

# Hopf Bifurcation in Coupled Cell Networks with Interior Symmetries

Fernando Antoneli<sup>1</sup>      Ana Paula S. Dias<sup>1,2</sup>  
Rui C. Paiva<sup>1,2</sup>

<sup>1</sup>Centro de Matemática da Universidade do Porto\*

<sup>2</sup>Departamento de Matemática Pura

Faculdade de Ciências da Universidade do Porto  
Rua do Campo Alegre, 687, 4169-007 Porto, Portugal

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## Abstract

We consider an important class of non-symmetric networks that lies between the class of general networks and the class of symmetric networks, where group theoretic methods still apply – namely, networks admitting “interior symmetries”. The main result of this paper is the full analogue of the Equivariant Hopf Theorem for networks with symmetries. We extend the result of Golubitsky, Pivato and Stewart (Interior symmetry and local bifurcation in coupled cell networks, *Dynamical Systems* **19** (4) (2004) 389–407) to obtain states whose linearizations on certain subsets of cells, near bifurcation, are superpositions of synchronous states with states having spatio-temporal symmetries.

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# 1 Introduction

Recently, a new framework for the dynamics of networks has been proposed, with particular attention to patterns of synchrony and associated bifurcations. See Stewart, Golubitsky and Pivato [4, 11], Golubitsky, Nicol and Stewart [3], and Golubitsky, Stewart and Török [9]. Here, a network is represented by a directed graph whose nodes and edges are classified according to associated labels or ‘types’. The nodes (or ‘cells’) of a network  $\mathcal{G}$  represent dynamical systems, and the edges (‘arrows’) represent couplings. Cells with the same label have ‘identical’ internal dynamics; arrows with the same label correspond to ‘identical’ couplings. The ‘input set’ of a cell is the set edges directed to that cell. Label-preserving bijections between ‘input sets’ of cells are called ‘input isomorphisms’ and they capture the ‘local’ symmetries of the network. The set of all these ‘local’ symmetries has the structure of a groupoid. (A groupoid is an algebraic structure similar to a group, except that products of elements may not always be defined).

Coupled cell systems are dynamical systems compatible with the architecture or topology of a directed graph representing the network. Formally, they are defined in the following way. Each cell  $c$  is equipped with a phase space  $P_c$ , and the total phase space of the network is the cartesian product  $P = \prod_c P_c$ . A vector field  $f$  is called ‘admissible’ if its component  $f_c$  for cell  $c$  depends only on variables associated with the input set of  $c$  (domain condition), and if its components for cells  $c, d$  that have isomorphic input sets are identical up to a suitable permutation of the relevant variables (pull-back condition).

In the study of network dynamics there is an important class of networks, namely, networks that possess a group of symmetries. In this context there is a group of permutations of the cells (and arrows) that preserves the network structure (including cell-types and arrow-types) and its action on  $P$  is by permutation of cell coordinates. Moreover, the coupled cell systems (ODE’s) are of the form

$$\frac{dx}{dt} = f(x)$$

where the vector field  $f$  is smooth ( $C^\infty$ ) and satisfies

$$f(\gamma x) = \gamma f(x) \quad \forall x \in X, \gamma \in \Gamma$$

That is,  $f$  is ‘equivariant’ under the action of the group  $\Gamma$  on phase space  $P$ .

The theory of equivariant dynamical systems (see Golubitsky *et al.* [6, 8]) can be applied to such dynamical systems. In this theory, a central role is played by the ‘fixed-point spaces’ of subgroups  $\Sigma \subseteq \Gamma$ , defined by

$$\text{Fix}(\Sigma) = \{x \in X : \sigma x = x \ \forall \sigma \in \Sigma\}$$

Fixed-point spaces have the important property of flow invariance: they are invariant under *every* smooth equivariant vector field  $f$ , so that

$$f(\text{Fix}(\Sigma)) \subseteq \text{Fix}(\Sigma)$$

See [6, Lemma XIII 2.1] or [8, Theorem 1.17] for the simple proof and the implications for symmetry-breaking. In this context, there are two main local bifurcation theorems. The *Equivariant Branching Lemma* (see Golubitsky *et al.* [8, Theorem XIII 3.3]) proves the existence of certain branches of symmetry-breaking steady states; the *Equivariant Hopf Theorem* (see Golubitsky *et al.* [8, Theorem XVI 4.1]) proves the existence of certain branches of spatio-temporal symmetry-breaking time-periodic states.

In between the class of general networks and the class of symmetric networks lies an interesting class of non-symmetric networks, where group theoretic methods still apply, namely, networks admitting ‘interior symmetries’. In this case there is a group of permutations of a subset  $\mathcal{S}$  of the cells (and edges directed to  $\mathcal{S}$ ) that partially preserves the network structure (including cell-types and edges-types) and its action on  $P$  is by permutation of cell coordinates. In other words, the cells in  $\mathcal{S}$  together with all the edges directed to them form a subnetwork which possesses a non-trivial group of symmetry  $\Sigma_{\mathcal{S}}$ . For example, network  $\mathcal{G}_1$  (Figure 1 (left)) has exact  $\mathbf{S}_3$ -symmetry, whereas network  $\mathcal{G}_2$  (Figure 1 (right)) has  $\mathbf{S}_3$ -interior symmetry. This notion was introduced and investigated by Golubitsky, Pivato and Stewart [4]. The presence of interior symmetries places some restrictions on the structure of the network.

The local bifurcations from a synchronous equilibrium can be classified into two types: ‘synchrony-breaking’ bifurcations and ‘synchrony-preserving’ bifurcations. The synchrony-breaking bifurcations occur when a synchronous state loses stability and bifurcates to a state with less synchrony. Such bifurcations can be considered to be a generalisation of symmetry-breaking bifurcations in symmetric coupled cell systems. Golubitsky, Pivato and Stewart [4] provided analogues of the Equivariant Branching Lemma and the Equivariant Hopf Theorem for coupled cell systems with interior symmetries. The

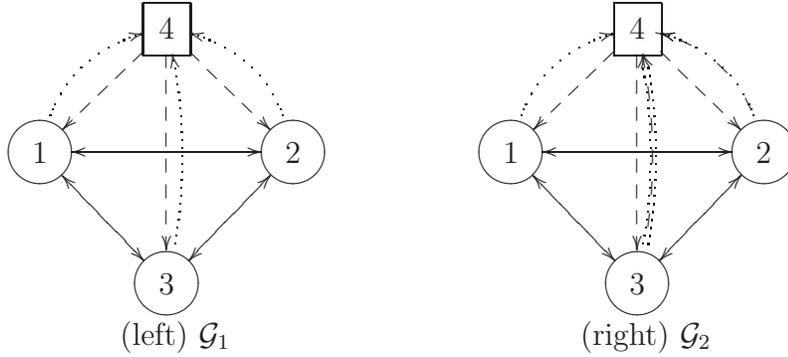


Figure 1: (left) Network  $\mathcal{G}_1$  with exact  $\mathbf{S}_3$ -symmetry. (right) Network  $\mathcal{G}_2$  with  $\mathbf{S}_3$ -interior symmetry.

analogue of the Equivariant Branching Lemma is a natural generalisation of the symmetric case, but the analogue of the Equivariant Hopf Theorem has novel and rather restrictive features. In particular, instead of proving the existence of states with certain spatio-temporal symmetries, they prove the existence of states whose linearizations on certain subsets of cells, near bifurcation, are superpositions of synchronous states with states having ‘spatial symmetries’.

The main result of this paper is the full analogue of the Equivariant Hopf Theorem for networks with symmetries (Theorem 4.8). We extend the result of Golubitsky, Pivato and Stewart [4] to obtain states whose linearizations on certain subsets of cells, near bifurcation, are superpositions of synchronous states with states having *spatio-temporal symmetries*, that is, corresponding to “interiorly”  $\mathbf{C}$ -axial subgroups of  $\Sigma_{\mathcal{G}} \times \mathbf{S}^1$ . This new version of the Hopf Theorem with interior symmetries includes the previous as a special case and is in complete analogy with the Equivariant Hopf Theorem (see Theorem 4.8). Our proof uses a modification of the Lyapunov-Schmidt reduction to arrive at a situation where the proof of the Standard Hopf Bifurcation Theorem can be applied. This completes the program of generalising the two main results from equivariant bifurcation theory to the class of networks with interior symmetries.

**Structure of the Paper** Section 2 recalls the formal definition of a coupled cell network and the associated dynamical systems, and states some basic fea-

tures, including the concept of a balanced equivalence relation (colouring). We also discuss the symmetry group of a network. Section 3 recalls the definition of interior symmetry given by Golubitsky, Pivato and Stewart [4] and gives an equivalent condition, in terms of symmetries of a subnetwork, which in some cases (no multiple edges and no self-connections) amounts to finding the symmetries of the subnetwork. We also analyse the structure of these networks and discuss some features of the admissible vector fields associated to such class of networks. Section 4 recalls the notion of synchrony-breaking bifurcation in coupled cell networks. Then we specialise to networks with interior symmetries where group theoretic concepts play a significant role, focusing on the important case of codimension-one synchrony-breaking bifurcations. The main part of this section gives the statement and proof of the Interior Symmetry-Breaking Hopf Bifurcation Theorem (Theorem 4.8) for networks with interior symmetries. We illustrate all the concepts and results by a running example of the simplest network with  $\mathbf{S}_3$ -interior symmetry and the closely related network with exact  $\mathbf{S}_3$ -symmetry (see Figure 1). Finally, we present a numerical simulation of the states provided by Theorem 4.8 in the case of our running example.

## 2 Network Formalism

First, we recall the formal definition of a coupled cell network and the associated dynamical systems. For a survey, overview and examples, see [7]. The initial definition of coupled cell network [11] was modified in [9] to permit multiple arrows and self-connections, which turns out to have major advantages. More recently, Stewart [10] extended the formalism introduced in [9] to include a large class of infinite networks – the so called *networks of finite type*.

### 2.1 Coupled Cell Networks

In this paper we consider *finite networks* and so employ the ‘finite multi-arrow’ formalism for consistency with the existing literature.

**Definition 2.1** ([9]) *A coupled cell network  $\mathcal{G}$  comprises:*

- (a) A finite set  $\mathcal{C}$  of *nodes* or *cells*.

- (b) An equivalence relation  $\sim_C$  on cells in  $\mathcal{C}$ , called *cell-equivalence*. The *type* or *cell label* of cell  $c$  is its  $\sim_C$ -equivalence class.
- (c) A finite set  $\mathcal{E}$  of *edges* or *arrows*.
- (d) An equivalence relation  $\sim_E$  on edges in  $\mathcal{E}$ , called *edge-equivalence* or *arrow-equivalence*. The *type* or *coupling label* of edge  $e$  is its  $\sim_E$ -equivalence class.
- (e) Two maps  $\mathcal{H} : \mathcal{E} \rightarrow \mathcal{C}$  and  $\mathcal{T} : \mathcal{E} \rightarrow \mathcal{C}$ . For  $e \in \mathcal{E}$  we call  $\mathcal{H}(e)$  the *head* of  $e$  and  $\mathcal{T}(e)$  the *tail* of  $e$ .

We also require a *consistency condition*:

- (f) Equivalent arrows have equivalent tails and heads:

$$\mathcal{H}(e_1) \sim_C \mathcal{H}(e_2) \quad \mathcal{T}(e_1) \sim_C \mathcal{T}(e_2)$$

for all  $e_1, e_2 \in \mathcal{E}$  with  $e_1 \sim_E e_2$ . ◇

**Example 2.2** We can represent abstract networks by labelled directed graphs. Figure 1 shows two examples. Here the node labels, drawn as the three circles and the square, indicate the cells; the symbols show that cells 1, 2, 3 have the same type, whereas cell 4 is different, in both cases. In the network  $\mathcal{G}_1$  there are three types of edge label, whereas in the network  $\mathcal{G}_2$  there are five types of edge label, drawn as different styles of arrows. The tail and head of each edge is, respectively, indicated by the absence or presence of a tip on one end of the arrow. When an arrow between cells  $c$  and  $d$  is drawn with tips in both ends then it represents two arrows of the same type with opposite orientation between cells  $c$  and  $d$ . ◇

## 2.2 Input Sets and the Symmetry Groupoid

Associated with each cell  $c \in \mathcal{C}$  is a canonical set of edges, namely, those that represent couplings into cell  $c$ :

**Definition 2.3** ([9]) If  $c \in \mathcal{C}$  then the *input set* of  $c$  is the finite set of edges directed to  $c$ ,

$$I(c) = \{e \in \mathcal{E} : \mathcal{H}(e) = c\} \tag{2.1}$$

◇

**Definition 2.4** ([9]) The relation  $\sim_I$  of *input equivalence* on  $\mathcal{C}$  is defined by  $c \sim_I d$  if and only if there exists a bijection

$$\beta : I(c) \rightarrow I(d) \quad (2.2)$$

such that for every  $i \in I(c)$ ,

$$i \sim_E \beta(i) \quad (2.3)$$

Any such bijection  $\beta$  is called an *input isomorphism* from cell  $c$  to cell  $d$ . The set  $B(c, d)$  denotes the collection of all input isomorphisms from cell  $c$  to cell  $d$ . The union

$$\mathcal{B}_{\mathcal{G}} = \bigcup_{c, d \in \mathcal{C}} B(c, d) \quad (2.4)$$

is the *symmetry groupoid* of the network  $\mathcal{G}$ . A coupled cell network is *homogeneous* if all input sets are isomorphic.  $\diamond$

The groupoid operation on  $\mathcal{B}_{\mathcal{G}}$  is composition of maps, and in general the composition  $\beta\alpha$  is defined only when  $\alpha \in B(a, b)$  and  $\beta \in B(b, c)$  for cells  $a, b, c$ . This is why  $\mathcal{B}_{\mathcal{G}}$  need not to be a group.

