma)$ and on $\vec{P}_V(\Sigma)$ are denoted, respectively, by

$$R_\Sigma^\Gamma : \mathcal{P}_V(\Sigma) \to \mathcal{P}_V(\Sigma) \qquad \text{and} \qquad \vec{R}_\Sigma^\Gamma : \vec{\mathcal{P}}_V(\Sigma) \to \vec{\mathcal{P}}_V(\Sigma)$$

and are given by

$$R_\Sigma^\Gamma(f)(x) \ = \ \int_{\Gamma/\Sigma} f(\gamma x) \qquad \text{and} \qquad \vec{R}_\Sigma^\Gamma(G)(x) \ = \ \int_{\Gamma/\Sigma} \gamma^{-1} G(\gamma x).$$

When Σ is a subgroup of finite index $[\Gamma : \Sigma] = n$, the formulas for the Reynolds operators become

$$R_\Sigma^\Gamma(f)(x) \ = \ \frac{1}{n} \sum_{\gamma\Sigma} f(\gamma x) \qquad \text{and} \qquad \vec{R}_\Sigma^\Gamma(G)(x) \ = \ \frac{1}{n} \sum_{\gamma\Sigma} \gamma^{-1} G(\gamma x)$$

and when Σ is the trivial group we suppress it from the notation for the Reynolds operators.

2.4 Hilbert-Poincaré Series and Molien Formulas

Recall that the ring of polynomial functions $\mathcal{P}_V$ on a finite-dimensional vector space V has a natural grading

$$\mathcal{P}_V \ = \ \bigoplus_{d=0}^{\infty} \mathcal{P}_V^d \ ,$$

where $\mathcal{P}_V^d$ is the space of homogeneous polynomial functions of degree d. By the linearity of the action of Γ on V, it induces a natural grading on $\mathcal{P}_V(\Gamma)$, that is,

$$\mathcal{P}_V(\Gamma) \ = \ \bigoplus_{d=0}^{\infty} \mathcal{P}_V^d(\Gamma) \ ,$$

where $\mathcal{P}_V^d(\Gamma) = \mathcal{P}_V(\Gamma) \cap \mathcal{P}_V^d$.

For W a finite-dimensional vector space, let us denote by $\vec{\mathcal{P}}_{V,W}$ the module over $\mathcal{P}_V$ of polynomial mappings from V to W. When $W = V$, we shall simply write $\vec{\mathcal{P}}_V$. Let us notice that $\vec{\mathcal{P}}_{V,W}$ has a natural grading

$$\vec{\mathcal{P}}_{V,W} \;=\; \bigoplus_{d=0}^{\infty} \vec{\mathcal{P}}_{V,W}^d \;,$$

where $\vec{\mathcal{P}}_{V,W}^d$ is the space of homogeneous polynomial mappings from V to W of degree d which induces a natural grading on $\vec{\mathcal{P}}_{V,W}(\Gamma)$, that is,

$$\vec{\mathcal{P}}_{V,W}(\Gamma) \;=\; \bigoplus_{d=0}^{\infty} \vec{\mathcal{P}}_{V,W}^d(\Gamma) \;,$$

where $\vec{\mathcal{P}}_{V,W}^d(\Gamma) = \vec{\mathcal{P}}_{V,W}(\Gamma) \cap \vec{\mathcal{P}}_{V,W}^d$.

The Hilbert-Poincaré series of $\mathcal{P}_V(\Gamma)$ is the generating function for the dimensions of the spaces of homogeneous polynomial invariants of each degree and is defined by

$$\Phi_V^\Gamma(t) = \sum_{d=0}^{\infty} \dim \mathcal{P}_V^d(\Gamma)\, t^d \;.$$

The following theorem provides an integral formula for the Hilbert-Poincaré series of $\mathcal{P}_V(\Gamma)$.

Theorem 2.6 (Molien) *Let (ρ, V) be a finite-dimensional real representation of a compact Lie group Γ. Then the Hilbert-Poincaré series of $\mathcal{P}_V(\Gamma)$ is given by*

$$\Phi_V^\Gamma(t) \;=\; \int_\Gamma \frac{1}{\det(1 - t\rho(\gamma))} \;.$$

Proof. See Sturmfels [26] for the proof in the case where Γ is a finite group and Sattinger [21] for the case where Γ is a general compact Lie group. $\square$

Similarly, the Hilbert-Poincaré series of $\vec{\mathcal{P}}_{V,W}(\Gamma)$ is the generating function for the dimensions of the spaces of homogeneous polynomial equivariant mappings of each degree and is defined by

$$\Psi_{V,W}^\Gamma(t) = \sum_{d=0}^{\infty} \dim \vec{\mathcal{P}}_{V,W}^d(\Gamma)\, t^d.$$

The generalization of the Molien's Theorem to the equivariant case when $(\rho, V) = (\eta, W)$ is given by Sattinger [21] and the case when (ρ, V) is not equivalent to (η, W) is given by Gatermann [12].

Theorem 2.7 *Let (ρ, V) and (η, W) be two finite-dimensional real representations of a compact Lie group Γ. Then the Hilbert-Poincaré series of $\vec{\mathcal{P}}_{V,W}(\Gamma)$ is given by*

$$\Psi^{\Gamma}_{V,W}(t) = \int_{\Gamma} \frac{\chi_W(\gamma)}{\det(1 - t\rho(\gamma))} \ ,$$

where χ_W is the character of (η, W). In particular, if $(\rho, V) = (\eta, W)$ then

$$\Psi^{\Gamma}_{V}(t) = \int_{\Gamma} \frac{\chi_V(\gamma)}{\det(1 - t\rho(\gamma))} \ ,$$

where χ_V is the character of (ρ, V).

Proof. See Gatermann [12, Theorem 12.2]. □

2.5 Character Formulas for Invariants and Equivariants

Let two representations of a group Γ be given on the vector spaces V and W, respectively. Then there is a natural representation of Γ on the tensor product $V \otimes W$ given by $\gamma(v \otimes w) = \gamma v \otimes \gamma w$. By iteration of this construction one obtains an action of Γ on the k-th tensor powers $V^{\otimes k}$ of a representation ρ of Γ on V. Finally, by restriction, one obtains a representation of Γ on the k-th symmetric tensor power $S^k V$, since it is an invariant subspace of $V^{\otimes k}$ under the action of Γ.

The following theorem (Sattinger [21, Theorem 5.10]) provides formulas, in terms of the character χ_V of (ρ, V), for the dimensions of $\mathcal{P}^d_V(\Gamma)$ and $\vec{\mathcal{P}}^d_V(\Gamma)$.

Theorem 2.8 (Sattinger) *Let Γ be a compact Lie group acting linearly on a finite-dimensional real vector space V. Let χ be the character afforded by V and $\chi_{(d)}$ the character afforded by the induced action of Γ on $S^d V$. Then*

$$\dim \mathcal{P}^d_V(\Gamma) = \int_{\Gamma} \chi_{(d)}(\gamma)$$

and

$$\dim \vec{\mathcal{P}}^d_V(\Gamma) = \int_{\Gamma} \chi_{(d)}(\gamma)\, \chi(\gamma) \ .$$

Proof. See Sattinger [21] or Antoneli et al. [1]. □

3 The Structure of the Anti-Invariants and Reversible-Equivariants

We start this section by giving the definitions of an anti-invariant function and a reversible-equivariant mapping under the action of a compact Lie group. We then present some basic facts about the reversible-equivariant theory. Next we obtain results about the structure of the spaces of anti-invariant functions and reversible-equivariant mappings. Finally, we generalize the character formulas of Theorem 2.8 to character formulas for the dimensions of the subspaces of homogeneous polynomials of each fixed degree in these spaces.

3.1 The Representation Theory

Let Γ be a compact Lie group. Consider a homomorphism of Lie groups

$$\sigma : \Gamma \to \mathbf{Z}_2 \cong \{\pm 1\} \ , \tag{3.1}$$

where $\mathbf{Z}_2$ is regarded as a zero-dimensional Lie group with respect to the discrete topology.

Note that σ defines a one-dimensional representation of Γ, by $x \mapsto \sigma(\gamma)x$ for $x \in \mathbf{R}$. Also, the character afforded by the homomorphism σ is the real-valued function whose value on $\gamma \in \Gamma$ is $\sigma(\gamma)$; in other words, it is σ itself, but regarded as a real-valued function, which we shall also denote by σ.

If σ in (3.1) is non-trivial, then its kernel $\Gamma_+ = \ker(\sigma)$ is a normal subgroup of Γ of index 2 and the complement Γ_- of Γ_+ in Γ is a non-trivial left-coset. If we choose some $\delta \in \Gamma_-$, then we have a decomposition of Γ as a disjoint union of left-cosets:

$$\Gamma = \Gamma_+ \dot{\cup} \, \Gamma_- = \Gamma_+ \dot{\cup} \ \delta \, \Gamma_+ \ .$$

In this case Γ is called a *reversing symmetry group*. An element of Γ_+ is called a *symmetry* in Γ and an element of Γ_- is called a *reversing symmetry* in Γ. Note that a product of two symmetries or two reversing symmetries is a symmetry and a product of a symmetry and a reversing symmetry is a reversing symmetry; if γ is a symmetry (reversing symmetry) then γ^{-1} is a symmetry (reversing symmetry).

Let (ρ, V) be a representation of Γ and consider σ in (3.1) a non-trivial homomorphism of Lie groups. We now give the following two definitions for the main objects of this work:

Definition 3.1 A polynomial mapping $\tilde{G} : V \to V$ is called Γ-*reversible-*

equivariant, or simply *reversible-equivariant*, if

$$\tilde{G}(\rho(\gamma)x) \;=\; \sigma(\gamma)\rho(\gamma)\tilde{G}(x) \tag{3.2}$$

for all $\gamma \in \Gamma$ and $x \in V$. The space of all reversible-equivariant polynomial mappings is a module over $\mathcal{P}_V(\Gamma)$ and is denoted by $\vec{\mathcal{Q}}_V(\Gamma)$. $\diamond$

Definition 3.2 A polynomial function $\tilde{f} : V \to \mathbf{R}$ is called Γ-*anti-invariant*, or simply *anti-invariant*, if

$$\tilde{f}(\rho(\gamma)x) \;=\; \sigma(\gamma)\tilde{f}(x) \ , \tag{3.3}$$

for all $\gamma \in \Gamma$ and $x \in V$. The space of all anti-invariant polynomial functions is a module over $\mathcal{P}_V(\Gamma)$ and is denoted by $\mathcal{Q}_V(\Gamma)$. $\diamond$

We proceed by defining the dual of a representation:

Definition 3.3 Let (ρ, V) be a representation of Γ and let $\sigma : \Gamma \to \mathbf{Z}_2$ be a homomorphism. The σ-*dual* of ρ is the representation ρ_σ of Γ on V defined by

$$\begin{array}{rcl} \rho_\sigma : \Gamma & \longrightarrow & \mathbf{GL}(V) \\ \gamma & \mapsto & \sigma(\gamma)\rho(\gamma) \end{array} .$$

The corresponding action of Γ on V is called *dual action* and it is written as $(\gamma, x) \mapsto \rho_\sigma(\gamma)x$. Note that $(\rho_\sigma)_\sigma = \rho$. $\diamond$

A representation (ρ, V) of Γ is said to be *self-dual* if it is Γ-isomorphic to (ρ_σ, V) or, equivalently, if there exists a reversible-equivariant linear isomorphism $L : V \to V$. In this case, we say that V is a *self-dual* vector space.

Remark 3.4 Using the σ-dual representation ρ_σ of ρ, condition (3.2) may be written as

$$\tilde{G}(\rho(\gamma)x) \;=\; \rho_\sigma(\gamma)\tilde{G}(x) \ . \tag{3.4}$$

In this way, we may regard a reversible-equivariant mapping on V as an equivariant mapping from (ρ, V) to (ρ_σ, V). Also, equation (3.3) is equivalent to the requirement that $\tilde{f}$ is an equivariant mapping from (ρ, V) to $(\sigma, \mathbf{R})$. Therefore, the existence of a finite generating set for both $\mathcal{Q}_V(\Gamma)$ and $\vec{\mathcal{Q}}_V(\Gamma)$ is guaranteed by Poénaru's Theorem (Theorem 2.5). $\diamond$

Remark 3.5 We make the following two observations:
(i) There is an important particular case in the reversible-equivariant theory, i. e., when Γ is a two-element group. This is the *purely reversible* framework, where there are no non-trivial symmetry and only one non-trivial reversing symmetry which is an involution.
(ii) When σ in (3.1) is the trivial homomorphism, (ρ, V) and (ρ_σ, V) are the same representation and so we encounter the *purely equivariant* framework. $\diamond$

In this subsection we relate the rings of invariant polynomial functions of Γ and Γ_+, the module of anti-invariants playing a fundamental role in this construction. We start by observing that $\mathcal{P}_V(\Gamma_+)$ (as well as $\mathcal{Q}_V(\Gamma)$) is a module over $\mathcal{P}_V(\Gamma)$.

Let us consider the relative Reynolds operator from Γ_+ to Γ on $\mathcal{P}_V(\Gamma_+)$, $R^{\Gamma}_{\Gamma_+} : \mathcal{P}_V(\Gamma_+) \to \mathcal{P}_V(\Gamma_+)$. In our particular case, it is simply given by

$$R^{\Gamma}_{\Gamma_+}(f)(x) \;=\; \frac{1}{2}\sum_{\gamma\Gamma_+} f(\gamma x) \;=\; \frac{1}{2}\Big(f(x) + f(\delta x)\Big) \;, \tag{3.5}$$

for an arbitrary (and fixed) $\delta \in \Gamma_-$.

Now we define the *relative Reynolds σ-operator* on $\mathcal{P}_V(\Gamma_+)$, $S^{\Gamma}_{\Gamma_+} : \mathcal{P}_V(\Gamma_+) \to \mathcal{P}_V(\Gamma_+)$, by

$$S^{\Gamma}_{\Gamma_+}(f)(x) \;=\; \frac{1}{2}\sum_{\gamma\Gamma_+} \sigma(\gamma) f(\gamma x).$$

So

$$S^{\Gamma}_{\Gamma_+}(f)(x) \;=\; \frac{1}{2}\Big(f(x) - f(\delta x)\Big) \;, \tag{3.6}$$

for an arbitrary (and fixed) $\delta \in \Gamma_-$.

Let us denote by $I_{\mathcal{P}_V(\Gamma_+)}$ the identity map on $\mathcal{P}_V(\Gamma_+)$. We then have:

Proposition 3.6 *The Reynolds operator $R^{\Gamma}_{\Gamma_+}$ and the Reynolds σ-operator $S^{\Gamma}_{\Gamma_+}$ satisfy the following properties:*

(i) They are homomorphisms of $\mathcal{P}_V(\Gamma)$-modules and

$$R^{\Gamma}_{\Gamma_+} + S^{\Gamma}_{\Gamma_+} \;=\; I_{\mathcal{P}_V(\Gamma_+)} \;. \tag{3.7}$$

(ii) They are idempotent projections with

$$\ker(R^{\Gamma}_{\Gamma_+}) \;=\; \mathcal{Q}_V(\Gamma) \qquad \textit{and} \qquad \ker(S^{\Gamma}_{\Gamma_+}) \;=\; \mathcal{P}_V(\Gamma) \;,$$

$$\mathrm{im}(R^{\Gamma}_{\Gamma_+}) \;=\; \mathcal{P}_V(\Gamma) \qquad \textit{and} \qquad \mathrm{im}(S^{\Gamma}_{\Gamma_+}) \;=\; \mathcal{Q}_V(\Gamma) \;.$$

(iii) The following decompositions (as direct sum of $\mathcal{P}_V(\Gamma)$-modules) hold:

$$\mathcal{P}_V(\Gamma_+) \;=\; \ker(R^{\Gamma}_{\Gamma_+}) \oplus \mathrm{im}(R^{\Gamma}_{\Gamma_+}) \;=\; \ker(S^{\Gamma}_{\Gamma_+}) \oplus \mathrm{im}(S^{\Gamma}_{\Gamma_+}) \;.$$

Proof. To prove (i) we just note that if $f \in \mathcal{P}_V(\Gamma_+)$ and $p \in \mathcal{P}_V(\Gamma)$, then

$$R^{\Gamma}_{\Gamma_+}(p\, f) \;=\; p\, R^{\Gamma}_{\Gamma_+}(f) \qquad \text{and} \qquad S^{\Gamma}_{\Gamma_+}(p\, f) \;=\; p\, S^{\Gamma}_{\Gamma_+}(f) \;.$$

Moreover, (3.7) follows directly from (3.5) and (3.6).

To prove (ii), we first show that

$$\mathcal{P}_V(\Gamma) \;=\; \{f \in \mathcal{P}_V(\Gamma_+) : f(\delta x) = f(x)\,,\ \forall\, x \in V\}$$

and

$$\mathcal{Q}_V(\Gamma) \;=\; \{f \in \mathcal{P}(\Gamma_+) : f(\delta x) = -f(x)\,,\ \forall\, x \in V\}.$$

The inclusions " $\subseteq$ " follow directly from the definitions of $\mathcal{P}_V(\Gamma)$ and $\mathcal{Q}_V(\Gamma)$, so we just show " $\supseteq$ ". Since $\Gamma_- = \delta\Gamma_+$ and Γ_+ is normal in Γ, it follows that for every $\gamma \in \Gamma_-$ we have $\gamma = \tilde{\gamma}\delta$, for some $\tilde{\gamma} \in \Gamma_+$. If $f \in \mathcal{P}_V(\Gamma_+)$ is such that $f(\delta x) = f(x)$ for all $x \in V$, then

$$f(\gamma x) \;=\; f(\tilde{\gamma}\delta x) \;=\; f(\delta x) \;=\; f(x)\,, \quad \forall\, x \in V\ ,$$

that is, $f \in \mathcal{P}_V(\Gamma)$. If $f \in \mathcal{P}_V(\Gamma_+)$ is such that $f(\delta x) = -f(x)$, for all $x \in V$, then

$$f(\gamma x) \;=\; f(\tilde{\gamma}\delta x) \;=\; f(\delta x) \;=\; -\,f(x) \;=\; \sigma(\gamma)f(x)\,, \quad \forall\, x \in V\ ,$$

that is, $f \in \mathcal{Q}_V(\Gamma)$.

Now we show that, for any $f \in \mathcal{P}_V(\Gamma_+)$, we have

$$R^{\Gamma}_{\Gamma_+}(f) \in \mathcal{P}_V(\Gamma) \qquad \text{and} \qquad S^{\Gamma}_{\Gamma_+}(f) \in \mathcal{Q}_V(\Gamma).$$

In fact, for any $\gamma \in \Gamma_+$, we have that $\delta\gamma = \tilde{\gamma}\delta$ for some $\tilde{\gamma} \in \Gamma_+$ and thus

$$f(\delta\gamma x) \;=\; f(\tilde{\gamma}\delta x) \;=\; f(\delta x)\,, \quad \forall\, x \in V\ .$$

Then, it is again direct from (3.5) and (3.6) that

$$R^{\Gamma}_{\Gamma_+}(f)(\gamma x) \;=\; R^{\Gamma}_{\Gamma_+}(f)(x) \qquad \text{and} \qquad S^{\Gamma}_{\Gamma_+}(f)(\gamma x) \;=\; S^{\Gamma}_{\Gamma_+}(f)(x)\ .$$

Also, since $\delta^2 \in \Gamma_+$,

$$R^{\Gamma}_{\Gamma_+}(f)(\delta x) \;=\; \frac{1}{2}(f(\delta x) + f(\delta^2 x)) \;=\; R^{\Gamma}_{\Gamma_+}(f)(x)$$

and

$$S^{\Gamma}_{\Gamma_+}(f)(\delta x) \;=\; \frac{1}{2}(f(\delta x) - f(\delta^2 x)) \;=\; -\,S^{\Gamma}_{\Gamma_+}(f)(x)\ .$$

Hence, $\mathrm{im}(R^{\Gamma}_{\Gamma_+}) \subseteq \mathcal{P}_V(\Gamma)$ and $\mathrm{im}(S^{\Gamma}_{\Gamma_+}) \subseteq \mathcal{Q}_V(\Gamma)$. The other inclusions are immediate, since

$$R^{\Gamma}_{\Gamma_+}|_{\mathcal{P}_V(\Gamma)} \;=\; I_{\mathcal{P}_V(\Gamma)} \qquad \text{and} \qquad S^{\Gamma}_{\Gamma_+}|_{\mathcal{Q}_V(\Gamma)} \;=\; I_{\mathcal{Q}_V(\Gamma).}$$

Finally, (iii) follows from (i) and (ii). $\square$

Corollary 3.7 *The following direct sum decomposition of modules over the ring $\mathcal{P}_V(\Gamma)$ holds:*

$$\mathcal{P}_V(\Gamma_+) \;=\; \mathcal{P}_V(\Gamma) \oplus \mathcal{Q}_V(\Gamma)\;.$$

Now we relate the modules of equivariant polynomial mappings under Γ and Γ_+. This can be done by a similar construction as above for the rings of invariant polynomial functions under Γ and Γ_+: we consider the relative Reynolds operator from Γ_+ to Γ on $\vec{\mathcal{P}}_V(\Gamma_+)$, $\vec{R}^{\Gamma}_{\Gamma_+} : \vec{\mathcal{P}}_V(\Gamma_+) \to \vec{\mathcal{P}}_V(\Gamma_+)$, which is, in the present case, given by

$$\vec{R}^{\Gamma}_{\Gamma_+}(G)(x) \;=\; \frac{1}{2}\sum_{\gamma\Gamma_+} \gamma^{-1}G(\gamma x) \;=\; \frac{1}{2}\Big(G(x) + \delta^{-1}G(\delta x)\Big)\;, \tag{3.8}$$

for an arbitrary $\delta \in \Gamma_-$.