**Example 2.5** In our running examples, shown in Figure 1, it is easy to see that both networks have only two input isomorphism classes of cells:  $\{1, 2, 3\}$  and  $\{4\}$ . The input sets of cells 1, 2, 3 are isomorphic, since each one of them contains three edges two of them drawn as a solid arrow with a circle in the tail and one of them drawn as a dashed arrow with a square in the tail.  $\diamond$

### 2.3 Admissible Vector Fields

We now explain how to interpret such diagrams as in Figure 1 as being representative of a class of vector fields.

For each cell in  $\mathcal{C}$  choose a *cell phase space*  $P_c$ , which we assume to be a nonzero finite-dimensional real vector space. We require

$$c \sim_{\mathcal{C}} d \quad \Rightarrow \quad P_c = P_d$$

and in this case we employ the same coordinate systems on  $P_c$  and  $P_d$ . The *total phase space* is then

$$P = \prod_{c \in \mathcal{C}} P_c$$

with a cell-based coordinate system

$$x = (x_c)_{c \in \mathcal{C}}$$

If  $\mathcal{D} \subseteq \mathcal{C}$  is any finite set of cells then we write

$$P_{\mathcal{D}} = \prod_{d \in \mathcal{D}} P_d$$

and

$$x_{\mathcal{D}} = (x_{c_1}, \dots, x_{c_\ell})$$

where  $x_c \in P_c$ .

For any  $\beta \in B(c, d)$  we define the *pull-back map*

$$\beta^* : P_{\mathcal{T}(I(d))} \rightarrow P_{\mathcal{T}(I(c))}$$

by

$$(\beta^* z)_{\mathcal{T}(i)} = z_{\mathcal{T}(\beta(i))} \tag{2.5}$$

for all  $i \in I(c)$  and  $z \in P_{\mathcal{T}(I(d))}$ .

We use pull-back maps to relate different components of a vector field associated with a given coupled cell network. Specifically, the class of vector fields that are encoded by a coupled cell network is given by the following definition.

**Definition 2.6** ([9]) A map  $f : P \rightarrow P$  is  $\mathcal{G}$ -*admissible* if:

- (a) *Domain condition:* For all  $c \in \mathcal{C}$  the component  $f_c(x)$  depends only on the internal phase space variables  $x_c$  and the coupling phase space variables  $x_{\mathcal{T}(I(c))}$ ; that is, there exists  $\hat{f}_c : P_c \times P_{\mathcal{T}(I(c))} \rightarrow P_c$  such that

$$f_c(x) = \hat{f}_c(x_c, x_{\mathcal{T}(I(c))}) \tag{2.6}$$

- (b) *Pull-back condition:* For all  $c, d \in \mathcal{C}$  and  $\beta \in B(c, d)$

$$\hat{f}_d(x_d, x_{\mathcal{T}(I(d))}) = \hat{f}_c(x_d, \beta^* x_{\mathcal{T}(I(d))}) \tag{2.7}$$

for all  $x \in P$ . ◇

**Example 2.7** For the networks  $\mathcal{G}_1$  and  $\mathcal{G}_2$  of Figure 1 the cell phase spaces  $P_1, P_2$  and  $P_3$  are identical and equal to  $\mathbf{R}^k$ , whereas  $P_4 = \mathbf{R}^l$ . The general form of the admissible vector fields (ODE's) encoded by the network  $\mathcal{G}_1$  is

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_2, x_3}, x_4) \\ \dot{x}_2 &= f(x_2, \overline{x_3, x_1}, x_4) \\ \dot{x}_3 &= f(x_3, \overline{x_1, x_2}, x_4) \\ \dot{x}_4 &= g(x_4, \overline{x_1, x_2, x_3}) \end{aligned} \tag{2.8}$$

where  $x_i \in \mathbf{R}^k$  ( $i = 1, 2, 3$ ),  $x_4 \in \mathbf{R}^l$ ,  $f : \mathbf{R}^{3k} \times \mathbf{R}^l \rightarrow \mathbf{R}^k$  is a smooth map, invariant under permutation of the second and third arguments and  $g : \mathbf{R}^{3k} \times \mathbf{R}^l \rightarrow \mathbf{R}^l$  is a smooth map, invariant under any permutation of the last three arguments. The general form of the admissible vector fields (ODE's) associated with the network  $\mathcal{G}_2$  is

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_2, x_3}, x_4) \\ \dot{x}_2 &= f(x_2, \overline{x_3, x_1}, x_4) \\ \dot{x}_3 &= f(x_3, \overline{x_1, x_2}, x_4) \\ \dot{x}_4 &= g(x_4, x_1, x_2, x_3) \end{aligned} \tag{2.9}$$

where  $x_i \in \mathbf{R}^k$  ( $i = 1, 2, 3$ ),  $x_4 \in \mathbf{R}^l$ ,  $f : \mathbf{R}^{3k} \times \mathbf{R}^l \rightarrow \mathbf{R}^l$  is a smooth map, invariant under permutation of the second and third argument and  $g : \mathbf{R}^{3k} \times \mathbf{R}^l \rightarrow \mathbf{R}^l$  is a general smooth map.  $\diamond$

## 2.4 Balanced Equivalence Relations

An equivalence relation  $\bowtie$  on  $\mathcal{C}$  determines a unique partition of  $\mathcal{C}$  into  $\bowtie$ -equivalence classes, which can be interpreted as a colouring of  $\mathcal{C}$  in which  $\bowtie$ -equivalent cells receive the same colour. Conversely, any partition (colouring) determines a unique equivalence relation. The corresponding *polydiagonal* is

$$\Delta_{\bowtie} = \{x \in P : c \bowtie d \Rightarrow x_c = x_d\} \tag{2.10}$$

A subspace  $V$  of  $P$  is called *admissibly flow-invariant* if  $f(V) \subset V$  for all admissible vector field  $f$  on  $P$ .

**Definition 2.8** ([9]) An equivalence relation  $\bowtie$  on  $\mathcal{C}$  is *balanced* if for every  $c, d \in \mathcal{C}$  with  $c \bowtie d$ , there exists  $\beta \in B(c, d)$  such that  $\mathcal{T}(i) \bowtie \mathcal{T}(\beta(i))$

for all  $i \in I(c)$ . The associated colouring is called a *balanced colouring*. In particular,  $B(c, d) \neq \emptyset$  implies  $c \sim_I d$ . Hence, balanced equivalence relations refine input equivalence.  $\diamond$

A crucial property of balanced equivalence relations is that they define admissibly flow-invariant subspaces, and conversely:

**Theorem 2.9 (Stewart *et al.* [11])** *Let  $\bowtie$  be an equivalence relation on a coupled cell network. Then  $\Delta_{\bowtie}$  is admissibly flow-invariant if and only if  $\bowtie$  is balanced.*

The proof of the above result for finite networks is given in [9, 11] and for networks of finite type in [10]. The dynamical implication of such flow-invariance is that  $\bowtie$  determines a *robust pattern of synchrony*: there exist trajectories  $x(t)$  of the ODE such that

$$c \bowtie d \Rightarrow x_c(t) = x_d(t) \forall t \in \mathbf{R}$$

Such trajectories arise when initial conditions  $x(0)$  lie in  $\Delta_{\bowtie}$ . Then the entire trajectory, for all positive and negative time, lies in  $\Delta_{\bowtie}$  and is a trajectory of the restriction  $f|_{\Delta_{\bowtie}}$ . The associated dynamics can be steady-state, periodic, even chaotic, depending on  $f$  and its restriction to  $\Delta_{\bowtie}$ . An example of synchronised chaos generated by this mechanism can be found in [7].

Since there is always a canonical balanced relation  $\sim_I$  on every network, let  $\Delta_I$  denote polydiagonal subspace of  $P$  associated to the input equivalence relation  $\sim_I$ , that is,

$$\Delta_I = \{x \in P : c \sim_I d \Rightarrow x_c = x_d\}$$

Then  $\Delta_I$  is a flow invariant subspace. Solution of admissible vector fields contained in  $\Delta_I$  represent the states of highest degree of synchrony allowed by the network.

**Remark 2.10** Whenever self-connections or multiple arrows do not occur it will be convenient to revert to the formalism of [11], but now considered as a specialisation of the multi-arrow formalism. Since no two distinct arrows have the same head and tail, we can identify an arrow  $e$  with the pair of cells  $(\mathcal{T}(e), \mathcal{H}(e))$ . Now the set  $\mathcal{E}$  of arrows identifies with a subset of  $\mathcal{C} \times \mathcal{C} \setminus \{(c, c) : c \in \mathcal{C}\}$ . Similarly the input set  $I(c)$  can be identified with the set of all tail cells of arrows  $e$  that have  $c$  as head cell.  $\diamond$

**Example 2.11** We continue with our running examples, the networks  $\mathcal{G}_1$  and  $\mathcal{G}_2$  of Figure 1. There is an equivalence relation  $\bowtie$  for which  $1 \bowtie 2$ ; its equivalence classes are  $\{1, 2\}$ ,  $\{3\}$  and  $\{4\}$ . The corresponding polydiagonal is

$$\Delta_{\bowtie} = \{x \in P : x_1 = x_2\} = \{(x, x, y, z)\}$$

On this subspace the differential equations become

$$\begin{aligned} \dot{x} &= f(x, \overline{x}, \overline{y}, z) \\ \dot{x} &= f(x, \overline{y}, \overline{x}, z) \\ \dot{y} &= f(y, \overline{x}, \overline{x}, z) \\ \dot{z} &= g(z, x, x, y) \end{aligned} \tag{2.11}$$

Since the first two equations are identical (recall that the bar over  $x, y$  means that they can be interchanged),  $\Delta_{\bowtie}$  is invariant under all admissible vector fields. The relation  $\bowtie$  is balanced. The only condition to verify is that cells 1 and 2, which are  $\bowtie$ -equivalent but distinct, have input sets that are isomorphic by an isomorphism that preserves  $\bowtie$ -equivalence classes for both networks. In both networks the input sets are:

$$I(1) = \{(2, 1), (3, 1), (4, 1)\} \quad \text{and} \quad I(2) = \{(1, 2), (3, 2), (4, 2)\}$$

where  $(c, d)$  denotes an arrow with tail  $c$  and head  $d$  (see Remark 2.10). The bijection  $\beta : I(1) \rightarrow I(2)$  with  $\beta((2, 1)) = (1, 2)$ ,  $\beta((3, 1)) = (3, 2)$  and  $\beta((4, 1)) = (4, 2)$  is an input isomorphism that preserves  $\bowtie$ -equivalence classes since  $1 \bowtie 2$ ,  $3 \bowtie 3$  and  $4 \bowtie 4$ . That is,  $\bowtie$  is balanced as claimed. There are two other balanced equivalence relations (different from  $\sim_I$ ) on the networks  $\mathcal{G}_1$  and  $\mathcal{G}_2$ . In one of them the equivalence classes are  $\{2, 3\}$ ,  $\{1\}$  and  $\{4\}$ . In the other the equivalence classes are  $\{1, 3\}$ ,  $\{2\}$  and  $\{4\}$ .  $\diamond$

## 2.5 Symmetry Groups of Networks

We now consider symmetries of networks in the group-theoretic ('global') sense.

**Definition 2.12** ([1]) Let  $\mathcal{G}$  be a network. A *symmetry* of  $\mathcal{G}$  consists of a pair of bijections  $\gamma_C : \mathcal{C} \rightarrow \mathcal{C}$  and  $\gamma_E : \mathcal{E} \rightarrow \mathcal{E}$  where  $\gamma_C$  preserves input equivalence and  $\gamma_E$  preserves edge equivalence, that is, for all  $c \in \mathcal{C}$  and  $e \in \mathcal{E}$ ,

$$\gamma_C(c) \sim_I c \quad \text{and} \quad \gamma_E(e) \sim_E e \tag{2.12}$$

In addition, the two bijections must satisfy the consistency conditions

$$\gamma_C(\mathcal{H}(e)) = \mathcal{H}(\gamma_E(e)) \quad \text{and} \quad \gamma_C(\mathcal{T}(e)) = \mathcal{T}(\gamma_E(e)) \quad (2.13)$$

for all  $e \in \mathcal{E}$ . The set of all  $\gamma = (\gamma_C, \gamma_E)$  forms a finite group  $\text{Aut}(\mathcal{G})$  called the *symmetry group* of the network of  $\mathcal{G}$ .  $\diamond$

Observe that a symmetry  $\gamma$  preserves input sets in a natural sense. Because of the way input sets are defined in the multi-arrow formalism the precise relation is

$$\gamma_E(I(c)) = I(\gamma_C(c))$$

where  $\gamma = (\gamma_C, \gamma_E) \in \text{Aut}(\mathcal{G})$ .

**Remark 2.13** When the network  $\mathcal{G}$  has no self-connections and multi-arrows there is a simplification of the notion of symmetry due to the following observation. Given a vertex permutation  $\gamma_C$ , there is a unique edge permutation  $\gamma_E$  satisfying the consistency condition (2.13), that is,  $\gamma_E$  is implicitly defined by  $\gamma_C$  since, by Remark 2.10, each arrow  $e$  can be identified with a pair of cells  $(\mathcal{T}(e), \mathcal{H}(e))$ . Thus a *symmetry* of  $\mathcal{G}$  is given by a permutation  $\gamma$  of  $\mathcal{C}$  such that

- (a)  $\gamma(c) \sim_I c$  for all  $c \in \mathcal{C}$ .
- (b)  $(\gamma(a), \gamma(b)) \in \mathcal{E} \Leftrightarrow (a, b) \in \mathcal{E}$ .
- (c)  $(\gamma(a), \gamma(b)) \sim_E (a, b) \quad \forall (a, b) \in \mathcal{E}$ .

In this case, the group  $\text{Aut}(\mathcal{G})$  of symmetries of the network  $\mathcal{G}$  is a subgroup of the group  $\text{Sym}(\mathcal{C})$  of permutations on the set of cells of the network. We shall adopt this convention throughout the remainder of the paper whenever the network under consideration has no self-connections and multi-arrows.  $\diamond$

**Example 2.14** Since the networks  $\mathcal{G}_1$  and  $\mathcal{G}_2$  of our running example of Figure 1 do not have multiple arrows and self-connections Remark 2.13 applies. The group  $\mathbf{S}_3 \subset \mathbf{S}_4$  consisting of the transpositions  $(1\ 2)$ ,  $(1\ 3)$ ,  $(2\ 3)$ , the 3-cycle permutations  $(1\ 2\ 3)$ ,  $(1\ 3\ 2)$  and the identity is the symmetry group of the network  $\mathcal{G}_1$ . Observe that cell 4 is fixed by the symmetry group. On the other hand, the network  $\mathcal{G}_2$  has only the identity permutation as a symmetry because the arrows  $(1, 4)$ ,  $(2, 4)$  and  $(3, 4)$  are all different amongst each other.  $\diamond$

This last example shows that the definition of symmetry of a network is very rigid. In the next section we will generalise the definition of symmetry of a network by introducing the notion of *interior symmetry*. In this new context the network  $\mathcal{G}_2$  of our example admits an action of the permutation group  $\mathbf{S}_3$  as a group of interior symmetries. This corresponds to the symmetry group of the subnetwork of  $\mathcal{G}_2$  obtained by ignoring the arrows  $(1, 4)$ ,  $(2, 4)$  and  $(3, 4)$  of  $\mathcal{G}_2$ .

### 3 Interior Symmetry

We present the notion of interior symmetry following Golubitsky *et al.* [4] and give an alternative characterisation in terms of the symmetries of a sub-network.

#### 3.1 Interior Symmetry Groups of Networks

**Definition 3.1** ([4]) Let  $\mathcal{G}$  be a coupled cell network. Let  $\mathcal{S} \subseteq \mathcal{C}$  be a subset of cells and put  $I(\mathcal{S}) = \{e \in \mathcal{E} : \mathcal{H}(e) \in \mathcal{S}\}$ . A pair of bijections  $\sigma_C : \mathcal{C} \rightarrow \mathcal{C}$  and  $\sigma_E : \mathcal{E} \rightarrow \mathcal{E}$  is an *interior symmetry* of  $\mathcal{G}$  (on the subset  $\mathcal{S}$ ) if:

- (a)  $\sigma_C : \mathcal{C} \rightarrow \mathcal{C}$  is an input equivalence preserving permutation which is the identity map on the complement  $\mathcal{C} \setminus \mathcal{S}$  of  $\mathcal{S}$  in  $\mathcal{C}$ ,
- (b)  $\sigma_E : \mathcal{E} \rightarrow \mathcal{E}$  is an edge equivalence preserving permutation which is the identity map on the complement  $\mathcal{E} \setminus I(\mathcal{S})$  of  $I(\mathcal{S})$  in  $\mathcal{E}$ ,
- (c) the consistency condition

$$\sigma_C(\mathcal{H}(e)) = \mathcal{H}(\sigma_E(e)) \quad \text{and} \quad \sigma_C(\mathcal{T}(e)) = \mathcal{T}(\sigma_E(e)) \quad (3.1)$$

is satisfied for every  $e \in I(\mathcal{S})$ .

The set of all interior symmetries of  $\mathcal{G}$  (on the subset  $\mathcal{S}$ ) forms a finite group  $\Sigma_{\mathcal{S}}$  called the *group of interior symmetries* of  $\mathcal{G}$  (on the subset  $\mathcal{S}$ ).  $\diamond$

Note that in Definition 3.1 if  $\mathcal{S} = \mathcal{C}$  then  $\Sigma_{\mathcal{S}} = \text{Aut}(\mathcal{G})$ . Hence, the definition of interior symmetry of a network is a generalisation of a symmetry of a network. That is why we refer to the elements of  $\text{Aut}(\mathcal{G})$  as *global symmetries* of  $\mathcal{G}$ . The most interesting case is when  $\text{Aut}(\mathcal{G})$  is trivial but  $\Sigma_{\mathcal{S}}$  is non-trivial for some  $\mathcal{S}$ .

**Example 3.2** We continue with our running example, the two networks  $\mathcal{G}_1$  and  $\mathcal{G}_2$  of Figure 1. We have seen that the network  $\mathcal{G}_1$  is  $\mathbf{S}_3$ -symmetric and the network  $\mathcal{G}_2$  has only the trivial symmetry. However, the group of permutations

$$\mathbf{S}_3 = \{\text{id}, (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$$

is the group of interior symmetries of the network  $\mathcal{G}_2$  on the subset  $\mathcal{S} = \{1, 2, 3\}$ . Observe that all elements of  $\mathbf{S}_3$  fix cell 4 and

$$I(\mathcal{S}) = \{(1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2), (4, 1), (4, 2), (4, 3)\}$$

If we assume that the permutations in  $\mathbf{S}_3$  act as identity on the set of arrows

$$\mathcal{E} \setminus I(\mathcal{S}) = \{(1, 4), (2, 4), (3, 4)\}$$

then  $\mathbf{S}_3$  is the group of interior symmetries of the network  $\mathcal{G}_2$  on the subset  $\mathcal{S} = \{1, 2, 3\}$ .  $\diamond$

There is an alternative characterisation of interior symmetries using the notion of symmetry of a network. The main idea is the following: by “ignoring” some arrows we find a subnetwork whose symmetry group is the group of interior symmetries of the original network.

Let us be more precise. Given a coupled cell network  $\mathcal{G}$  and a subset  $\mathcal{S} \subset \mathcal{C}$  of cells define  $\mathcal{G}_{\mathcal{S}} = (\mathcal{C}, I(\mathcal{S}), \sim_C, \sim_E)$  to be the subnetwork of  $\mathcal{G}$  whose set of cells is  $\mathcal{C}$  (together with its cell-equivalence  $\sim_C$ ) and whose set of arrows is  $I(\mathcal{S})$ . The edge-equivalence on  $I(\mathcal{S})$  is obtained by the restriction of the edge-equivalence  $\sim_E$  on  $\mathcal{E}$ .

**Proposition 3.3** *Let  $\mathcal{G}$  be a coupled cell network and  $\mathcal{S} \subset \mathcal{C}$  be a subset of cells of the set of cells of  $\mathcal{G}$ . Consider the network  $\mathcal{G}_{\mathcal{S}}$  as defined above. Then the group of interior symmetries of the network  $\mathcal{G}$  (on the subset  $\mathcal{S}$ ) can be canonically identified with the group of symmetries of the network  $\mathcal{G}_{\mathcal{S}}$ :*

$$\Sigma_{\mathcal{S}} \cong \text{Aut}(\mathcal{G}_{\mathcal{S}})$$

**Proof.** We start by proving that  $\Sigma_{\mathcal{S}}$  can be canonically identified with a subset of  $\text{Aut}(\mathcal{G}_{\mathcal{S}})$ . Let  $\sigma = (\sigma_C, \sigma_E) \in \Sigma_{\mathcal{S}}$  be an interior symmetry of  $\mathcal{G}$  (on the subset  $\mathcal{S}$ ) as in Definition 3.1. Then, because both  $\sigma_C$  and  $\sigma_E$  are, respectively, the identity map on  $\mathcal{C} \setminus \mathcal{S}$  and  $\mathcal{E} \setminus I(\mathcal{S})$ , it follows that  $\sigma$  is a

symmetry of  $\mathcal{G}_{\mathcal{S}}$ , according to Definition 2.12. Now we show that the above identification is surjective. Let  $\gamma = (\gamma_C, \gamma_E) \in \text{Aut}(\mathcal{G}_{\mathcal{S}})$  be a symmetry of  $\mathcal{G}_{\mathcal{S}}$  (in the sense of Definition 2.12), that is,  $\gamma_E$  is a permutation on the set  $I(\mathcal{S})$ . Now we can extend  $\gamma_E$  to a permutation  $\sigma_E$  on  $\mathcal{E}$  which acts as identity on  $\mathcal{E} \setminus I(\mathcal{S})$ . The pair  $\sigma = (\sigma_C, \sigma_E)$  where  $\sigma_C = \gamma_C$  is an interior symmetry of  $\mathcal{G}$  (on the subset  $\mathcal{S}$ ) according to Definition 3.1.  $\square$

The characterisation of interior symmetry provided by Proposition 3.3 is particularly useful when the network does not have multiple arrows and/or self-connections, since by Remark 2.13, a symmetry is simply a permutation on the set vertices of the underlying graph.

**Example 3.4** Consider the two networks  $\mathcal{G}_1$  and  $\mathcal{G}_2$  of Figure 1. Let  $\mathcal{S} = \{1, 2, 3\}$ . Note that the network  $\mathcal{G}_{\mathcal{S}}$  obtained from  $\mathcal{G}_1$  is the same as the one obtained from  $\mathcal{G}_2$ . In Figure 2 we show these three networks. Observe that for the three networks the sets of arrows coming from the set  $\mathcal{S} = \{1, 2, 3\}$  and directed to the complement  $\mathcal{C} \setminus \mathcal{S} = \{4\}$  are different  $\diamond$

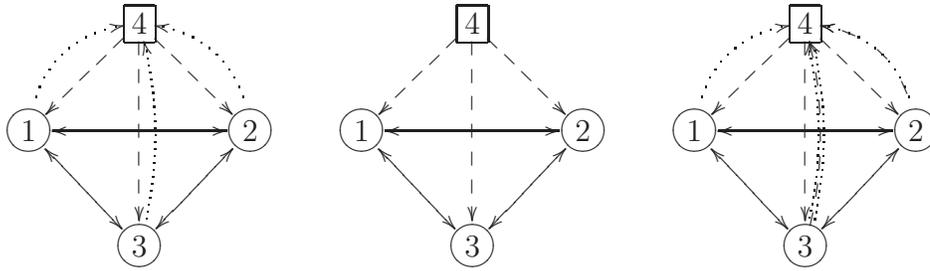


Figure 2: (left) Network  $\mathcal{G}_1$ . (center) Network  $\mathcal{G}_{\mathcal{S}}$  where  $\mathcal{S} = \{1, 2, 3\}$ . (right) Network  $\mathcal{G}_2$ .