Now we define the *relative Reynolds σ-operator* from Γ_+ to Γ on $\vec{\mathcal{P}}_V(\Gamma_+)$, $\vec{S}^{\Gamma}_{\Gamma_+} : \vec{\mathcal{P}}_V(\Gamma_+) \to \vec{\mathcal{P}}_V(\Gamma_+)$, by

$$\vec{S}^{\Gamma}_{\Gamma_+}(G)(x) \;=\; \frac{1}{2}\sum_{\gamma\Gamma_+} \sigma(\gamma)\gamma^{-1}G(\gamma x).$$

So

$$\vec{S}^{\Gamma}_{\Gamma_+}(G)(x) \;=\; \frac{1}{2}\Big(G(x) - \delta^{-1}G(\delta x)\Big)\;, \tag{3.9}$$

for an arbitrary $\delta \in \Gamma_-$.

Let us denote by $I_{\vec{\mathcal{P}}_V(\Gamma_+)}$ the identity map on $\vec{\mathcal{P}}_V(\Gamma_+)$. We then have:

Proposition 3.8 *The Reynolds operator $\vec{R}^{\Gamma}_{\Gamma_+}$ and the Reynolds σ-operator $\vec{S}^{\Gamma}_{\Gamma_+}$ satisfy the following properties:*

(i) They are homomorphisms of $\mathcal{P}_V(\Gamma)$-modules and

$$\vec{R}^{\Gamma}_{\Gamma_+} + \vec{S}^{\Gamma}_{\Gamma_+} \;=\; I_{\vec{\mathcal{P}}_V(\Gamma_+)}\;. \tag{3.10}$$

(ii) They are idempotent projections with

$$\ker(\vec{R}^{\Gamma}_{\Gamma_+}) \;=\; \vec{\mathcal{Q}}_V(\Gamma) \qquad \textit{and} \qquad \ker(\vec{S}^{\Gamma}_{\Gamma_+}) \;=\; \vec{\mathcal{P}}_V(\Gamma)\;,$$

$$\mathrm{im}(\vec{R}^{\Gamma}_{\Gamma_+}) \;=\; \vec{\mathcal{P}}_V(\Gamma) \qquad \textit{and} \qquad \mathrm{im}(\vec{S}^{\Gamma}_{\Gamma_+}) \;=\; \vec{\mathcal{Q}}_V(\Gamma)\;.$$

(iii) The following decompositions (as direct sum of $\mathcal{P}_V(\Gamma)$-modules) hold:

$$\vec{\mathcal{P}}_V(\Gamma_+) \;=\; \ker(\vec{S}^{\Gamma}_{\Gamma_+}) \oplus \mathrm{im}(\vec{S}^{\Gamma}_{\Gamma_+}) \;=\; \ker(\vec{R}^{\Gamma}_{\Gamma_+}) \oplus \mathrm{im}(\vec{R}^{\Gamma}_{\Gamma_+})\;.$$

Proof. It is analogous to the proof of Proposition 3.6. $\square$

We are now in position to state the following corollary which, together with Corollary 3.7, forms our first main result.

Corollary 3.9 *The following direct sum decomposition of modules over the ring $\mathcal{P}_V(\Gamma)$ holds:*

$$\vec{\mathcal{P}}_V(\Gamma_+) \;=\; \vec{\mathcal{P}}_V(\Gamma) \oplus \vec{\mathcal{Q}}_V(\Gamma)\ .$$

We end this subsection with a remark.

Remark 3.10 The notion of anti-invariant polynomial function is a particular case of the definition of *relative invariant* of a group Γ. Stanley [25] carried out a study of the structure of relative invariants when Γ is a finite group generated by pseudo-reflections. Smith [23] has focused attention to the invariant theory by using results on relative invariants when Γ is also an arbitrary finite group. Let us switch to the notation of Smith. Let $\mathbb{F}$ be a field, $\rho : \Gamma \to \mathbf{GL}(n, \mathbb{F})$ a representation of a finite group Γ and $\chi : \Gamma \to \mathbb{F}^\times$ be a linear character. Therein, the ring of invariants of Γ is denoted by

$$\mathbb{F}[V]^\Gamma \;=\; \{f \in \mathbb{F}[V] : f(\gamma x) = f(x)\,,\ \forall\, \gamma \in \Gamma\}$$

and the module of χ-*relative invariants* of Γ by

$$\mathbb{F}[V]^\Gamma_\chi \;=\; \{f \in \mathbb{F}[V] : f(\gamma x) = \chi(\gamma)\, f(x)\,,\ \forall\, \gamma \in \Gamma\}\,.$$

Now let Σ be a subgroup of Γ of index $m = [\Gamma : \Sigma]$. Then,

$$\mathbb{F}[V]^\Sigma = \;\mathbb{F}[V]^\Gamma \oplus \mathbb{F}[V]^\Gamma_\chi \oplus \cdots \oplus \mathbb{F}[V]^\Gamma_{\chi^{m-1}}\ . \tag{3.11}$$

If $\gamma_\chi\Sigma$ generates the cyclic group Γ/Σ, then the subspaces $\mathbb{F}[V]^\Gamma_{\chi^j}$, for $j = 0, \ldots, m-1$, are the eigenspaces associated with the eigenvalues $\chi(\gamma_\chi)^j$ of the action of γ_χ on $\mathbb{F}[V]^\Sigma$. In particular, when Γ is generated by pseudo-reflections, Smith [23, Theorem 2.7] shows that $\mathbb{F}[V]^\Gamma_\chi$ is a free module over $\mathbb{F}[V]^\Gamma$ on a single generator which can be constructed from the action of Γ on $\mathbb{F}^n$. Switching back to our notation, the module $\mathcal{Q}_V(\Gamma)$ of anti-invariants is the module of σ-relative invariants $\mathbf{R}[V]^\Gamma_\sigma$. Thus, when Γ is a finite group, the ring $\mathcal{P}_V(\Gamma)$ and the module $\mathcal{Q}_V(\Gamma)$ are the eigenspaces associated to the eigenvalues 1 and -1, respectively, of the action of $\delta \in \Gamma_-$ on $\mathcal{P}_V(\Gamma_+)$. Therefore, decomposition (3.11) is a generalization of the Corollary 3.7 (where $m = 2$ and χ is a real-valued linear character). On the other hand, Corollary 3.7 is an extension of the decomposition (3.11) for arbitrary compact Lie groups. $\diamond$

As well as in the case of invariants and equivariants it is possible to define Hilbert-Poincaré Series for the modules $\mathcal{Q}_V(\Gamma)$ and $\vec{\mathcal{Q}}_V(\Gamma)$ and establish Molien formulas for them. In the same vein, we prove character formulas for the dimensions of their homogeneous components. See Sattinger [21] or Antoneli et al. [1] for the case of invariants and purely equivariants.

We begin by observing that $\mathcal{Q}_V(\Gamma)$ is a subspace of $\mathcal{P}_V$ and $\vec{\mathcal{Q}}_V(\Gamma)$ is a subspace of $\vec{\mathcal{P}}_V$ and so they have natural gradings

$$\mathcal{Q}_V(\Gamma) \;=\; \bigoplus_{d=0}^{\infty} \mathcal{Q}_V^d(\Gamma) \qquad \text{and} \qquad \vec{\mathcal{Q}}_V(\Gamma) \;=\; \bigoplus_{d=0}^{\infty} \vec{\mathcal{Q}}_V^d(\Gamma) \;,$$

where $\mathcal{Q}_V^d(\Gamma) = \mathcal{Q}_V(\Gamma) \cap \mathcal{P}_V^d$ and $\vec{\mathcal{Q}}_V^d(\Gamma) = \vec{\mathcal{Q}}_V(\Gamma) \cap \vec{\mathcal{P}}_V^d$, respectively. It follows from Corollaries 3.7 and 3.9 that we have the direct sum decomposition of vector spaces

$$\mathcal{P}_V^d(\Gamma_+) \;=\; \mathcal{P}_V^d(\Gamma) \oplus \mathcal{Q}_V^d(\Gamma)$$

and

$$\vec{\mathcal{P}}_V^d(\Gamma_+) \;=\; \vec{\mathcal{P}}_V^d(\Gamma) \oplus \vec{\mathcal{Q}}_V^d(\Gamma)$$

for every degree $d \in \mathbf{N}$.

The *Hilbert-Poincaré series* for $\mathcal{Q}_V(\Gamma)$ and $\vec{\mathcal{Q}}_V(\Gamma)$ can be defined in the same way as for $\mathcal{P}_V(\Gamma)$ and $\vec{\mathcal{P}}_V(\Gamma)$ and are denoted by

$$\tilde{\Phi}_V^\Gamma(t) \;=\; \sum_{d=0}^{\infty} \dim \mathcal{Q}_V^d(\Gamma)\, t^d \qquad \text{and} \qquad \tilde{\Psi}_V^\Gamma(t) \;=\; \sum_{d=0}^{\infty} \dim \vec{\mathcal{Q}}_V^d(\Gamma)\, t^d \;.$$

The general Molien formula (Theorem 2.7) applied to the *Hilbert-Poincaré series* of $\mathcal{Q}_V(\Gamma)$ and $\vec{\mathcal{Q}}_V(\Gamma)$ gives

$$\tilde{\Phi}_V^\Gamma(t) \;=\; \int_\Gamma \frac{\sigma(\gamma)}{\det(1 - t\rho(\gamma))} \qquad \text{and} \qquad \tilde{\Psi}_V^\Gamma(t) \;=\; \int_\Gamma \frac{\sigma(\gamma)\chi_V(\gamma)}{\det(1 - t\rho(\gamma))} \;,$$

where χ_V is the character of (ρ, V).

Now let $L_s^d(V)$ denote the space of real-valued symmetric d-multilinear functions on V^d. Then there is a canonical isomorphism

$$L_s^d(V) \;\cong\; \mathcal{P}_V^d \;. \tag{3.12}$$

Let $L_s^d(V,W)$ denote the space of W-valued symmetric d-multilinear mappings on V^d. Then there is a canonical isomorphism

$$L_s^d(V,W) \;\cong\; \vec{\mathcal{P}}_{V,W}^d \;. \tag{3.13}$$

We also have the following canonical isomorphism (see [15, p. 621]):

$$\mathrm{Hom}(S^d V, W) \cong L_s^d(V, W) \; . \tag{3.14}$$

Since $V^* = \mathrm{Hom}(V, \mathbf{R})$ and $V^* \otimes W \cong \mathrm{Hom}(V, W)$ (see [15, p. 618]), we have that

$$L_s^d(V, W) \;\cong\; \mathrm{Hom}(S^d V, W) \;\cong\; (S^d V)^* \otimes W \; ,$$

which, together with (3.13), gives

$$\vec{\mathcal{P}}^d_{V,W} \;\cong\; (S^d V)^* \otimes W. \tag{3.15}$$

If (ρ, V) and (η, W) are two finite-dimensional representations of Γ, then the group Γ acts naturally on $\vec{\mathcal{P}}_{V,W}$ by

$$\Big(\gamma \cdot G\Big)(x) \;=\; \eta(\gamma^{-1})\, G(\rho(\gamma)x)\, , \quad \forall\, \gamma \in \Gamma,\, x \in V,\, G \in \vec{\mathcal{P}}_{V,W}$$

and so we have

$$\vec{\mathcal{P}}^d_{V,W}(\Gamma) \;=\; \mathrm{Fix}_{\vec{\mathcal{P}}^d_{V,W}}(\Gamma) \; . \tag{3.16}$$

Combining (3.15) with (3.16) we get

$$\vec{\mathcal{P}}^d_{V,W}(\Gamma) \;\cong\; \mathrm{Fix}_{(S^d V)^* \otimes W}(\Gamma) \; . \tag{3.17}$$

We then obtain:

Theorem 3.11 *Let Γ be a compact Lie group. Let (ρ, V) and (η, W) be two finite-dimensional representations of Γ with corresponding characters χ_V and χ_W. Then*

$$\dim \vec{\mathcal{P}}^d_{V,W}(\Gamma) \;=\; \int_\Gamma \chi_{V(d)}(\gamma)\chi_W(\gamma) \; ,$$

where $\chi_{V(d)}$ is the character afforded by the induced action of Γ on $S^d V$.

Proof. The Trace Formula (Proposition 2.1) combined with equation (3.17) leads to

$$\begin{aligned}
\dim \vec{\mathcal{P}}^d_{V,W}(\Gamma) \;&=\; \dim \mathrm{Fix}_{(S^d V)^* \otimes W}(\Gamma) \\
&=\; \int_\Gamma \chi_{(S^d V)^* \otimes W}(\gamma) \\
&=\; \int_\Gamma \chi_{(S^d V)^*}(\gamma)\chi_W(\gamma) \\
&=\; \int_\Gamma \chi_{(S^d V)}(\gamma^{-1})\chi_W(\gamma) \\
&=\; \int_\Gamma \chi_{V(d)}(\gamma)\chi_W(\gamma) \; ,
\end{aligned}$$

as desired. $\Box$

Now recalling Remark 3.4 and applying the above theorem we obtain:

Corollary 3.12 *Let Γ be a compact Lie group. Let (ρ, V) be a finite-dimensional representation of Γ with corresponding character χ. Then*

$$\dim \mathcal{Q}_V^d(\Gamma) = \int_\Gamma \sigma(\gamma)\chi_{(d)}(\gamma)$$

and

$$\dim \vec{\mathcal{Q}}_V^d(\Gamma) = \int_\Gamma \sigma(\gamma)\chi_{(d)}(\gamma)\chi(\gamma) \ ,$$

where $\chi_{(d)}$ is the character afforded by the induced action of Γ on S^dV.

In order to evaluate these character formulas it is necessary to compute the character $\chi_{(d)}$ of S^dV. There is a well known recursive formula, whose proof can be found in Antoneli et al. [1, Section 4]:

$$d\,\chi_{(d)}(\gamma) = \sum_{i=0}^{d-1} \chi(\gamma^{d-i})\chi_{(i)}(\gamma) \ , \tag{3.18}$$

where $\chi_{(0)} = 1$. See [1, Section 6] for several examples of calculations with these formulas.

We note that Theorem 2.8 is a particular case of Theorem 3.11, where $W = \mathbf{R}$ under the trivial action of Γ in the invariant case and where $(\eta, W) = (\rho, V)$ in the purely equivariant case.

Remark 2.3 applied to the character formulas provides useful integral formulas for the dimensions of the spaces of invariants, anti-invariants, equivariants and reversible-equivariants: for an arbitrary (and fixed) $\delta \in \Gamma_-$,

$$\dim \mathcal{P}_V^d(\Gamma) = \frac{1}{2}\Big[\int_{\Gamma_+} \chi_{(d)}(\gamma) + \int_{\Gamma_+} \chi_{(d)}(\delta\gamma)\Big] \ ,$$

$$\dim \mathcal{Q}_V^d(\Gamma) = \frac{1}{2}\Big[\int_{\Gamma_+} \chi_{(d)}(\gamma) - \int_{\Gamma_+} \chi_{(d)}(\delta\gamma)\Big] \ ,$$

$$\dim \vec{\mathcal{P}}_V^d(\Gamma) = \frac{1}{2}\Big[\int_{\Gamma_+} \chi_{(d)}(\gamma)\chi_V(\gamma) + \int_{\Gamma_+} \chi_{(d)}(\delta\gamma)\chi_V(\delta\gamma)\Big] \ ,$$

$$\dim \vec{\mathcal{Q}}_V^d(\Gamma) = \frac{1}{2}\Big[\int_{\Gamma_+} \chi_{(d)}(\gamma)\chi_V(\gamma) - \int_{\Gamma_+} \chi_{(d)}(\delta\gamma)\chi_V(\delta\gamma)\Big] \ .$$

From the above formulas we get

$$\dim \mathcal{P}_V^d(\Gamma) + \dim \mathcal{Q}_V^d(\Gamma) = \int_{\Gamma_+} \chi_{(d)}(\gamma) = \dim \mathcal{P}_V^d(\Gamma_+) \tag{3.19}$$

and

$$\dim \vec{\mathcal{P}}_V^d(\Gamma) + \dim \vec{\mathcal{Q}}_V^d(\Gamma) = \int_{\Gamma_+} \chi_{(d)}(\gamma)\chi_V(\gamma) = \dim \vec{\mathcal{P}}_V^d(\Gamma_+) \ , \tag{3.20}$$

in agreement with Corollaries 3.7 and 3.9.

Next we present two necessary conditions for a representation (ρ, V) of Γ to be self-dual.

Proposition 3.13 *Let Γ be a compact Lie group acting on a finite-dimensional vector space V as a reversing symmetry group. If V is self-dual, then every reversing symmetry has vanishing trace.*

Proof. Let (ρ, V) denote the given representation of Γ and (ρ_σ, V) its dual for a non-trivial homomorphism $\sigma : \Gamma \to \mathbf{Z}_2$. For every $\gamma \in \Gamma$,

$$\operatorname{tr}(\rho(\gamma)) = \operatorname{tr}(\rho_\sigma(\gamma)) = \operatorname{tr}(\sigma(\gamma)\rho(\gamma)) = \sigma(\gamma)\operatorname{tr}(\rho(\gamma)) .$$

If $\gamma \in \Gamma_-$, then $\operatorname{tr}(\rho(\gamma)) = -\operatorname{tr}(\rho(\gamma))$, that is, $\operatorname{tr}(\rho(\gamma)) = 0$. □

Proposition 3.14 *Let Γ be a compact Lie group acting on a finite-dimensional vector space V as a reversing symmetry group. If V is self-dual, then the Hilbert-Poincaré series of $\vec{\mathcal{P}}_V(\Gamma)$ e $\vec{\mathcal{Q}}_V(\Gamma)$ are equal. Moreover, every coefficient of the Hilbert-Poincaré series of $\vec{\mathcal{P}}_V(\Gamma_+)$ is even.*

Proof. By Proposition 3.13, we have $\chi_V(\gamma) = 0$ for all $\gamma \in \Gamma_-$ and so $\dim \vec{\mathcal{P}}^d_V(\Gamma) = \dim \vec{\mathcal{Q}}^d_V(\Gamma)$, for every $d \in \mathbf{N}$. Thus

$$\Psi^\Gamma_V(t) = \sum_{d=0}^{\infty} \dim \vec{\mathcal{P}}^d_V(\Gamma)\, t^d = \sum_{d=0}^{\infty} \dim \vec{\mathcal{Q}}^d_V(\Gamma)\, t^d = \tilde{\Psi}^\Gamma_V(t) .$$

Moreover, from (3.20) it follows that $\dim \vec{\mathcal{P}}^d_V(\Gamma_+) = 2\dim \vec{\mathcal{P}}^d_V(\Gamma)$, which is even for every $d \in \mathbf{N}$. □

Remark 3.15 If V is self-dual, then from Proposition 3.13 and (3.18) it follows that $\chi_{(d)}(\gamma) = 0$ for all $\gamma \in \Gamma_-$ whenever d is odd. Therefore, by a similar argument as applied above to the equivariants and reversible-equivariants, we conclude that

$$\dim \mathcal{P}^d_V(\Gamma) = \dim \mathcal{Q}^d_V(\Gamma), \quad \text{if } d \text{ is odd.}$$

From (3.19), $\dim \mathcal{P}^d_V(\Gamma_+)$ is even whenever d is odd. ◇

We finish this section with some examples.

Example 3.16 ($\Gamma = \mathbf{S}_2$) Consider the action of the group $\Gamma = \mathbf{S}_2$ on $\mathbf{R}^2$ by permutation of the coordinates. This action is generated by the matrix $\delta = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, which we take to be a reversing symmetry. So $\Gamma_+ = \{\mathbf{1}\}$ and $\Gamma_- = \{\delta\}$. Hilbert bases for $\mathcal{P}_{\mathbf{R}^2}(\mathbf{S}_2)$ and $\mathcal{P}_{\mathbf{R}^2}$ are $\{x+y, xy\}$ and $\{x, y\}$, respectively. From (3.19) we have that

$$\dim \mathcal{Q}^1_{\mathbf{R}^2}(\mathbf{S}_2) = \dim \mathcal{Q}^2_{\mathbf{R}^2}(\mathbf{S}_2) = 1, \quad \dim \mathcal{Q}^3_{\mathbf{R}^2}(\mathbf{S}_2) = \dim \mathcal{Q}^4_{\mathbf{R}^2}(\mathbf{S}_2) = 2 .$$

Note that $x^n - y^n$ are anti-invariants for all $n \in \mathbf{N}$. From the identity

$$x^n - y^n \;=\; (x+y)(x^{n-1} - y^{n-1}) - xy(x^{n-2} - y^{n-2})$$

and cumbersome calculations it is possible to show that $\mathcal{Q}_{\mathbf{R}^2}(\mathbf{S}_2)$ is generated by $u = x - y$. However, we delay this proof to the next section by a simple and direct use of Algorithm 4.3.