Let  $\mathcal{G}$  be a network and fix a phase space  $P$ . Suppose that  $\mathcal{G}$  admits non-trivial interior symmetries  $\Sigma_{\mathcal{S}}$  on a subset of cells  $\mathcal{S}$ . Then we can decompose the phase space  $P$  as a cartesian product  $P = P_{\mathcal{S}} \times P_{\mathcal{C} \setminus \mathcal{S}}$  where

$$P_{\mathcal{S}} = \prod_{s \in \mathcal{S}} P_s \quad \text{and} \quad P_{\mathcal{C} \setminus \mathcal{S}} = \prod_{c \in \mathcal{C} \setminus \mathcal{S}} P_c$$

For any  $x \in P$  we write  $x = (x_{\mathcal{S}}, x_{\mathcal{C} \setminus \mathcal{S}})$  where  $x_{\mathcal{S}} \in P_{\mathcal{S}}$  and  $x_{\mathcal{C} \setminus \mathcal{S}} \in P_{\mathcal{C} \setminus \mathcal{S}}$ . If  $\sigma = (\sigma_C, \sigma_E) \in \Sigma_{\mathcal{S}}$  then  $\sigma_C$  permutes the cells of  $\mathcal{S}$  and induces an action of

$\Sigma_{\mathcal{S}}$  on  $P$  by permuting the cell coordinates

$$\sigma(x_c)_{c \in \mathcal{C}} = (x_{\sigma_C^{-1}(c)})_{c \in \mathcal{C}}$$

Since  $\Sigma_{\mathcal{S}}$  fixes all cells in  $\mathcal{C} \setminus \mathcal{S}$  we can write

$$\sigma(x_{\mathcal{S}}, x_{\mathcal{C} \setminus \mathcal{S}}) = (\sigma x_{\mathcal{S}}, x_{\mathcal{C} \setminus \mathcal{S}}) \quad (3.2)$$

As in the case of symmetric networks we can construct (some) balanced equivalence relations on a network  $\mathcal{G}$  from subgroups of the interior symmetry group. Suppose that  $K \subseteq \Sigma_{\mathcal{S}}$  is a subgroup. Then

$$\text{Fix}_P(K) = \{(x_{\mathcal{S}}, x_{\mathcal{C} \setminus \mathcal{S}}) : \sigma x_{\mathcal{S}} = x_{\mathcal{S}} \ \forall \sigma \in K\}$$

Define the relation  $\bowtie_K$  on the cells in  $\mathcal{C}$  by

$$c \bowtie_K d \iff \exists \sigma = (\sigma_C, \sigma_E) \in K : \sigma_C(c) = d$$

Then the  $\bowtie_K$ -classes are the  $K$ -orbits on the cells in  $\mathcal{S}$  and the corresponding polydiagonal is

$$\Delta_K = \Delta_{\bowtie_K} = \text{Fix}_P(K)$$

The following proposition from Golubitsky *et al.* [4, Proposition 1, p. 397] is fundamental in the study of coupled cell networks with interior symmetries.

**Proposition 3.5 (Golubitsky *et al.* [4])** *Let  $\mathcal{G}$  be a network admitting a non-trivial interior symmetry group  $\Sigma_{\mathcal{S}}$  and fix a phase space  $P$ . Let  $K$  be any subgroup of  $\Sigma_{\mathcal{S}}$ . Then  $\bowtie_K$  is a balanced relation on  $\mathcal{G}$ . In particular,  $\text{Fix}_P(K)$  is a flow invariant subspace for all  $\mathcal{G}$ -admissible vector fields.*

**Proof.** Let  $s_1$  and  $s_2$  be two cells on the same  $K$ -orbit. Then there exists an element  $\sigma = (\sigma_C, \sigma_E)$  of  $K$  such that  $\sigma_C(s_1) = s_2$  and by the consistency condition (3.1) it follows that the restriction

$$\sigma_E|_{I(s_1)} : I(s_1) \rightarrow I(s_2)$$

is an input isomorphism. Since the  $\bowtie_K$ -equivalence classes are exactly the  $K$ -orbits on  $\mathcal{C}$  it follows that the input isomorphism  $\sigma_E|_{I(s_1)}$  preserves the  $\bowtie_K$  equivalence relation. Hence, by Theorem 2.9 it follows that  $\Delta_H = \text{Fix}_P(K)$  is a flow invariant subspace for all  $\mathcal{G}$ -admissible vector fields.  $\square$

**Example 3.6** Consider the networks  $\mathcal{G}_1$  and  $\mathcal{G}_2$  of Figure 1 and fix a phase space  $P$  for both networks. There are two non-trivial conjugacy classes of subgroups of  $\mathbf{S}_3$ . The first conjugacy class is represented for example by the subgroup generated by a 3-cycle

$$\mathbf{Z}_3 = \langle (1\ 2\ 3) \rangle$$

The associated balanced relation has two equivalence classes  $\{1, 2, 3\}$  and  $\{4\}$  given by the three orbits of  $\mathbf{Z}_3$  on the set of cells  $\mathcal{C}$ . The fixed-point subspace of  $\mathbf{Z}_3$  is

$$\text{Fix}_P(\mathbf{Z}_3) = \{(x, x, x, y) : x \in P_{\mathcal{S}}, y \in P_{\mathcal{C} \setminus \mathcal{S}}\} = \text{Fix}_P(\mathbf{S}_3)$$

The second conjugacy class of subgroups is represented for example by the subgroup generated by a transposition

$$\mathbf{Z}_2 = \langle (1\ 2) \rangle$$

The associated balanced relation has three equivalence classes  $\{1, 2\}$ ,  $\{3\}$  and  $\{4\}$  given by the three orbits of  $\mathbf{Z}_2$  on the set of cells  $\mathcal{C}$ . The fixed-point subspace of  $\mathbf{Z}_2$  is

$$\text{Fix}_P(\mathbf{Z}_2) = \{(x, x, y, z) : x, y \in P_{\mathcal{S}}, z \in P_{\mathcal{C} \setminus \mathcal{S}}\}$$

The other two subgroups in the conjugacy class of  $\langle (1\ 2) \rangle$  are the ones generated by  $(1\ 3)$  and  $(2\ 3)$ . Observe that these three balanced equivalence relations given by orbits of subgroups are exactly the same balanced equivalence relations previously found by direct methods (Example 2.11). Therefore, in our running example all flow-invariant subspaces can be given as fixed-point subspaces of subgroups.  $\diamond$

**Remark 3.7** It is not true, even for symmetric networks, that all balanced equivalence relations are given by orbits of subgroups of the symmetry group of the network. Balanced equivalence relations that are not of this type are called *exotic*. For examples of exotic balanced relations see Antoneli and Stewart [1, 2].  $\diamond$

## 3.2 Admissible Vector Fields with Interior Symmetry

Let  $\mathcal{G}$  be a network with a non-trivial interior symmetry group  $\Sigma_{\mathcal{S}}$  on a subset of cells  $\mathcal{S}$  and fix a phase space  $P$ . We have a natural decomposition

$$P = P_{\mathcal{S}} \oplus P_{\mathcal{C} \setminus \mathcal{S}} \tag{3.3}$$

with coordinates  $(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}})$ . If  $f : P \rightarrow P$  is a  $\mathcal{G}$ -admissible vector field then we can write  $f = (f_{\mathcal{S}}, f_{\mathcal{C}\setminus\mathcal{S}})$  where  $f_{\mathcal{S}} : P \rightarrow P_{\mathcal{S}}$  and  $f_{\mathcal{C}\setminus\mathcal{S}} : P \rightarrow P_{\mathcal{C}\setminus\mathcal{S}}$ . Groupoid-equivariance of the coupled cell system implies that

$$\sigma f_{\mathcal{S}}(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) = f_{\mathcal{S}}(\sigma x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) \quad (3.4)$$

for all  $\sigma \in \Sigma_{\mathcal{S}}$ .

A  $\mathcal{G}$ -admissible vector field  $f$  can be written as

$$f(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) = \begin{bmatrix} f_{\mathcal{S}}(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) \\ \tilde{f}_{\mathcal{C}\setminus\mathcal{S}}(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) \end{bmatrix} + \begin{bmatrix} 0 \\ h(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) \end{bmatrix} \quad (3.5)$$

where  $\tilde{f}_{\mathcal{C}\setminus\mathcal{S}}, h : P \rightarrow P_{\mathcal{C}\setminus\mathcal{S}}$  and  $f_{\mathcal{C}\setminus\mathcal{S}} = \tilde{f}_{\mathcal{C}\setminus\mathcal{S}} + h$ . The vector field  $\tilde{f} = (f_{\mathcal{S}}, \tilde{f}_{\mathcal{C}\setminus\mathcal{S}})$  is the  $\Sigma_{\mathcal{S}}$ -equivariant part of  $f$ , that is, for all  $\sigma \in \Sigma_{\mathcal{S}}$

$$\sigma \tilde{f}(x) = \tilde{f}(\sigma x)$$

or more explicitly,

$$\begin{bmatrix} \sigma f_{\mathcal{S}}(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) \\ \tilde{f}_{\mathcal{C}\setminus\mathcal{S}}(x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) \end{bmatrix} = \begin{bmatrix} f_{\mathcal{S}}(\sigma x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) \\ \tilde{f}_{\mathcal{C}\setminus\mathcal{S}}(\sigma x_{\mathcal{S}}, x_{\mathcal{C}\setminus\mathcal{S}}) \end{bmatrix} \quad (3.6)$$

since  $\Sigma_{\mathcal{S}}$  acts trivially on  $P_{\mathcal{C}\setminus\mathcal{S}}$ . Equation (3.5) can be seen as a decomposition of the vector field  $f$  as the sum of a  $\Sigma_{\mathcal{S}}$ -equivariant vector field and a non-equivariant ‘‘perturbation’’ with null components in  $\mathcal{S}$ .

**Example 3.8** Consider the network  $\mathcal{G}_2$  of Figure 1. Recall from Example 2.7 the general form of the ODE’s associated with the network  $\mathcal{G}_2$ . Using the decomposition (3.3) we have  $x_{\mathcal{S}} = (x_1, x_2, x_3)$  and  $x_{\mathcal{C}\setminus\mathcal{S}} = (x_4)$  where  $x_i \in \mathbf{R}^k$  ( $i = 1, 2, 3$ ),  $x_4 \in \mathbf{R}^l$ . Then by (3.5) we can write a general ODE for the network  $\mathcal{G}_2$  as

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_2, x_3}, x_4) \\ \dot{x}_2 &= f(x_2, \overline{x_3, x_1}, x_4) \\ \dot{x}_3 &= f(x_3, \overline{x_1, x_2}, x_4) \\ \dot{x}_4 &= g(x_4, \overline{x_1, x_2, x_3}) + h(x_4, x_1, x_2, x_3) \end{aligned}$$

where  $f : \mathbf{R}^{3k} \times \mathbf{R}^l \rightarrow \mathbf{R}^k$  is a smooth map invariant under permutation of the second and third argument,  $g : \mathbf{R}^l \times \mathbf{R}^{3k} \rightarrow \mathbf{R}^l$  is  $\mathbf{S}_3$ -invariant with respect to  $(x_1, x_2, x_3)$  and  $h : \mathbf{R}^l \times \mathbf{R}^{3k} \rightarrow \mathbf{R}^l$  is a general smooth map.  $\diamond$

Now we introduce another set of coordinates on  $P$ , adapted to the action of the interior symmetry group. By Proposition 3.5 the subspace  $\text{Fix}_P(\Sigma_{\mathcal{S}})$  is flow-invariant. Since  $\text{Fix}_P(\Sigma_{\mathcal{S}})$  is  $\Sigma_{\mathcal{S}}$ -invariant and  $\Sigma_{\mathcal{S}}$  acts trivially on the cells in  $\mathcal{C} \setminus \mathcal{S}$  we have that  $P_{\mathcal{C} \setminus \mathcal{S}} \subset \text{Fix}_P(\Sigma_{\mathcal{S}})$ . Let

$$U = \text{Fix}_P(\Sigma_{\mathcal{S}}) \quad (3.7)$$

The action of the group  $\Sigma_{\mathcal{S}}$  decomposes the set  $\mathcal{S}$  as

$$\mathcal{S} = \mathcal{S}_1 \cup \dots \cup \mathcal{S}_k$$

where the sets  $\mathcal{S}_i$  ( $i = 1, \dots, k$ ) are the orbits of the  $\Sigma_{\mathcal{S}}$ -action. Let

$$W = \left\{ x \in P : x_c = 0 \ \forall c \in \mathcal{C} \setminus \mathcal{S} \text{ and } \sum_{s \in \mathcal{S}_i} x_s = 0 \text{ for } 1 \leq i \leq k \right\} \quad (3.8)$$

Since  $W$  is a  $\Sigma_{\mathcal{S}}$ -invariant subspace of  $P_{\mathcal{S}}$  and  $W \cap U = \{0\}$  we can decompose the phase space  $P$  as a direct sum of  $\Sigma_{\mathcal{S}}$ -invariant subspaces

$$P = W \oplus U \quad (3.9)$$

In particular, (3.8) implies that vectors in  $W$ , when written in coupled cell coordinates, have zero components on all cells in  $\mathcal{C} \setminus \mathcal{S}$ .