By Theorem 2.7, the Hilbert-Poincaré series of $\vec{\mathcal{P}}_{\mathbf{R}^2}(\mathbf{S}_2)$ is $\Psi^{\Gamma}_{\mathbf{R}^2}(t) = \frac{1}{(1-t)^2}$. Furthermore, it is easy to check that

$$L \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is an $\mathbf{S}_2$-reversible-equivariant linear isomorphism and so this representation of $\mathbf{S}_2$ on $\mathbf{R}^2$ is self-dual. Therefore, by Proposition 3.14, the Hilbert-Poincaré series of $\vec{\mathcal{Q}}_{\mathbf{R}^2}(\mathbf{S}_2)$ is $\tilde{\Psi}^{\Gamma}_{\mathbf{R}^2}(t) = \frac{1}{(1-t)^2}$. $\diamond$

Example 3.17 ($\Gamma = \mathbb{O}$) Consider the action of the octahedral group $\mathbb{O}$ on $\mathbb{R}^3$ generated by

$$\kappa_x \;=\; \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_x \;=\; \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad S_y \;=\; \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Here, κ_x is a reflection on the plane $x = 0$ and S_x, S_y are rotations of $\pi/2$ about the axis x e y, respectively. Let $\delta = \kappa_x$ act as reversing symmetry and S_x, S_y act as symmetries. Hence $\Gamma_+ = \langle S_x, S_y \rangle \cong \mathbf{S}_4$ and $\Gamma_- = \kappa_x \, \Gamma_+$. Using the software SINGULAR to compute the generators of $\mathcal{P}_{\mathbf{R}^3}(\mathbf{S}_4)$ we find

$$u_1 \;=\; x^2 + y^2 + z^2 \,, \qquad u_2 \;=\; x^2y^2 + y^2z^2 + x^2z^2 \,, \qquad u_3 \;=\; x^2y^2z^2,$$

$$u_4 \;=\; x^3(yz^5 - y^5z) + y^3(x^5z - xz^5) + z^3(xy^5 - x^5y).$$

By Melbourne [20, Lemma A.1] u_1, u_2 and u_3 generate $\mathcal{P}_{\mathbf{R}^3}(\mathbb{O})$. Note that u_4 is an homogeneous $\mathbb{O}$-anti-invariant (of degree 9) and, from (3.19), that

$$\dim \mathcal{Q}^i_{\mathbf{R}^3}(\mathbb{O}) \;=\; 0 \,, \; \forall \, i = 1, \ldots, 8 \qquad \text{and} \qquad \dim \mathcal{Q}^9_{\mathbf{R}^3}(\mathbb{O}) \;=\; 1 \,.$$

Hence, it follows that $\mathcal{Q}_{\mathbf{R}^3}(\mathbb{O})$ admits only one generator of degree 9, which can be taken to be u_4. Finally, using the software GAP to compute the Hilbert-Poincaré series of $\vec{\mathcal{P}}_{\mathbf{R}^3}(\mathbf{S}_4)$ we obtain

$$\begin{aligned} \Psi^{\mathbf{S}_4}_{\mathbf{R}^3}(t) \;&=\; \frac{t - t^2 + t^3}{(1-t)(1-t^2)(1-t^4)} \\ &=\; t + 2t^3 + t^4 + 4t^5 + 2t^6 + 6t^7 + 4t^8 + \cdots \,. \end{aligned}$$

Since $\dim \vec{\mathcal{P}}^1_{\mathbf{R}^3}(\mathbf{S}_4) \;=\; 1$ is odd, it follows from Proposition 3.14 that a such representation of the octahedral group is non self-dual. $\diamond$

4 The Algorithms

In this section we prove the second main result of this paper, namely the algorithmic way to compute anti-invariants and reversible-equivariants. The basic idea is to take advantage of the direct sum decompositions from Corollaries 3.7 and 3.9 to transfer the appropriate basis and generating sets from one ring or module to another. We finish this section by applying these results to some important examples.

4.1 Algorithm for Computing the Anti-Invariants

In Corollary 3.7 we consider the ring $\mathcal{P}_V(\Gamma_+)$ of Γ_+-invariant polynomial functions as a module over its subring $\mathcal{P}_V(\Gamma)$ of Γ-invariant polynomial functions and obtain a direct sum decomposition into two $\mathcal{P}_V(\Gamma)$-modules, where one of them is $\mathcal{P}_V(\Gamma)$ itself and the other is the module $\mathcal{Q}_V(\Gamma)$ of anti-invariant polynomial functions. In this subsection we show how to obtain generating sets of $\mathcal{Q}_V(\Gamma)$ and $\mathcal{P}_V(\Gamma_+)$ as modules over $\mathcal{P}_V(\Gamma)$ from a Hilbert basis of $\mathcal{P}_V(\Gamma_+)$.

We start by introducing the multi-index notation in order to write expressions like (2.1), in more compact way. A *multi-index* α of dimension s is an s-tuple $\alpha = (\alpha_1, \ldots, \alpha_s)$ of non-negative integers. The *degree* of a multi-index α is

$$|\alpha| \ = \ \sum_{i=1}^{s} \alpha_i \ .$$

For $u = (u_1, \ldots, u_s)$ an s-tuple of indeterminates we write

$$u^\alpha \ = \ u_1^{\alpha_1} \ldots u_s^{\alpha_s} \ ,$$

so (2.1) is simply rewritten as

$$f(x) \ = \ \sum_{\alpha} a_\alpha \, u^\alpha(x) \ ,$$

with coefficients $a_\alpha \in \mathbf{R}$. Now if we introduce a new indeterminate u_{s+1}, then a polynomial in the indeterminates u and u_{s+1} may be written as

$$\sum_{\alpha, \alpha_{s+1}} a_{\alpha, \alpha_{s+1}} \, u^\alpha u_{s+1}^{\alpha_{s+1}} \ .$$

Note that this is simply a polynomial in $s+1$ indeterminates with the last indeterminate being explicitly singled out.

Theorem 4.1 *Let Γ be a compact Lie group acting on V. Let $\{u_1, \ldots, u_s\}$ be* *a Hilbert basis of the ring $\mathcal{P}_V(\Gamma_+)$. Set*

$$\tilde{u}_j \;=\; S^{\Gamma}_{\Gamma_+}(u_j)\ .$$

Then $\{\tilde{u}_1, \ldots, \tilde{u}_s\}$ is a generating set of the module $\mathcal{Q}_V(\Gamma)$ over $\mathcal{P}_V(\Gamma)$.

Proof. We need to show that every polynomial function $\tilde{f} \in \mathcal{Q}_V(\Gamma)$ can be written as

$$\tilde{f}(x) \;=\; \sum_{j=1}^{s} p_j(x)\tilde{u}_j(x)\,, \quad \forall\, x \in V\ ,$$

where $p_j \in \mathcal{P}_V(\Gamma)$ and $\tilde{u}_j = S^{\Gamma}_{\Gamma_+}(u_j)$. We prove this by induction on the cardinality s of the set $\{u_1, \ldots, u_s\}$.

Let us fix $\delta \in \Gamma_-$ and let $\{u_1, \ldots, u_s\}$ be a Hilbert basis of $\mathcal{P}_V(\Gamma_+)$. From Proposition 3.6, there exists $f \in \mathcal{P}_V(\Gamma_+)$ such that $S^{\Gamma}_{\Gamma_+}(f) = \tilde{f}$. First we write

$$f(x) \;=\; \sum_{\alpha} a_\alpha\, u^\alpha(x)\ ,$$

with $a_\alpha \in \mathbf{R}$, and compute

$$\tilde{f}(x) \;=\; S^{\Gamma}_{\Gamma_+}(f)(x) \;=\; \frac{1}{2}(f(x) - f(\delta x))\ .$$

Thus, we get that

$$\tilde{f}(x) \;=\; \sum_{\alpha} a_\alpha\,(u^\alpha(x) - u^\alpha(\delta x))\ .$$

• Assume $s = 1$. Then we may write

$$\tilde{f}(x) \;=\; \sum_{i} a_i\,(u^i(x) - u^i(\delta x))\ , \tag{4.1}$$

where $a_i \in \mathbf{R}$ and $i \in \mathbf{N}$. Recall the well-known polynomial identity

$$u^i(x) - u^i(\delta x) \;=\; (u(x) - u(\delta x))\left(\sum_{j=0}^{i-1} u^j(x)u^{(i-1)-j}(\delta x)\right)$$

from which we extract the following polynomial:

$$p_i(x) \;=\; \sum_{j=0}^{i-1} u^j(x)u^{(i-1)-j}(\delta x)\ .$$

We show now that $p_i \in \mathcal{P}_V(\Gamma)$. Since Γ_+ is a normal subgroup of Γ, it follows that for every $\gamma \in \Gamma_+$ there exists $\tilde{\gamma} \in \Gamma_+$ such that

$$u(\delta\gamma x) \;=\; u(\tilde{\gamma}\delta x) \;=\; u(\delta x)\,, \quad \forall\, x \in V\ .$$

Thus for all $\gamma \in \Gamma_+$ we have

$$p_i(\gamma x) \;=\; \sum_{j=0}^{i-1} u^j(\gamma x)\, u^{(i-1)-j}(\delta\gamma x) \;=\; \sum_{j=0}^{i-1} u^j(x)\, u^{(i-1)-j}(\delta x) \;=\; p_i(x)\ .$$

Furthermore, since $\delta^2 \in \Gamma_+$,

$$p_i(\delta x) \;=\; \sum_{j=0}^{i-1} u^j(\delta x)\, u^{(i-1)-j}(\delta^2 x) \;=\; \sum_{j=0}^{i-1} u^j(\delta x)\, u^{(i-1)-j}(x) \;=\; p_i(x)\ .$$

Therefore, (4.1) becomes

$$\tilde{f}(x) \;=\; \sum_i a_i\, p_i(x)(u(x) - u(\delta x)) \;=\; p(x)\tilde{u}(x)\ ,$$

where $p = 2\sum_i a_i\, p_i \in \mathcal{P}_V(\Gamma)$.

- Assume that for all sets $\{u_1, \ldots, u_\ell\}$ with $1 \leqslant \ell \leqslant s$ we have

$$\sum_\alpha a_\alpha \left(u^\alpha(x) - u^\alpha(\delta x)\right) \;=\; \sum_{j=1}^{\ell} p_j(x)\tilde{u}_j(x), \tag{4.2}$$

where $p_j \in \mathcal{P}_V(\Gamma)$ and $\alpha \in \mathbf{N}^\ell$.