We can choose coordinates  $(w, u)$  with  $w \in W$  and  $u \in U$  adapted to the decomposition (3.9) and write any admissible vector field  $f$  as

$$f(w, u) = \begin{bmatrix} f_W(w, u) \\ f_U(w, u) \end{bmatrix} + \begin{bmatrix} 0 \\ h(w, u) \end{bmatrix} \quad (3.10)$$

where  $f_U, h : P \rightarrow U$  and  $f_W : P \rightarrow W$  satisfies

$$\sigma f_W(w, u) = f_W(\sigma w, u) \quad \forall \sigma \in \Sigma_{\mathcal{S}}$$

With respect to the decomposition (3.9), the equivariant part of  $f$  is written as  $\tilde{f}(w, u) = (f_W(w, u), f_U(w, u))$  and for all  $\sigma \in \Sigma_{\mathcal{S}}$  we have

$$\begin{bmatrix} \sigma f_W(w, u) \\ f_U(w, u) \end{bmatrix} = \begin{bmatrix} f_W(\sigma w, u) \\ f_U(\sigma w, u) \end{bmatrix} \quad (3.11)$$

since  $\Sigma_{\mathcal{S}}$  acts trivially on  $U = \text{Fix}_P(\Sigma_{\mathcal{S}})$ .

**Example 3.9** Consider the network  $\mathcal{G}_2$  of Figure 1. With respect to the decomposition (3.3) adapted to the network structure, the total phase space  $P$  has coordinates  $x_{\mathcal{S}} = (x_1, x_2, x_3)$  and  $x_{\mathcal{C} \setminus \mathcal{S}} = (x_4)$  where  $x_i \in \mathbf{R}^k$  ( $i = 1, 2, 3$ ),  $x_4 \in \mathbf{R}^l$ . Now with respect to the decomposition (3.9) adapted to the  $\mathbf{S}_3$ -action on  $P$  we have that

$$W = \{(w_1, w_2, -w_1 - w_2, 0) : w_1, w_2 \in \mathbf{R}^k\}$$

and

$$U = \text{Fix}_P(\mathbf{S}_3) = \{(u_1, u_1, u_1, u_2) : u_1 \in \mathbf{R}^k, u_2 \in \mathbf{R}^l\}$$

◇

In the linear case, we may choose a basis of  $P$  adapted to the decomposition (3.9) and then a  $\mathcal{G}$ -admissible linear vector field  $L$  can be written as

$$L = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix} \quad (3.12)$$

where  $B = L|_U : U \rightarrow U$ ,  $C : W \rightarrow U$  and  $A : W \rightarrow W$  satisfies (by (3.11))

$$A\sigma = \sigma A \quad \forall \sigma \in \Sigma_{\mathcal{S}}$$

The spectral properties of  $L$  in (3.12) are given by Golubitsky *et al.* [4, Lemma 1, p. 399]. Since we will use these results several times we reproduce it here.

**Lemma 3.10 (Golubitsky *et al.* [4])** *Let  $\mathcal{G}$  be a network admitting a non-trivial group of interior symmetries  $\Sigma_{\mathcal{S}}$  and fix a total phase space  $P$ . Let  $L : P \rightarrow P$  be a  $\mathcal{G}$ -admissible linear vector field and consider the decomposition of  $L$  given by (3.12). Then*

- (i) *The eigenvalues of  $L$  are the eigenvalues of  $A$  together with the eigenvalues of  $B$ .*
- (ii) *A vector  $u \in U = \text{Fix}_P(\Sigma_{\mathcal{S}})$  is an eigenvector of  $B$  with eigenvalue  $\nu$  if and only if  $u$  is an eigenvector of  $L$  with eigenvalue  $\nu$ .*
- (iii) *If  $w \in W$  is an eigenvector of  $A$  with eigenvalue  $\mu$ , then there exists an eigenvector  $v$  of  $L$  with eigenvalue  $\mu$  of the form*

$$v = w + u$$

*where  $u \in U = \text{Fix}_P(\Sigma_{\mathcal{S}})$ .*

(iv) All eigenspaces of  $A$  are  $\Sigma_S$ -invariant.

**Proof.** Parts (i) (ii) and (iii) are consequences of the block form (3.12) of  $L$ . Part (iv) follows from the  $\Sigma_S$ -equivariance of  $A$ .  $\square$

**Example 3.11** We continue our running example, the networks  $\mathcal{G}_1$  and  $\mathcal{G}_2$  of Figure 1. The general form of the admissible linear mappings associated with the networks  $\mathcal{G}_1$  and  $\mathcal{G}_2$  of Figure 1 are (in cell coordinates)

$$L_1 = \begin{pmatrix} a & b & b & d \\ b & a & b & d \\ b & b & a & d \\ e & e & e & c \end{pmatrix} \quad \text{and} \quad L_2 = \begin{pmatrix} a & b & b & d \\ b & a & b & d \\ b & b & a & d \\ e_1 & e_2 & e_3 & c \end{pmatrix}$$

where  $a, b$  are  $k \times k$  matrices,  $c$  is a  $l \times l$  matrix,  $d$  is a  $k \times l$  matrix and  $e, e_1, e_2, e_3$  are  $l \times k$  matrices. Choosing adequate bases for  $W$  and  $U$  the linear mappings  $L_1$  and  $L_2$  can be written as

$$L_1 = \begin{pmatrix} a-b & 0 & 0 & 0 \\ 0 & a-b & 0 & 0 \\ 0 & 0 & a+2b & d \\ 0 & 0 & 3e & c \end{pmatrix}$$

and

$$L_2 = \begin{pmatrix} a-b & 0 & 0 & 0 \\ 0 & a-b & 0 & 0 \\ 0 & 0 & a+2b & d \\ e_1-e_3 & e_2-e_3 & e_1+e_2+e_3 & c \end{pmatrix}$$

$\diamond$

## 4 Synchrony Breaking Bifurcations

Now we study local bifurcations in coupled cell networks with non-trivial interior symmetries. We are interested in codimension-one synchrony-breaking bifurcations. Steady-state and Hopf of bifurcations in coupled cell networks with interior symmetries were studied by Golubitsky *et al.* [4].

## 4.1 Local Bifurcations in Coupled Cell Systems

Let  $\mathcal{G}$  be a coupled cell network and fix a phase space  $P$ . Let  $f : P \times \mathbf{R}^k \rightarrow P$  be a smooth  $k$ -parameter family of  $\mathcal{G}$ -admissible vector fields in  $P$  and assume that the ODE

$$\frac{dx}{dt} = f(x, \lambda) \tag{4.1}$$

has a synchronous equilibrium  $x_0$  in  $\Delta_I$  (the polydiagonal subspace of  $P$  associated with the input equivalence relation  $\sim_I$ ). In the present context we may assume that

$$f(x_0, \lambda) \equiv 0$$

and that a bifurcation occurs at  $\lambda = 0$ . Let  $L = (df)_{(x_0, 0)}$  be the linearization of  $f$  at  $(x_0, 0)$  and denote by  $E^c$  the center subspace of  $L$ .

Local bifurcations in coupled cell networks can be divided into two types according to  $E^c$  is contained or not into the flow-invariant subspace  $\Delta_I$ .

**Definition 4.1** We say that a coupled cell system (4.1) undergoes a *synchrony-preserving* bifurcation at a synchronous equilibrium in  $\Delta_I$  if  $E^c \subset \Delta_I$  and that (4.1) undergoes a *synchrony-breaking* bifurcation if  $E^c \not\subset \Delta_I$ .  $\diamond$

Now we specialise to codimension-one bifurcations, that is,  $f : P \times \mathbf{R} \rightarrow P$  is a smooth 1-parameter family of  $\mathcal{G}$ -admissible vector fields in  $P$ . These bifurcations fall into two classes: *steady-state* bifurcations ( $L|_{E^c}$  has a zero eigenvalue) and *Hopf* bifurcations ( $L|_{E^c}$  has a pair of purely imaginary eigenvalues). The new steady-states and periodic solutions that emanate from the synchrony-preserving bifurcations are themselves synchronous solutions. For the remainder of this paper we will focus on codimension-one synchrony-breaking bifurcations from a synchronous equilibrium.

## 4.2 Local Bifurcations with Interior Symmetry

Interior symmetries introduce genuine restrictions on the form of the linearization and this structure can be used to study certain kind of synchrony-breaking bifurcations, namely, the bifurcations that break the interior symmetry.

Let  $\mathcal{G}$  be a network admitting a non-trivial group of interior symmetries  $\Sigma_{\mathcal{S}}$  on  $\mathcal{S}$  and fix a phase space  $P$ . First, note that the polydiagonal subspace

$\Delta_I$  associated to the input equivalence relation  $\sim_I$  satisfies

$$\Delta_I \subseteq \text{Fix}_P(\Sigma_S)$$

Since we are interested in synchrony-breaking bifurcations that also break the interior symmetry we may assume that  $x_0 \in \text{Fix}_P(\Sigma_S)$  and that the center subspace  $E^c(L)$  associated to the critical eigenvalues satisfies

$$E^c(L) \not\subseteq \text{Fix}_P(\Sigma_S) \quad (4.2)$$

However, this is not enough to exclude the possibility of having critical eigenvectors in  $\text{Fix}_P(\Sigma_S)$  in a synchrony-breaking bifurcation. That is, we could have a situation where some critical eigenvectors belong to  $\text{Fix}_P(\Sigma_S)$  and the others are outside  $\text{Fix}_P(\Sigma_S)$ . Indeed, it is well known [3] that (non-symmetric) coupled cell systems generically can exhibit mode interaction in codimension-one bifurcations. In this paper we make a stronger assumption. We assume

$$E^c(L) \cap \text{Fix}_P(\Sigma_S) = \{0\} \quad (4.3)$$

and so we exclude the possibility of having eigenvectors in  $\text{Fix}_P(\Sigma_S)$ . This situation corresponds to a synchrony-breaking bifurcation that “breaks *only* the interior symmetry”.

**Definition 4.2** Let  $f : P \rightarrow P$  be a  $\mathcal{G}$ -admissible vector field and let  $L = (df)_{(x_0)}$  be the linearization of  $f$  at  $x_0$ . Consider the decomposition (3.9) of  $P$  adapted to the  $\Sigma_S$ -action and write  $L$  in block form as

$$L = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}$$

Then the matrix  $A$  is called the  $\Sigma_S$ -equivariant sub-block of  $L$ .  $\diamond$

If we write  $f$  using coordinates  $(w, u)$  adapted to the decomposition  $P = W \oplus U$  as

$$f(w, u) = \begin{bmatrix} f_W(w, u) \\ f_U(w, u) \end{bmatrix} + \begin{bmatrix} 0 \\ h(w, u) \end{bmatrix}$$

then

$$A = (d_{(1)}f_W)_{(w_0)}$$

where  $x_0 = (w_0, u_0)$  and

$$(d_{(1)}f_W)_{(w_0)} \cdot w = (df_W)_{(w_0, u_0)} \cdot (w, 0)$$

for all  $w \in W$ .

**Remark 4.3** It can be shown that the following three conditions are equivalent:

- (a)  $E^c(L) \cap \text{Fix}_P(\Sigma_{\mathcal{S}}) = \{0\}$ .
- (b)  $\dim E^c(L) = \dim E^c(A)$ .
- (c) All the critical eigenvalues of  $L$  come from the  $\Sigma_{\mathcal{S}}$ -equivariant sub-block  $A$  of  $L$ .

It is obvious that (a) implies both (b) and (c). On the other hand, to prove that (b) implies (a), we observe that by Lemma 3.10 (iii), we always have  $\dim E^c(A) \leq \dim E^c(L)$ . Finally, to prove that (c) implies (a), we observe that the block form of  $L$  guarantees that no generalised eigenvector associated to an eigenvalue coming from sub-block  $A$  belong to  $\text{Fix}_P(\Sigma_{\mathcal{S}})$ .  $\diamond$

In general  $f$  is not  $\Sigma_{\mathcal{S}}$ -equivariant and  $L$  does not commute with  $\Sigma_{\mathcal{S}}$ . In particular,  $E^c(L) \not\subset W$ . However, the block matrix  $A$  does commute with  $\Sigma_{\mathcal{S}}$  and thus  $E^c(A) \subset W$  is  $\Sigma_{\mathcal{S}}$ -invariant. Moreover, if  $A$  has purely imaginary eigenvalues there is a natural action of  $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$  on  $E^c(A)$ , where  $\mathbf{S}^1$  acts by  $\exp(sA^t)$ .

**Definition 4.4** Consider a 1-parameter family of coupled cell systems (4.1) with interior symmetry group  $\Sigma_{\mathcal{S}}$  on  $\mathcal{S}$  undergoing a codimension-one synchrony-breaking bifurcation at a synchronous equilibrium  $x_0$  when  $\lambda = 0$ . We say that  $f$  undergoes a *codimension-one interior symmetry-breaking bifurcation* if the following conditions hold:

- (a) All the critical eigenvalues  $\mu$  of  $L$  come from the  $\Sigma_{\mathcal{S}}$ -equivariant sub-block  $A$  of  $L$ .
- (b) The critical eigenvalues  $\mu$  extend uniquely and smoothly to eigenvalues  $\mu(\lambda)$  of  $(df)_{(x_0, \lambda)}$  for  $\lambda$  near 0.
- (c) The *eigenvalue crossing condition*:

$$\left. \frac{d}{d\lambda} \text{Re}(\mu(\lambda)) \right|_{\lambda=0} \neq 0 \quad (4.4)$$

More specifically, the bifurcation problem (4.1) is called

- (1) A *codimension-one interior symmetry-breaking steady-state bifurcation* if, in addition to the conditions (a), (b), (c) above, the matrix  $A$  has a zero eigenvalue and the associated center subspace is given by

$$E_0(A) = \ker(A) \tag{4.5}$$

- (2) A *codimension-one interior symmetry-breaking Hopf bifurcation* if, in addition to the conditions (a), (b), (c) above, the matrix  $A$  is non-singular and (after rescaling time if necessary) all the critical eigenvalues have the form  $\pm i$  and the associated center subspace is given by

$$E_i(A) = \{x \in P : (A^2 + 1)x = 0\} \tag{4.6}$$

◇

**Example 4.5** Consider the networks  $\mathcal{G}_1$  and  $\mathcal{G}_2$  of Figure 1. Suppose that for all cells  $c$  we choose the internal phase space to be  $P_c = \mathbf{C}$  and so the total phase space is  $P = \mathbf{C}^4$ . Consider the decomposition of  $P = W \oplus U$  adapted to the  $\mathbf{S}_3$ -action. Then

$$W = \{(w_1, w_2, -w_1 - w_2, 0) : w_1, w_2 \in \mathbf{C}\},$$

$$U = \text{Fix}_P(\mathbf{S}_3) = \{(u_1, u_1, u_1, u_2) : u_1, u_2 \in \mathbf{C}\}$$

and  $W$  is a  $\mathbf{S}_3$ -simple representation ( $W = W_1 \oplus W_2$  where  $W_1, W_2$  are two isomorphic  $\mathbf{S}_3$ -absolutely irreducible spaces). Now consider a 1-parameter family  $f : P \times \mathbf{R} \rightarrow P$  of  $\mathcal{G}$ -admissible vector fields on  $P$  undergoing a codimension-one interior symmetry-breaking Hopf bifurcation at an equilibrium point  $x_0$  when  $\lambda = 0$ . Since  $W$  is a  $\mathbf{S}_3$ -simple representation, one necessarily have that  $E^c(A) = W$ . Moreover, the action of the circle group  $\mathbf{S}^1$  defined by  $\exp(sA^t)$  is equivalent to the standard action of  $\mathbf{S}^1$  on  $\mathbf{C}^4$ , that is,

$$\theta \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\theta} z_2) \tag{4.7}$$

for all  $\theta \in \mathbf{S}^1$  and  $z_1, z_2 \in \mathbf{C}$ . ◇

### 4.3 Interior Symmetry-Breaking Hopf Theorem

The Hopf Bifurcation Theorem concerns periodic solutions to differential equations near a point where the linearization has purely imaginary eigenvalues.

Let  $\mathcal{G}$  be a coupled cell network admitting a non-trivial group of interior symmetries  $\Sigma_{\mathcal{S}}$  on a subset  $\mathcal{S}$  of cells and choose a total phase space  $P$ . Consider a smooth 1-parameter family  $f : P \times \mathbf{R} \rightarrow P$  of  $\mathcal{G}$ -admissible vector fields on  $P$  and assume that

$$\frac{dx}{dt} = f(x, \lambda) \tag{4.8}$$

has an equilibrium  $x_0$ , such that for  $\lambda = 0$  the linearization  $L = (df)_{(x_0, 0)}$  of  $f$  at  $(x_0, 0)$  is non-singular but has purely imaginary eigenvalues.

Before stating the next theorem let us introduce an important concept which generalises the notion of  $\mathbf{C}$ -axial subgroup from equivariant bifurcation theory.

**Definition 4.6** Let  $\mathcal{G}$  be a coupled cell network admitting a non-trivial group of interior symmetries  $\Sigma_{\mathcal{S}}$  on a subset  $\mathcal{S}$ . Let  $P$  denote the total phase space and consider the decomposition (3.9) of  $P$  adapted to the  $\Sigma_{\mathcal{S}}$ -action. Suppose that there is an action of circle group  $\mathbf{S}^1$  on  $W$  which commutes with the action of  $\Sigma_{\mathcal{S}}$ . Let  $E \subset W$  be a  $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ -invariant subspace. An isotropy subgroup  $\Delta \subseteq \Sigma_{\mathcal{S}} \times \mathbf{S}^1$  is called *interiorly  $\mathbf{C}$ -axial (on  $E$ )* if

$$\dim_{\mathbf{R}} \text{Fix}_E(\Delta) = 2$$

◇

Now suppose that the family (4.8) undergoes a codimension-one interior symmetry-breaking Hopf bifurcation at the equilibrium  $x_0$  when  $\lambda = 0$ . Then the center subspace  $E^c(A)$  of the  $\Sigma_{\mathcal{S}}$ -equivariant sub-block of the linearization  $L = (df)_{(x_0, 0)}$  of  $f$  at  $(x_0, 0)$  is a  $\Sigma_{\mathcal{S}}$ -invariant subspace of  $W$ . Therefore, the action of the circle group  $\mathbf{S}^1$  defined by  $\exp(sA^t)$  commutes with the action of  $\Sigma_{\mathcal{S}}$  and so there is a well-defined action of  $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$  on  $W$  and  $E^c(A)$  is a  $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ -invariant subspace.

**Example 4.7** Consider the networks  $\mathcal{G}_1$  and  $\mathcal{G}_2$  of Figure 1. Suppose that for all cells  $c$  we choose the internal phase space to be  $P_c = \mathbf{C}$  and so the total phase space is  $P = \mathbf{C}^4$ . Suppose that a smooth 1-parameter family  $f : P \times \mathbf{R} \rightarrow P$  of  $\mathcal{G}$ -admissible vector fields on  $P$  undergoes a codimension-one interior symmetry-breaking Hopf bifurcation at the equilibrium  $x_0 = 0$  when  $\lambda = 0$ . Then  $E_i(A) = W$ , where  $A$  is the  $\Sigma_{\mathcal{S}}$ -equivariant sub-block of the linearization  $L = (df)_{(0, 0)}$  of  $f$  at  $(0, 0)$ . In Example 4.5 we observed

that the action of  $\mathbf{S}^1$  on  $W$ , given by  $\exp(sA^t)$ , can be identified with the standard action of  $\mathbf{S}^1$  on  $\mathbf{C}^4$ . There are three non-trivial conjugacy classes of isotropy subgroups of  $\mathbf{S}_3 \times \mathbf{S}^1$  acting on  $W$ . The first conjugacy class of subgroups is represented for example by the subgroup

$$\mathbf{Z}_2 = \langle ((1\ 2), \mathbf{1}) \rangle$$

The fixed-point subspace of  $\mathbf{Z}_2$  is

$$\text{Fix}_W(\mathbf{Z}_2) = \{(-w, -w, 2w, 0) : w \in \mathbf{C}\}$$

The second conjugacy class of subgroups is represented for example by the subgroup

$$\tilde{\mathbf{Z}}_2 = \langle ((1\ 2), \pi) \rangle$$

The fixed-point subspace of  $\tilde{\mathbf{Z}}_2$  is

$$\text{Fix}_W(\tilde{\mathbf{Z}}_2) = \{(w, -w, 0, 0) : w \in \mathbf{C}\}$$

The third conjugacy class of subgroups is represented for example by the subgroup

$$\tilde{\mathbf{Z}}_3 = \langle ((1\ 2\ 3), \frac{2\pi}{3}) \rangle$$

The fixed-point subspace of  $\tilde{\mathbf{Z}}_3$  is

$$\text{Fix}_W(\tilde{\mathbf{Z}}_3) = \{(w, e^{i\frac{2\pi}{3}}w, e^{i\frac{4\pi}{3}}w, 0) : w \in \mathbf{C}\}$$

◇

The main result of this paper is the interior symmetry-breaking Hopf bifurcation Theorem.

**Theorem 4.8** *Let  $\mathcal{G}$  be a coupled cell network admitting a non-trivial group of interior symmetries  $\Sigma_{\mathcal{S}}$  relative to a subset  $\mathcal{S}$  of cells and fix a phase space  $P$ . Consider (4.8) where  $f : P \times \mathbf{R} \rightarrow P$  is a smooth 1-parameter family of  $\mathcal{G}$ -admissible vector fields on  $P$ . Suppose that a codimension-one interior symmetry-breaking Hopf bifurcation (see Definition 4.4) occurs at an equilibrium point  $x_0 \in \text{Fix}_P(\Sigma_{\mathcal{S}})$  when  $\lambda = 0$ . Let  $\Delta \subset \Sigma_{\mathcal{S}} \times \mathbf{S}^1$  be an interiorly  $\mathbf{C}$ -axial subgroup (on  $E^c(A)$ ). Then generically there exists a family of small amplitude periodic solutions of (4.8) bifurcating from  $(x_0, 0)$*

and having period near  $2\pi$ . Moreover, to lowest order in the bifurcation parameter  $\lambda$ , the solution  $x(t)$  is of the form

$$x(t) \approx w(t) + u(t) \quad (4.9)$$

where  $w(t) = \exp(tL)w_0$  ( $w_0 \in \text{Fix}_W(\Delta)$ ) has exact spatio-temporal symmetry  $\Delta$  on the cells in  $\mathcal{S}$  and  $u(t) = \exp(tL)u_0$  ( $u_0 \in \text{Fix}_P(\Sigma_{\mathcal{S}})$ ) is synchronous on the  $\Sigma_{\mathcal{S}}$ -orbits of cells in  $\mathcal{S}$ .

We call such a state a *synchronously modulated  $\Delta$ -symmetric wave* on  $\mathcal{S}$ .

**Remarks 4.9**

- (a) The above theorem asserts no restriction on  $u_j(t)$  when  $j \in \mathcal{C} \setminus \mathcal{S}$ .
- (b) Theorem 4.8 generalises the interior symmetry Hopf Theorem of Golubitsky *et al.* [4, Theorem 3]. Given a subgroup  $\Delta \subseteq \Sigma_{\mathcal{S}} \times \mathbf{S}^1$  we define the *spatial subgroup* of  $\Delta$  to be  $K = \Delta \cap \Sigma_{\mathcal{S}}$ . A subgroup  $\Delta$  is called *spatially  $\mathbf{C}$ -axial* if

$$\dim_{\mathbf{R}} \text{Fix}_{E_i(A)}(\Delta) = \dim_{\mathbf{R}} \text{Fix}_{E_i(A)}(K) = 2$$

where  $K$  is the spatial subgroup of  $\Delta$ . Obviously every spatially  $\mathbf{C}$ -axial subgroup is interiorly  $\mathbf{C}$ -axial. Since the Hopf Theorem of [4] is proved for all spatially  $\mathbf{C}$ -axial subgroups, it is a special case of Theorem 4.8.

- (c) Theorem 4.8 holds if the assumption (4.6) is generalised to: the matrix  $A$  is non-singular, semi-simple and (after rescaling time if necessary) all the critical eigenvalues have the form  $k_l i$  ( $k_l \in \mathbf{Z}$ ).  $\diamond$

The proof of Theorem 4.8 follows from a couple of lemmas that we state and prove below. We start by setting up the framework.

Let  $C_{2\pi}^0(P)$  be the space consisting of all continuous  $2\pi$ -periodic mappings from  $\mathbf{R}$  to  $P$  endowed with the  $C^0$  norm and  $C_{2\pi}^1(P)$  be the space consisting of all continuous differentiable  $2\pi$ -periodic mappings from  $\mathbf{R}$  to  $P$  endowed with the  $C^1$  norm.