Now consider the set $\{u_1, \ldots, u_\ell\} \cup \{u_{\ell+1}\}$ and let $\tilde{f} \in \mathcal{Q}_V(\Gamma)$. Then we may write

$$\begin{aligned}\tilde{f}(x) &= \sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}}(u^\alpha(x)u_{\ell+1}^{\alpha_{\ell+1}}(x) - u^\alpha(\delta x)u_{\ell+1}^{\alpha_{\ell+1}}(\delta x)) \\ &= \sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}}\Big[u^\alpha(x)(u_{\ell+1}^{\alpha_{\ell+1}}(x) - u_{\ell+1}^{\alpha_{\ell+1}}(\delta x)) + u_{\ell+1}^{\alpha_{\ell+1}}(\delta x)(u^\alpha(x) - u^\alpha(\delta x))\Big],\end{aligned}$$

with u^α, $u_{\ell+1}^{\alpha_{\ell+1}} \in \mathcal{P}_V(\Gamma_+)$, $\alpha \in \mathbf{N}^\ell$ and $\alpha_{\ell+1} \in \mathbf{N}$. By Corollary 3.7 we can decompose

$$u^\alpha(x) \;=\; v_\alpha(x) + w_\alpha(x) \qquad \text{and} \qquad u_{\ell+1}^{\alpha_{\ell+1}}(x) \;=\; v_{\alpha_{\ell+1}}(x) + w_{\alpha_{\ell+1}}(x),$$

with v_α, $v_{\alpha_{\ell+1}} \in \mathcal{P}_V(\Gamma)$ and w_α, $w_{\alpha_{\ell+1}} \in \mathcal{Q}_V(\Gamma)$. Then

$$u^\alpha(\delta x) \;=\; v_\alpha(x) - w_\alpha(x) \qquad \text{and} \qquad u_{\ell+1}^{\alpha_{n+1}}(\delta x) \;=\; v_{\alpha_{\ell+1}}(x) - w_{\alpha_{\ell+1}}(x)\ .$$

By subtracting, we get

$$u^\alpha(x) - u^\alpha(\delta x) \;=\; 2w_\alpha(x) \qquad \text{and} \qquad u_{\ell+1}^{\alpha_{\ell+1}}(x) - u_{\ell+1}^{\alpha_{\ell+1}}(\delta x) \;=\; 2w_{\alpha_{\ell+1}}(x),$$

Therefore,

$$
\begin{aligned}
\tilde{f}(x) &= \sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}} \Big[(v_\alpha(x)+w_\alpha(x))2w_{\alpha_{\ell+1}}(x) + (v_{\alpha_{\ell+1}}(x)-w_{\alpha_{\ell+1}}(x))2w_\alpha(x)\Big] \\
&= \sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}} (2v_\alpha(x)w_{\alpha_{\ell+1}}(x) + 2v_{\alpha_{\ell+1}}(x)w_\alpha(x)) \\
&= \sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}} \Big[v_\alpha(x)(u_{\ell+1}^{\alpha_{\ell+1}}(x) - u_{\ell+1}^{\alpha_{\ell+1}}(\delta x)) + v_{\alpha_{\ell+1}}(x)(u^\alpha(x)-u^\alpha(\delta x))\Big] \\
&= \sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}} v_\alpha(x)(u_{\ell+1}^{\alpha_{\ell+1}}(x) - u_{\ell+1}^{\alpha_{\ell+1}}(\delta x)) \\
&\qquad\qquad + \sum_{\alpha_{\ell+1}} v_{\alpha_{\ell+1}}(x) \left(\sum_\alpha a_{\alpha,\alpha_{\ell+1}}(u^\alpha(x)-u^\alpha(\delta x))\right) .
\end{aligned}
$$

By the induction hypothesis (4.2) we can write

$$
(u_{\ell+1}^{\alpha_{\ell+1}}(x) - u_{\ell+1}^{\alpha_{\ell+1}}(\delta x)) \ = \ p_{\ell+1,\alpha_{\ell+1}}(x)\tilde{u}_{\ell+1}(x)
$$

and

$$
\sum_\alpha a_{\alpha,\alpha_{\ell+1}}(u^\alpha(x)-u^\alpha(\delta x)) \ = \ \sum_{j=1}^{\ell} p_{j,\alpha_{\ell+1}}(x)\tilde{u}_j(x) \ ,
$$

with $p_{\ell+1,\alpha_{\ell+1}},\ p_{j,\alpha_{\ell+1}} \in \mathcal{P}_V(\Gamma)$. Then

$$
\begin{aligned}
\tilde{f}(x) &= \sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}} v_\alpha(x)(p_{\ell+1,\alpha_{\ell+1}}(x)\tilde{u}_{\ell+1}(x)) + \sum_{\alpha_{\ell+1}} v_{\alpha_{\ell+1}}(x)\left(\sum_{j=1}^{\ell} p_{j,\alpha_{\ell+1}}(x)\tilde{u}_j(x)\right) \\
&= \left(\sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}} v_\alpha(x) p_{\ell+1,\alpha_{\ell+1}}(x)\right)\tilde{u}_{\ell+1}(x) + \sum_{j=1}^{\ell}\left(\sum_{\alpha_{\ell+1}} v_{\alpha_{\ell+1}}(x)p_{j,\alpha_{\ell+1}}(x)\right)\tilde{u}_j(x).
\end{aligned}
$$

But now we observe that

$$
p_j \ = \ \sum_{\alpha_{\ell+1}} v_{\alpha_{\ell+1}}\, p_{j,\alpha_{\ell+1}} \ \in \mathcal{P}_V(\Gamma)
$$

and

$$
p_{\ell+1} \ = \ \sum_{\alpha,\alpha_{\ell+1}} a_{\alpha,\alpha_{\ell+1}} v_\alpha\, p_{\ell+1,\alpha_{\ell+1}} \ \in \mathcal{P}_V(\Gamma) \ .
$$

Therefore,

$$
\tilde{f}(x) \ = \ \sum_{j=1}^{\ell+1} p_j(x)\tilde{u}_j(x) \ . \qquad \square
$$

Corollary 4.2 *Let Γ be a compact Lie group acting on V. Let $\{u_1,\ldots,u_s\}$ be a Hilbert basis of the ring $\mathcal{P}_V(\Gamma_+)$. Set $\tilde{u}_i = S^{\Gamma}_{\Gamma_+}(u_i)$, for $i=1,\ldots,s$. Then $\{1,\tilde{u}_1,\ldots,\tilde{u}_s\}$ is a generating set of the module $\mathcal{P}_V(\Gamma_+)$ over $\mathcal{P}_V(\Gamma)$.*

Here is the procedure to find a generating set of the module of anti-invariants $\mathcal{Q}_V(\Gamma)$ over the ring $\mathcal{P}_V(\Gamma)$:

Algorithm 4.3 (Generating Set of Anti-Invariants)

INPUT: $\cdot$ *compact Lie group* $\Gamma \subset \mathbf{O}(n)$
$\cdot$ *normal subgroup* $\Gamma_+ \subset \Gamma$ *of index 2*
$\cdot$ $\delta \in \Gamma \setminus \Gamma_+$
$\cdot$ *Hilbert basis* $\{u_1, \ldots, u_s\}$ *of* $\mathcal{P}_V(\Gamma_+)$
OUTPUT: *generating set* $\{\tilde{u}_1, \ldots, \tilde{u}_s\}$ *of the module of anti-invariants* $\mathcal{Q}_V(\Gamma)$ *over the ring* $\mathcal{P}_V(\Gamma)$
PROCEDURE:
for i from 1 *to s do*
$\tilde{u}_i(x) := \frac{1}{2}(u_i(x) - u_i(\delta x))$
end
return $\{\tilde{u}_1, \ldots, \tilde{u}_s\}$

Remark 4.4 It is a straightforward consequence of the algorithm above that the $\mathbf{S}_2$-anti-invariant $u = x - y$ in Example 3.16 generates $\mathcal{Q}_{\mathbf{R}^2}(\mathbf{S}_2)$ as a $\mathcal{P}_{\mathbf{R}^2}(\mathbf{S}_2)$-module. Also, the $\mathbb{O}$-anti-invariant

$$u_4 \;=\; x^3(yz^5 - y^5z) + y^3(x^5z - xz^5) + z^3(xy^5 - x^5y).$$

in Example 3.17 generates $\mathcal{Q}_{\mathbf{R}^3}(\mathbb{O})$ as a $\mathcal{P}_{\mathbf{R}^3}(\mathbb{O})$-module. $\diamond$

4.2 Algorithm for Computing the Reversible-Equivariants

In Corollary 3.9 we consider the module $\vec{\mathcal{P}}_V(\Gamma_+)$ of Γ_+-equivariant polynomial mappings as a module over the ring $\mathcal{P}_V(\Gamma)$ of invariant polynomials and obtain a direct sum decomposition into two $\mathcal{P}_V(\Gamma)$-modules, where one of them is the module $\vec{\mathcal{P}}_V(\Gamma)$ of purely equivariant polynomial mappings and the other is the module $\vec{\mathcal{Q}}_V(\Gamma)$ of reversible-equivariant polynomial mappings. In this subsection we show how to obtain generating sets of $\vec{\mathcal{P}}_V(\Gamma_+)$ and $\vec{\mathcal{Q}}_V(\Gamma)$ as modules over $\mathcal{P}_V(\Gamma)$ from a Hilbert basis of $\mathcal{P}_V(\Gamma_+)$ together with a generating set of $\vec{\mathcal{P}}_V(\Gamma_+)$ as a module over $\mathcal{P}_V(\Gamma_+)$.

In particular, we achieve our ultimate goal, which is to show how the construction of a generating set for the module $\vec{\mathcal{Q}}_V(\Gamma)$ of reversible-equivariant polynomial mappings over $\mathcal{P}_V(\Gamma)$ can be reduced to a problem in standard invariant theory, whose solution is well known in several important cases.

Lemma 4.5 *Let* Γ *be a compact Lie group acting on* V*. Let* $\{u_1, \ldots, u_s\}$ *be a Hilbert basis of* $\mathcal{P}_V(\Gamma_+)$*. Let* $\{\tilde{u}_0 \equiv 1, \tilde{u}_1, \ldots, \tilde{u}_s\}$ *be the generating set of the module* $\mathcal{P}_V(\Gamma_+)$ *over the ring* $\mathcal{P}_V(\Gamma)$ *obtained from* $\{u_1, \ldots, u_s\}$ *as in Corollary 4.2 and* $\{H_0, \ldots, H_r\}$ *a generating set of the module* $\vec{\mathcal{P}}_V(\Gamma_+)$ *over the ring*

$\mathcal{P}_V(\Gamma_+)$. *Then*

$$\{H_{ij} \;=\; \tilde{u}_i H_j : i = 0, \ldots, s\,;\; j = 0, \ldots, r\}$$

is a generating set of the module $\vec{\mathcal{P}}_V(\Gamma_+)$ *over the ring* $\mathcal{P}_V(\Gamma)$.