By introducing a perturbed period parameter  $\tau$  we can re-scale time again, from  $t$  to  $s(1 + \tau)t$ , and consider the operator  $\mathcal{F} : C_{2\pi}^1(P) \times \mathbf{R} \times \mathbf{R} \rightarrow C_{2\pi}^0(P)$  given by

$$\mathcal{F}(x, \lambda, \tau) = (1 + \tau) \frac{dx}{ds}(s) - f(x(s), \lambda) \quad (4.10)$$

The  $2\pi$ -periodic solutions of the equation  $\mathcal{F}(x, \lambda, \tau) = 0$  near  $(0, 0, 0)$  correspond bijectively to the small amplitude periodic solutions of (4.8) near  $x_0$  and with period near  $2\pi$ . As it is well known, the operator  $\mathcal{F}$  is  $\mathbf{S}^1$ -equivariant with respect to the *phase shift action* of  $\mathbf{S}^1$  on the spaces  $C_{2\pi}^1(P)$  and  $C_{2\pi}^0(P)$ , that is, if  $x \in C_{2\pi}^0(P)$  and  $\theta \in \mathbf{S}^1$  then

$$(\theta \cdot x)(s) = x(s + \theta)$$

and thus

$$\theta \cdot \mathcal{F}(x, \tau, \lambda) = \mathcal{F}(\theta \cdot x, \tau, \lambda)$$

The linearization of  $\mathcal{F}$  about the origin is

$$\mathcal{L}(x) = \frac{dx}{ds}(s) - Lx(s) \tag{4.11}$$

and  $\ker(\mathcal{L})$  consists of all functions  $\operatorname{Re}(e^{is}v)$  where  $v$  is an eigenvector of  $L$  associated to the eigenvalue  $i$ .

In the standard Hopf Bifurcation Theorem [5, Theorem VIII 3.1]  $\ker(\mathcal{L})$  is two-dimensional and Lyapunov-Schmidt reduction in the presence of symmetry leads to a reduced equation that can be solved for a unique branch of  $2\pi$ -periodic solutions as long as the eigenvalues crossing condition is valid. In the equivariant context,  $\ker(\mathcal{L})$  may be higher-dimensional – generically  $\ker(\mathcal{L})$  is a  $\Gamma$ -simple representation. The proof of the Equivariant Hopf Bifurcation Theorem [8, Theorem XVI 4.1] proceeds by restricting the Lyapunov-Schmidt reduced equation to the fixed-point subspace  $\operatorname{Fix}_{E_i(L)}(\Delta)$  of a  $\mathbf{C}$ -axial subgroup  $\Delta$ , which is two-dimensional. Then the proof is completed as in the standard Hopf Bifurcation Theorem.

That approach does not work in the context of interior symmetries since in general there is no action of  $\Sigma_S \times \mathbf{S}^1$  on  $E_i(L)$ , because the original vector field  $f$  (and its linearization  $L$ ) is not  $\Sigma_S$ -equivariant. Nevertheless, we shall introduce a “modified Lyapunov-Schmidt procedure” that does work in the context of interior symmetries.

The decomposition in (3.9) induces the decompositions

$$C_{2\pi}^0(P) = C_{2\pi}^0(W) \oplus C_{2\pi}^0(\operatorname{Fix}_P(\Sigma_S))$$

and

$$C_{2\pi}^1(P) = C_{2\pi}^1(W) \oplus C_{2\pi}^1(\operatorname{Fix}_P(\Sigma_S))$$

**Lemma 4.10** *For any subgroup  $\Delta$  of  $\Sigma_S \times \mathbf{S}^1$  we have the decompositions*

$$\text{Fix}_{C_{2\pi}^0(P)}(\Delta) = C_{2\pi}^0(\text{Fix}_W(\Delta)) \oplus C_{2\pi}^0(\text{Fix}_P(\Sigma_S))$$

and

$$\text{Fix}_{C_{2\pi}^1(P)}(\Delta) = C_{2\pi}^1(\text{Fix}_W(\Delta)) \oplus C_{2\pi}^1(\text{Fix}_P(\Sigma_S))$$

**Proof.** Let  $x \in C_{2\pi}^0(P)$  be written as  $x(s) = (w(s), u(s))$  where  $w \in C_{2\pi}^0(W)$  and  $u \in C_{2\pi}^0(\text{Fix}_P(\Sigma_S))$ . If  $w(s) \in \text{Fix}_W(\Delta)$  for all  $s$  then it is clear that

$$C_{2\pi}^0(\text{Fix}_W(\Delta)) \oplus C_{2\pi}^0(\text{Fix}_P(\Sigma_S)) \subseteq \text{Fix}_{C_{2\pi}^0(P)}(\Delta)$$

On the other hand, let  $(\delta, \theta) \in \Delta$  and suppose  $x \in \text{Fix}_{C_{2\pi}^0(P)}(\Delta)$ . Then

$$(\delta, \theta) \cdot x(s) = (\delta x)(s + \theta) = x(s)$$

for all  $s$ . The decomposition  $x(s) = (w(s), u(s))$  yields

$$((\delta w)(s + \theta), u(s + \theta)) = (w(s), u(s))$$

that is,  $w(s) \in \text{Fix}_W(\Delta)$  and so we have

$$\text{Fix}_{C_{2\pi}^0(P)}(\Delta) \subseteq C_{2\pi}^0(\text{Fix}_W(\Delta)) \oplus C_{2\pi}^0(\text{Fix}_P(\Sigma_S))$$

Therefore,

$$\text{Fix}_{C_{2\pi}^0(P)}(\Delta) = C_{2\pi}^0(\text{Fix}_W(\Delta)) \oplus C_{2\pi}^0(\text{Fix}_P(\Sigma_S))$$

The same argument with  $C_{2\pi}^1$  instead of  $C_{2\pi}^0$  gives the other equality.  $\square$

**Lemma 4.11** *Let  $L : P \rightarrow P$  be a  $\mathcal{G}$ -admissible linear mapping. Let  $\mathcal{L} : C_{2\pi}^1(P) \times \mathbf{R} \times \mathbf{R} \rightarrow C_{2\pi}^0(P)$  be the linear operator given by equation (4.11) and  $\Delta \subset \Sigma_S \times \mathbf{S}^1$  be a subgroup. Then we have that*

$$\mathcal{L}(C_{2\pi}^1(\text{Fix}_P(\Sigma_S))) \subseteq C_{2\pi}^0(\text{Fix}_P(\Sigma_S))$$

$$\mathcal{L}(C_{2\pi}^1(\text{Fix}_W(\Delta))) \subseteq (C_{2\pi}^0(\text{Fix}_W(\Delta)) \oplus C_{2\pi}^0(\text{Fix}_P(\Sigma_S)))$$

and

$$\mathcal{L}(\text{Fix}_{C_{2\pi}^1(P)}(\Delta)) \subseteq \text{Fix}_{C_{2\pi}^0(P)}(\Delta) \tag{4.12}$$

In particular, we can define a linear operator

$$\mathcal{L}_\Delta : \text{Fix}_{C_{2\pi}^1(P)}(\Delta) \longrightarrow \text{Fix}_{C_{2\pi}^0(P)}(\Delta) \tag{4.13}$$

by restriction.

**Proof.** Note that since the circle group  $\mathbf{S}^1$  acts on the domain of the mappings all the decompositions above are  $\mathbf{S}^1$ -invariant.

First suppose  $x(s) = (0, u(s))$  with  $u(s) \in C_{2\pi}^1(\text{Fix}_P(\Sigma_S))$ . Then

$$\mathcal{L}(x) = \frac{du}{ds}(s) - L(0, u(s))$$

If  $\sigma \in \Sigma_S$  then

$$\begin{aligned} \sigma \mathcal{L}(x) &= \sigma \frac{du}{ds}(s) - \sigma L(0, u(s)) \\ &= \frac{d\sigma u}{ds}(s) - L(0, u(s)) \\ &= \frac{du}{ds}(s) - L(0, u(s)) \\ &= \mathcal{L}(x) \end{aligned}$$

The second equality above follows from the fact that

$$\sigma(L(0, u)) = L(0, u)$$

for all  $\sigma \in \Sigma_S$ . Therefore, we have  $\mathcal{L}(x(s)) \in C_{2\pi}^0(\text{Fix}_P(\Sigma_S))$ .

Next suppose that  $x(s) = (w(s), 0)$  with  $w(s) \in C_{2\pi}^1(\text{Fix}_W(\Delta))$ . Since  $w(s) \in W$  for all  $s \in \mathbf{R}$ , we have that

$$(\delta, \theta) \cdot w(s) = \delta w(s + \theta) = w(s)$$

for all  $(\delta, \theta) \in \Delta$ ,  $s \in \mathbf{R}$ . Write

$$\mathcal{L}(x) = ([\mathcal{L}(x)]_1(s), [\mathcal{L}(x)]_2(s))$$

with

$$[\mathcal{L}(x)]_1(s) \in W \quad \text{for all } s \in \mathbf{R}$$

and

$$[\mathcal{L}(x)]_2(s) \in \text{Fix}_P(\Sigma_S) \quad \text{for all } s \in \mathbf{R}$$

Then

$$[\mathcal{L}(x)]_1(s) = \frac{dw}{ds}(s) - Aw(s)$$

and

$$[\mathcal{L}(x)]_2(s) = -[C w(s) + B 0] = -C w(s)$$

Clearly,  $[\mathcal{L}(x)]_2(s) \in \text{Fix}_P(\Sigma_S)$ . Let  $(\delta, \theta) \in \Delta$  then

$$\begin{aligned} (\delta, \theta) \cdot [\mathcal{L}(x)]_1(s) &= (\delta, \theta) \cdot \frac{dw}{ds}(s) - (\delta, \theta) \cdot A w(s) \\ &= \delta \frac{dw}{ds}(s + \theta) - \delta A w(s + \theta) \\ &= \frac{d\delta w}{ds}(s + \theta) - A \delta w(s + \theta) \\ &= \frac{dw}{ds}(s) - A w(s) \\ &= [\mathcal{L}(x)]_1(s) \end{aligned}$$

and thus  $[\mathcal{L}(x)]_1(s) \in \text{Fix}_W(\Delta)$ . Therefore

$$\mathcal{L}(x) \in C_{2\pi}^0(\text{Fix}_W(\Delta)) \oplus C_{2\pi}^0(\text{Fix}_P(\Sigma_S))$$

Thus by linearity of  $\mathcal{L}$  and Lemma 4.10 we have

$$\mathcal{L}(\text{Fix}_{C_{2\pi}^1(P)}(\Delta)) \subseteq \text{Fix}_{C_{2\pi}^0(P)}(\Delta)$$

□

Consider now a 1-parameter family of  $\mathcal{G}$ -admissible vector fields  $f(x, \lambda)$  such that  $L = (df)_{(x_0, 0)}$  satisfies the conditions of the definition of interior symmetry-breaking Hopf bifurcation (Definition 4.4 (2)), where  $A$  is the  $\Sigma_S$ -equivariant sub-block of  $L$ .

**Lemma 4.12** *Let  $\mathcal{L}_\Delta : \text{Fix}_{C_{2\pi}^1(P)}(\Delta) \rightarrow \text{Fix}_{C_{2\pi}^0(P)}(\Delta)$  be the operator given by formula (4.13), where  $L = (df)_{(x_0, 0)}$ . Let  $\Delta \subset \Sigma_S \times \mathbf{S}^1$  be a subgroup. Then*

$$\dim_{\mathbf{R}} \ker(\mathcal{L}_\Delta) = \dim_{\mathbf{R}} \text{Fix}_{E_i(A)}(\Delta)$$

**Proof.** By Lemma 3.10 and assumption (4.6),  $\ker(\mathcal{L}_\Delta)$  consists of all functions  $\text{Re}(e^{is} v_0)$  where  $v_0$  is an eigenvector of  $L$  associated to the eigenvalue  $i$  which can be decomposed as

$$v_0 = w_0 + u_0$$

where  $u_0 \in \text{Fix}_P(\Sigma_S)$  is uniquely determined by an eigenvector  $w_0 \in \text{Fix}_W(\Delta)$  of  $A$  with purely imaginary eigenvalue and

$$(\delta, \theta) \cdot \text{Re}(e^{is}w_0) = \text{Re}(e^{i(s+\theta)}\delta w_0) = \text{Re}(e^{is}w_0)$$

for all  $(\delta, \theta) \in \Delta$ . Hence

$$w_0 \in \text{Fix}_W(\Delta) \cap E_i(A) = \text{Fix}_{E_i(A)}(\Delta)$$

By uniqueness of the decomposition  $v_0 = w_0 + u_0$  and the dimension condition (b) of Remark 4.3 we have

$$\dim_{\mathbf{R}} \ker(\mathcal{L}_\Delta) = \dim_{\mathbf{R}} \text{Fix}_{E_i(A)}(\Delta)$$

□

**Lemma 4.13** *Let us write the 1-parameter family of admissible vector fields  $f(x, \lambda)$  in the form*

$$f(x, \lambda) = \begin{bmatrix} f_S(x, \lambda) \\ \tilde{f}_{C \setminus S}(x, \lambda) \end{bmatrix} + \begin{bmatrix} 0 \\ h(x, \lambda) \end{bmatrix} \quad (4.14)$$

where

$$\tilde{f}(x, \lambda) = \begin{bmatrix} f_S(x, \lambda) \\ \tilde{f}_{C \setminus S}(x, \lambda) \end{bmatrix}$$

is the  $\Sigma_S$ -equivariant part of  $f$ . Let  $\mathcal{F}, \tilde{\mathcal{F}}$  be operators on  $C_{2\pi}^1(P) \times \mathbf{R} \times \mathbf{R} \rightarrow C_{2\pi}^0(P)$  defined by formula (4.10) using  $f$  and  $\tilde{f}$ , respectively. Define

$$\mathcal{H}(x, \tau, \lambda) = h(x(s), \lambda)$$

so that

$$\mathcal{F}(x, \tau, \lambda) = \tilde{\mathcal{F}}(x, \tau, \lambda) - \mathcal{H}(x, \tau, \lambda)$$

Then

$$\mathcal{F}(\text{Fix}_{C_{2\pi}^1(P)}(\Delta) \times \mathbf{R} \times \mathbf{R}) \subseteq \text{Fix}_{C_{2\pi}^0(P)}(\Delta)$$

In particular, we may define the operator

$$\mathcal{F}_\Delta : \text{Fix}_{C_{2\pi}^1(P)}(\Delta) \times \mathbf{R} \times \mathbf{R} \longrightarrow \text{Fix}_{C_{2\pi}^0(P)}(\Delta) \quad (4.15)$$

by restriction and the linearization of  $\mathcal{F}_\Delta$  about the origin is the linear operator  $\mathcal{L}_\Delta$  given by the formula (4.13), where  $L = (\text{df})_{(x_0, 0)}$ .