Proof. Let $G \in \vec{\mathcal{P}}_V(\Gamma_+)$. Then

$$G \;=\; \sum_{j=0}^{r} p_j\, H_j \;, \tag{4.3}$$

with $p_j \in \mathcal{P}_V(\Gamma_+)$. Since $\{\tilde{u}_0, \ldots, \tilde{u}_s\}$ is a generating set of the module $\mathcal{P}_V(\Gamma_+)$ over the ring $\mathcal{P}_V(\Gamma)$, it follows that

$$p_j = \;\sum_{i=0}^{s} p_{ij}\, \tilde{u}_i \;, \tag{4.4}$$

with $p_{ij} \in \mathcal{P}_V(\Gamma)$. From (4.3) and (4.4) we get

$$G \;=\; \sum_{j=0}^{r} \left(\sum_{i=0}^{s} p_{ij}\, \tilde{u}_i \right) H_j \;=\; \sum_{i,j=0}^{s,r} p_{ij} \left(\tilde{u}_i H_j \right) \;,$$

as desired. $\square$

Lemma 4.6 *Let* Γ *be a compact Lie group acting on* V. *Let* $\{H_{00}, \ldots, H_{sr}\}$ *be a generating set of the module* $\vec{\mathcal{P}}_V(\Gamma_+)$ *over the ring* $\mathcal{P}_V(\Gamma)$ *given as in Lemma 4.5. Then*

$$\{\tilde{H}_{ij} \;=\; \vec{S}^{\Gamma}_{\Gamma_+}(H_{ij}) : i = 0, \ldots, s\,;\; j = 0, \ldots, r\}$$

is a generating set of the module $\vec{\mathcal{Q}}_V(\Gamma)$ *over the ring* $\mathcal{P}_V(\Gamma)$.

Proof. Let $\{H_{00}, \ldots, H_{sr}\}$ be a generating set of the $\mathcal{P}_V(\Gamma)$-module $\vec{\mathcal{P}}_V(\Gamma_+)$ and let $\tilde{G} \in \vec{\mathcal{Q}}_V(\Gamma)$. From Proposition 3.8, there exists $G \in \vec{\mathcal{P}}_V(\Gamma_+)$ such that $\tilde{G} \;= \vec{S}^{\Gamma}_{\Gamma_+}(G)$. Now write

$$G \;=\; \sum_{i,j=0}^{s,r} p_{ij}\, H_{ij}$$

with $p_{ij} \in \mathcal{P}_V(\Gamma)$. Applying $\vec{S}^{\Gamma}_{\Gamma_+}$ on both sides, we get

$$\tilde{G} \;=\; \sum_{i,j=0}^{s,r} p_{ij}\, \vec{S}^{\Gamma}_{\Gamma_+}(H_{ij}) \;=\; \sum_{i,j=0}^{s,r} p_{ij}\, \tilde{H}_{ij} \;,$$

as desired. $\square$

It is now immediate from the two lemmas above the following result:

Theorem 4.7 *Let Γ be a compact Lie group acting on V. Let $\{u_1, \ldots, u_s\}$ be a Hilbert basis of $\mathcal{P}_V(\Gamma_+)$ and $\{H_0, \ldots, H_r\}$ be a generating set of the module $\vec{\mathcal{P}}_V(\Gamma_+)$ over the ring $\mathcal{P}_V(\Gamma_+)$. Let $\{\tilde{u}_0 \equiv 1, \tilde{u}_1, \ldots, \tilde{u}_s\}$ be the generating set of the module $\mathcal{P}_V(\Gamma_+)$ over the ring $\mathcal{P}_V(\Gamma)$ obtained from $\{u_1, \ldots, u_s\}$ as in Corollary 4.2. Then*

$$\{\tilde{H}_{ij} \;=\; \vec{S}^{\Gamma}_{\Gamma_+}(\tilde{u}_i H_j) : i = 0, \ldots, s\,;\; j = 0, \ldots, r\}$$

is a generating set of the module $\vec{\mathcal{Q}}_V(\Gamma)$ over $\mathcal{P}_V(\Gamma)$.

Here is the procedure to find a generating set of the module $\vec{\mathcal{Q}}_V(\Gamma)$ over the ring $\mathcal{P}_V(\Gamma)$:

Algorithm 4.8 (Generating Set of Reversible-Equivariants)

INPUT: · *compact Lie group* $\Gamma \subset \mathbf{O}(n)$
· *normal subgroup* $\Gamma_+ \subset \Gamma$ *of index* 2
· $\delta \in \Gamma \setminus \Gamma_+$
· *Hilbert basis* $\{u_1, \ldots, u_s\}$ *of* $\mathcal{P}_V(\Gamma_+)$
· *generating set* $\{H_0, \ldots, H_r\}$ *of* $\vec{\mathcal{P}}_V(\Gamma_+)$ *over* $\mathcal{P}_V(\Gamma_+)$

OUTPUT: *generating set* $\{\tilde{H}_{00}, \ldots, \tilde{H}_{sr}\}$ *of the module of reversible-equivariants* $\vec{\mathcal{Q}}_V(\Gamma)$ *over the ring* $\mathcal{P}_V(\Gamma)$

PROCEDURE:

$\tilde{u}_0(x) := 1$
for i *from* 1 *to* s *do*
  $\tilde{u}_i(x) := \frac{1}{2}(u_i(x) - u_i(\delta x))$
  for j *from* 0 *to* r *do*
    $H_{0j}(x) := H_j(x)$
    $H_{ij}(x) := \tilde{u}_i(x)\, H_j(x)$
    $\tilde{H}_{ij}(x) := \frac{1}{2}(H_{ij}(x) - \delta^{-1} H_{ij}(\delta x))$
  end
end
return $\{\tilde{H}_{00}, \ldots, \tilde{H}_{sr}\}$

4.3 Examples

We illustrate now the methods obtained in this paper with some examples. Before presenting the examples, we should point out that despite our methods are completely general, they are only necessary when the representation at hand is *non self-dual*. Otherwise, when the representation is self-dual, there is a much more efficient procedure which takes advantage of the existence of a reversible-equivariant linear isomorphism $L : V \to V$ to obtain the generators of $\vec{\mathcal{Q}}_V(\Gamma)$ from those of $\vec{\mathcal{P}}_V(\Gamma)$ by the "pull-back" of L. See Baptistelli and Manoel [4] for details.

Example 4.9 **($\Gamma = \mathbf{Z}_2$)** Consider the action of the group $\Gamma = \mathbf{Z}_2$ on $\mathbf{R}^2$ generated by the reflection $\delta = \left(\begin{smallmatrix}-1 & 0\\ 0 & -1\end{smallmatrix}\right)$, which we take to be a reversing symmetry. So $\Gamma_+ = \{\mathbf{1}\}$ and $\Gamma_- = \{\delta\}$. It is clear that $u_1(x,y) = x$ and $u_2(x,y) = y$ constitute a Hilbert basis of $\mathcal{P}_{\mathbf{R}^2}$. By Algorithm 4.3 we have that

$$\tilde{u}_1(x,y) \;=\; x \qquad \text{and} \qquad \tilde{u}_2(x,y) \;=\; y$$

constitute a generating set of $\mathcal{Q}_{\mathbf{R}^2}(\mathbf{Z}_2)$. Now set $\tilde{u}_0(x,y) = 1$ and let

$$H_0(x,y) \;=\; (1,0) \qquad \text{and} \qquad H_1(x,y) \;=\; (0,1)$$

be a generating set of the module $\vec{\mathcal{P}}_{\mathbf{R}^2}$ over the ring $\mathcal{P}_{\mathbf{R}^2}$. By Algorithm 4.8 we get

$$\begin{aligned}
H_{00}(x,y) &= H_0(x,y) \;=\; (1,0)\,,\\
H_{01}(x,y) &= H_1(x,y) \;=\; (0,1)\,,\\
H_{10}(x,y) &= \tilde{u}_1(x,y)H_0(x,y) \;=\; (x,0)\,,\\
H_{11}(x,y) &= \tilde{u}_1(x,y)H_1(x,y) \;=\; (0,x)\,,\\
H_{20}(x,y) &= \tilde{u}_2(x,y)H_0(x,y) \;=\; (y,0)\,,\\
H_{21}(x,y) &= \tilde{u}_2(x,y)H_1(x,y) \;=\; (0,y)\,.
\end{aligned}$$

First, we note that $H_{ij} \in \vec{\mathcal{P}}_{\mathbf{R}^2}(\mathbf{Z}_2)$ for $i = 1,2$. Hence $\tilde{H}_{ij} \equiv 0$ for $i = 1,2$ and

$$\tilde{H}_{00}(x,y) \;=\; (1,0) \qquad \text{and} \qquad \tilde{H}_{01}(x,y) \;=\; (0,1)$$

constitute a generating set of the module $\vec{\mathcal{Q}}_{\mathbf{R}^2}(\mathbf{Z}_2)$ over the ring $\mathcal{P}_{\mathbf{R}^2}(\mathbf{Z}_2)$.

$\diamond$

Example 4.10 **($\Gamma = \mathbf{Z}_2 \oplus \mathbf{Z}_2$)** Consider the action of the group $\Gamma = \mathbf{Z}_2 \oplus \mathbf{Z}_2$ on $\mathbf{R}^2$ generated by the reflections

$$\kappa_1 \;=\; \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} \qquad \text{and} \qquad \kappa_2 \;=\; \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}.$$

Let us take $\delta = \kappa_1 \in \Gamma_-$ and $\kappa_2 \in \Gamma_+$. So $\Gamma_+ = \mathbf{Z}_2(\kappa_2) = \{\mathbf{1}, \kappa_2\}$ and $\Gamma_- = \{\kappa_1, -\mathbf{1}\}$. It is well known that $u_1(x,y) = x^2$ and $u_2(x,y) = y$ constitute a Hilbert basis of $\mathcal{P}_{\mathbf{R}^2}(\mathbf{Z}_2(\kappa_2))$. By Algorithm 4.3 we have that

$$\tilde{u}_1(x,y) \;=\; \tfrac{1}{2}(u_1(x,y) - u_1(\kappa_1(x,y))) \;=\; 0,$$

$$\tilde{u}_2(x,y) \;=\; \tfrac{1}{2}(u_2(x,y) - u_2(\kappa_1(x,y))) \;=\; y,$$

that is, $\mathcal{Q}_{\mathbf{R}^2}(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$ is generated by $\{y\}$ over $\mathcal{P}_{\mathbf{R}^2}(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$. Now set $\tilde{u}_0(x,y) = 1$ and take $\{H_0(x,y) = (x,0), H_1(x,y) = (0,1)\}$ as the generating set of

$\vec{\mathcal{P}}_{\mathbf{R}^2}(\mathbf{Z}_2(\kappa_2))$. By Algorithm 4.8 we get

$$\begin{aligned}
H_{00}(x,y) &= H_0(x,y) = (x,0)\,, \\
H_{01}(x,y) &= H_1(x,y) = (0,1)\,, \\
H_{11}(x,y) &= \tilde{u}_1(x,y)H_0(x,y) = (0,0)\,, \\
H_{12}(x,y) &= \tilde{u}_1(x,y)H_1(x,y) = (0,0)\,, \\
H_{20}(x,y) &= \tilde{u}_2(x,y)H_0(x,y) = (xy,0)\,, \\
H_{21}(x,y) &= \tilde{u}_2(x,y)H_1(x,y) = (0,y)\;.
\end{aligned}$$

Note that H_{00}, $H_{21} \in \vec{\mathcal{P}}_{\mathbf{R}^2}(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$ and so $\tilde{H}_{00} = \tilde{H}_{21} \equiv 0$. Hence,

$$\tilde{H}_{01}(x,y) = (0,1) \qquad \text{and} \qquad \tilde{H}_{20}(x,y) = (xy,0)$$

constitute a generating set of the module $\vec{\mathcal{Q}}_{\mathbf{R}^2}(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$ over $\mathcal{P}_{\mathbf{R}^2}(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$.

If we choose $\Gamma_+ = \mathbf{Z}_2(\kappa_1) = \{\mathbf{1}, \kappa_1\}$ and $\Gamma_- = \{\kappa_2, -\mathbf{1}\}$, then $\mathcal{Q}_{\mathbf{R}^2}(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$ is generated by $\{x\}$ and $\vec{\mathcal{Q}}_{\mathbf{R}^2}(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$ is generated by $\{(1,0),(0,xy)\}$ over $\mathcal{P}_{\mathbf{R}^2}(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$. $\diamond$

Example 4.11 ($\Gamma = \mathbf{D}_n$) Consider the action of the dihedral group $\Gamma = \mathbf{D}_n$ on $\mathbf{C}$ generated by the complex conjugation κ and the rotation $R_{2\pi/n}$:

$$\kappa z = \bar{z} \qquad \text{and} \qquad R_{\frac{2\pi}{n}} z = e^{i\frac{2\pi}{n}} z\;.$$

Let us take $\delta = R_{2\pi/n} \in \Gamma_-$ and $\kappa \in \Gamma_+$. We observe that this is only possible for n even (see [3, p. 249]). Then

$$\Gamma_+ = \langle \kappa, R^2_{\frac{2\pi}{n}} \rangle = \mathbf{D}_{\frac{n}{2}}\;.$$

It is well known (Golubitsky et al. [14, Section XII 4]) that

$$u_1(z) = z\bar{z} \qquad \text{and} \qquad u_2(z) = z^{\frac{n}{2}} + \bar{z}^{\frac{n}{2}}$$

constitute a Hilbert basis for $\mathcal{P}_{\mathbf{C}}(\mathbf{D}_{n/2})$. Application of Algorithm 4.3 gives

$$\begin{aligned}
\tilde{u}_1(z) &= \tfrac{1}{2}(u_1(z) - u_1(e^{i\frac{2\pi}{n}}z)) = 0 \\
\tilde{u}_2(z) &= \tfrac{1}{2}(u_2(z) - u_2(e^{i\frac{2\pi}{n}}z)) = z^{\frac{n}{2}} + \bar{z}^{\frac{n}{2}}\;,
\end{aligned}$$

that is, $\tilde{u}_2$ is the only generator of $\mathcal{Q}_{\mathbf{C}}(\mathbf{D}_n)$. In order to obtain the generators of $\vec{\mathcal{Q}}_{\mathbf{C}}(\mathbf{D}_n)$ we set $\tilde{u}_0(z) = 1$ and pick a generating set for $\vec{\mathcal{P}}(\mathbf{D}_{n/2})$ (Golubitsky et al. [14, Section XII 5]):

$$H_0(z) = z \qquad \text{and} \qquad H_1(z) = \bar{z}^{\frac{n}{2}-1}\;.$$

Now applying Algorithm 4.8 we get

$$
\begin{aligned}
H_{00}(z) &= H_0(z) = z\,, \\
H_{01}(z) &= H_1(z) = \bar{z}^{\frac{n}{2}-1}\,, \\
H_{10}(z) &= \tilde{u}_1(z)H_0(z) = 0\,, \\
H_{11}(z) &= \tilde{u}_1(z)H_1(z) = 0\,, \\
H_{20}(z) &= \tilde{u}_2(z)H_0(z) = z^{\frac{n}{2}+1} + (z\bar{z})\bar{z}^{\frac{n}{2}-1}\,, \\
H_{21}(z) &= \tilde{u}_2(z)H_1(z) = (z\bar{z})^{\frac{n}{2}-1}z + \bar{z}^{n-1}\,.
\end{aligned}
$$

Note that H_{00} and $H_{21} \in \vec{\mathcal{P}}_{\mathbf{C}}(\mathbf{D}_n)$ and so $\tilde{H}_{00} = \tilde{H}_{21} \equiv 0$. Moreover,

$$
\begin{aligned}
\tilde{H}_{01}(z) &= \bar{z}^{\frac{n}{2}-1}\,, \\
\tilde{H}_{20}(z) &= z^{\frac{n}{2}+1} + (z\bar{z})\bar{z}^{\frac{n}{2}-1}\,.
\end{aligned}
$$

Since $z\bar{z}$ is a $\mathbf{D}_n$-invariant, we obtain $\{\bar{z}^{n/2-1}, z^{n/2+1}\}$ as a generating set of the module $\vec{\mathcal{Q}}_{\mathbf{C}}(\mathbf{D}_n)$ over the ring $\mathcal{P}_{\mathbf{C}}(\mathbf{D}_n)$. Let us observe that Baptistelli and Manoel [3, p. 249] use a different approach to deal with this example, obtaining the same set of generators by direct calculations. $\diamond$

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References

[1] F. Antoneli, A.P.S. Dias, P.C. Matthews, Invariants, Equivariants and Characters in Symmetric Bifurcation Theory. *Proceedings of the Royal Society of Edinburgh. Section A, Mathematics*, To appear.

[2] V.I. Arnol'd, Reversible Systems. *Nonlinear and Turbulent Processes in Physics* **3** (1984) 1161–1174.

[3] P.H. Baptistelli, M.G. Manoel, Some Results on Reversible-Equivariant Vector Fields. *Cadernos de Matemática* **6** (2005) 237–263.

[4] P.H. Baptistelli, M.G. Manoel, The Classification of Reversible-Equivariant Steady-State Bifurcations on Self-Dual Spaces. *Cadernos de Matemática* **8** (2007) 43–71.

[5] G.D. Birkhoff, The Restricted Problem of Three Bodies. *Rend. Circ. Mat. Palermo* **39** (1915) 265–334.

[6] T. Bröcker, T tom Dieck, *Representations of Compact Lie Groups.* Graduate Texts in Mathematics **98**, Springer-Verlag, New York, 1995.

[7] P.-L. Buono, J.S.W. Lamb, R.M. Roberts, Bifurcation and Branching of Equilibria of Reversible Equivariant Vector Fields. In preparation.

[8] R.L Devaney, Reversible Diffeomorphisms and Flows. *Trans. Amer. Math. Soc.* **218** (1976) 89–113.

[9] R. DeVogelaere, On the Structure of Periodic Solutions of Conservative Systems, with Applications. *Contribution to the Theory of Nonlinear Oscillations* **4** (1958) 53–84. Princeton University Press, Princeton.

[10] W. Fulton, J. Harris, *Representation Theory. A First Course.* Graduate texts in Mathematics **129**, Readings in Mathematics, Springer-Verlag, New York, 1991.

[11] The GAP Group: GAP – Groups, Algorithms, and Programming. Version 4.4.9, 2006. (`http://www.gap-system.org`)

[12] K. Gatermann, Semi-invariants, Equivariants and Algorithms. *Appl. Algebra Eng. Commun. Comput.* **7** (1996) 105–124.

[13] K. Gatermann, *Computer Algebra Methods for Equivariant Dynamical Systems* Lecture Notes in Mathematics **1728**, Springer-Verlag, Berlin-Heidelberg, 2000.

[14] M. Golubitsky, I. Stewart, D. Schaeffer, *Singularities e Groups in Bifurcation Theory, Vol. II.* Appl. Math. Sci. **69**, Springer-Verlag, New York, 1985.

[15] R. Goodman, N.R. Wallach, *Representations and Invariants of the Classical Groups.* Cambridge University Press, Cambridge, 2001.

[16] G.-M. Greuel, G. Pfister, H. Schönemann, SINGULAR 3.0. A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern (2005). (`http://www.singular.uni-kl.de`)

[17] G. James, M. Liebeck, *Representations and Characters of Groups. 2nd ed.* Cambridge University Press, New York, 2001.

[18] J.S.W. Lamb, J.A.G. Roberts, Time-reversal Symmetry in Dynamical Systems: A Survey. *Physica* **D 112** (1998) 1–39.

[19] J.S.W Lamb, R.M. Roberts, Reversible Equivariant Linear Systems. *J. Diff. Eq.* **159** (1999) 239–279.

[20] I. Melbourne, A Singularity Theory Analysis of Bifurcation Problems with Octahedral Symmetry. *Dynamics and Stability of Systems* **1** (4) (1986) 293–321.

[21] D.H. Sattinger, *Group Theoretic Methods in Bifurcation Theory.* Lecture Notes in Mathematics **762**, Springer-Verlag, New York, 1979.

[22] M.B. Sevryuk, *Reversible Systems.* Lecture Notes in Mathematics **1211**, Springer-Verlag, New York, 1986.

[23] L. Smith, Free Modules of Relative Invariants and Some Rings of Invariants that are Cohen-Macaulay. *Proc. Amer. Math. Soc.* **134** (2006) 2205–2212.

[24] L. Smith, Polynomial Invariants of Finite Groups. A Survey of Recent Developments. *Bull. Amer. Math. Soc. (NS)* **34** (1997) 211–250.

[25] R.P. Stanley, Relative Invariants of Finite Groups Generated by Pseudo-reflections. *J. of Algebra* **49** (1977) 134–148.

[26] B. Sturmfels, *Algorithms in Invariant Theory.* Springer-Verlag, New York, 1993.