**Proof.** The  $\Sigma_{\mathcal{S}}$ -equivariance of  $\tilde{f}$  implies that  $\tilde{\mathcal{F}}$  is  $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ -equivariant (see [8, Lemma XVI 3.2]). It follows then that

$$\tilde{\mathcal{F}}(\text{Fix}_{C_{2\pi}^1(P)}(\Delta) \times \mathbf{R} \times \mathbf{R}) \subseteq \text{Fix}_{C_{2\pi}^0(P)}(\Delta)$$

Then it is enough to show that

$$\mathcal{H}(\text{Fix}_{C_{2\pi}^1(P)}(\Delta) \times \mathbf{R} \times \mathbf{R}) \subseteq \text{Fix}_{C_{2\pi}^0(P)}(\Delta)$$

Now let  $x(s) \in \text{Fix}_{C_{2\pi}^1(P)}(\Delta)$ . Recall that  $h : P \rightarrow P_{\mathcal{C} \setminus \mathcal{S}}$  and  $P_{\mathcal{C} \setminus \mathcal{S}} \subset \text{Fix}_P(\Sigma_{\mathcal{S}})$ . Therefore,

$$\mathcal{H}(x, \tau, \lambda)(s) = h(x(s), \lambda) \in \text{Fix}_P(\Sigma_{\mathcal{S}}) \quad (s \in \mathbf{R})$$

for all  $\lambda, \tau \in \mathbf{R}$ . By Lemma 4.10 we have that

$$C_{2\pi}^0(\text{Fix}_P(\Sigma_{\mathcal{S}})) \subset C_{2\pi}^0(\text{Fix}_W(\Delta)) \oplus C_{2\pi}^0(\text{Fix}_P(\Sigma_{\mathcal{S}})) = \text{Fix}_{C_{2\pi}^0(P)}(\Delta)$$

and the result follows.  $\square$

**Remark 4.14** Equation (4.12) of Lemma 4.11 can be derived directly from the above lemma.  $\diamond$

**Proof of Theorem 4.8.** Consider the operator

$$\mathcal{F}_{\Delta} : \text{Fix}_{C_{2\pi}^1(P)}(\Delta) \times \mathbf{R} \times \mathbf{R} \longrightarrow \text{Fix}_{C_{2\pi}^0(P)}(\Delta)$$

The linearization of  $\mathcal{F}_{\Delta}$  about the origin is the linear operator  $\mathcal{L}_{\Delta}$ . Now we invoke the assumption that  $\Delta$  is  $\mathbf{C}$ -axial for the natural  $\Sigma_{\mathcal{S}} \times \mathbf{S}^1$ -action on  $E_i(A)$ , which together with Lemma 4.12 implies that

$$\dim_{\mathbf{R}} \ker(\mathcal{L}_{\Delta}) = 2$$

Now we may proceed as in the proof of the standard Hopf Bifurcation Theorem. If we identify  $\ker(\mathcal{L}_{\Delta}) \cong \mathbf{C}$  and then the action of  $\mathbf{S}^1$  on  $\ker(\mathcal{L}_{\Delta})$  is equivalent to the standard action of  $\mathbf{S}^1$  on  $\mathbf{C}$ . The Lyapunov-Schmidt reduction applied to  $\mathcal{F}_{\Delta}$  yields a  $\mathbf{S}^1$ -equivariant bifurcation equation

$$\phi : \mathbf{C} \times \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{C}$$

Moreover, the assumptions of the definition of codimension-one interior symmetry-breaking bifurcation are exactly the conditions necessary to carry out the proof.  $\square$

**Example 4.15** Consider the network  $\mathcal{G}_2$  of Figure 1. Suppose that for all cells  $c$  we choose the internal phase space to be  $P_c = \mathbf{C}$ . Then the total phase space is  $P = \mathbf{C}^4$ . Suppose that a smooth 1-parameter family  $f : P \times \mathbf{R} \rightarrow P$  of  $\mathcal{G}$ -admissible vector fields on  $P$  undergoes a codimension-one interior symmetry-breaking Hopf bifurcation at the equilibrium  $x_0 = 0$  when  $\lambda = 0$ . Then  $E_i(A) = W$ , where  $A$  is the  $\Sigma_S$ -equivariant sub-block of the linearization  $L = (df)_{(0,0)}$  of  $f$  at  $(0,0)$ . By Theorem 4.8 there are three branches of synchronously modulated  $\Delta$ -symmetric waves associated to the three conjugacy classes of interiorly  $\mathbf{C}$ -axial subgroups of  $\Sigma_S \times \mathbf{S}^1$  (see Table 1). Observe that the first periodic state of Table 1 is associated to a spatially  $\mathbf{C}$ -axial subgroup and so is predicted by [4, Theorem 3]. The third periodic state of Table 1 is an approximate rotating wave.  $\diamond$

Subgroup	Form of solution to lowest order in $\lambda$
$\mathbf{Z}_2$	$(w_1(t) + u(t), w_1(t) + u(t), w_2(t) + u(t), v(t))$
$\tilde{\mathbf{Z}}_2$	$(w_1(t) + u(t), w_1(t + \frac{1}{2}) + u(t), \hat{w}(t) + u(t), v(t))$
$\tilde{\mathbf{Z}}_3$	$(w_1(t) + u(t), w_1(t + \frac{1}{3}) + u(t), w_1(t + \frac{2}{3}) + u(t), v(t))$

Table 1: Branches of synchronously modulated  $\Delta$ -symmetric waves supported by the network  $\mathcal{G}_2$ . The hat  $\hat{\cdot}$  indicates that  $\hat{w}$  has twice the frequency.

## 4.4 Numerical Simulation

In this last section we illustrate the conclusions of Example 4.15 with a numerical simulation. In order to write down an explicit coupled cell system associated to network  $\mathcal{G}_2$  we choose the internal phase space of all four cells to be  $P_c = \mathbf{C} \cong \mathbf{R}^2$ .

Consider the coupled cell system

$$\begin{aligned}
 \dot{x}_1 &= g(x_1, x_2, x_3) + 2x_4 \\
 \dot{x}_2 &= g(x_2, x_3, x_1) + 2x_4 \\
 \dot{x}_3 &= g(x_3, x_1, x_2) + 2x_4 \\
 \dot{x}_4 &= -x_4 + e_1x_1 + e_2x_2 + e_3x_3
 \end{aligned} \tag{4.16}$$

where  $g : (\mathbf{R}^2)^3 \rightarrow \mathbf{R}^2$  is given by

$$g(x, y, z) = -x + (a - 2b_2)x\|x\|^2 + b_1(y + z) + b_2(y\|y\|^2 + z\|z\|^2) \\ + a(x\|y\|^2 + x\|z\|^2) + b_3(y\|y\|^4 + z\|z\|^4)$$

and  $a, b_1(\lambda), b_2, b_3, e_1, e_2, e_3$  are  $2 \times 2$  matrices with  $b_1$  depending smoothly on a parameter  $\lambda$ . Let  $f$  be the vector field defined by (4.16). Observe that the origin is an equilibrium point for all  $\lambda$

$$f(0, \lambda) \equiv 0$$

The linearization of  $f$  at  $(0, \lambda)$  is given by (as  $2 \times 2$  block matrix)

$$L(\lambda) = \begin{pmatrix} -1 & b_1 & b_1 & 1 \\ b_1 & -1 & b_1 & 1 \\ b_1 & b_1 & -1 & 1 \\ e_1 & e_2 & e_3 & -1 \end{pmatrix}$$

where  $\pm 1$  represents  $\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ .

We need to choose the coefficients  $b_1$  and  $e_1, e_2, e_3$  in order to have purely imaginary eigenvalues for some  $\lambda$  coming from the sub-block  $A$  when  $L$  is written in the form (3.12). The following values will do the work:

$$b_1(\lambda) = \begin{pmatrix} -1 - \lambda & -1.5 \\ 1.5 & -1 \end{pmatrix}$$

and any values between  $-1$  and  $1$  for the entries of the matrices  $e_1, e_2, e_3$ .

The spectrum of the matrix  $L(\lambda)$  has the following properties:

- (1) For  $\lambda < 0$  all eigenvalues of  $L(\lambda)$  have negative real parts.
- (2) For  $\lambda = 0$  the matrix  $L = L(0)$  has two pairs of eigenvalues  $\pm i$  and the remaining eigenvalues have negative real parts. Moreover, the eigenvectors associated to the purely imaginary eigenvalues are not in  $\text{Fix}(\mathbf{S}_3)$ .
- (3) For  $\lambda > 0$  all eigenvalues of  $L(\lambda)$  whose associated eigenvectors are in  $\text{Fix}(\mathbf{S}_3)$  have negative real parts and the remaining eigenvalues have positive real parts.

Thus (4.16) undergoes a interior symmetry-breaking Hopf bifurcation when  $\lambda = 0$  giving rise to one branch of periodic solutions for each one of the three interiorly  $\mathbf{C}$ -axial subgroups of  $\mathbf{S}_3 \times \mathbf{S}^1$  as in Table 1, when  $\lambda > 0$ . However, depending on the choice of the coefficients  $a$ ,  $b_2$  and  $b_3$  of  $g$ , one can make at least of these periodic solutions to be stable. In our simulations we have chosen the following coefficients:

$$a = \begin{pmatrix} -0.5 & 0 \\ 0 & -0.5 \end{pmatrix}$$

in all simulations and

- (1) to obtain the solution with symmetry  $\tilde{\mathbf{Z}}_3$ :

$$b_2 = \begin{pmatrix} 0.6 & 2 \\ 2 & 0.6 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

- (2) to obtain the solution with (interior) symmetry  $\tilde{\mathbf{Z}}_2$ :

$$b_2 = \begin{pmatrix} -0.6 & 1 \\ 1 & -0.6 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0.2 & -0.7 \\ -0.7 & 0.2 \end{pmatrix}$$

- (3) to obtain the solution with (interior) symmetry  $\mathbf{Z}_2$ :

$$b_2 = \begin{pmatrix} -0.6 & 0 \\ 0 & -0.6 \end{pmatrix}, \quad b_3 = \begin{pmatrix} 0 & 0.7 \\ 0.7 & 0 \end{pmatrix}$$

The coefficients  $e_1$ ,  $e_2$  and  $e_3$  represent the coupling that break the  $\mathbf{S}_3$ -symmetry. If  $e_1 = e_2 = e_3$  then the coupled cell system (4.16) is admissible for the network  $\mathcal{G}_1$  of Figure 1 and so it is  $\mathbf{S}_3$ -symmetric. On the other hand, if  $e_1 \neq e_2 \neq e_3$  then the coupled cell system (4.16) is admissible for the network  $\mathcal{G}_2$  of Figure 1 and have genuine  $\mathbf{S}_3$ -interior symmetry.

In the following we present the results of numerical simulations obtaining the three types of periodic solutions mentioned above, for both of the networks  $\mathcal{G}_1$  and  $\mathcal{G}_2$  of our running example. In Figures 3, 4 and 5 we superimpose the time series of all four cells, which are identified by colours:

$$1 = \text{blue} \quad 2 = \text{red} \quad 3 = \text{green} \quad 4 = \text{black}$$

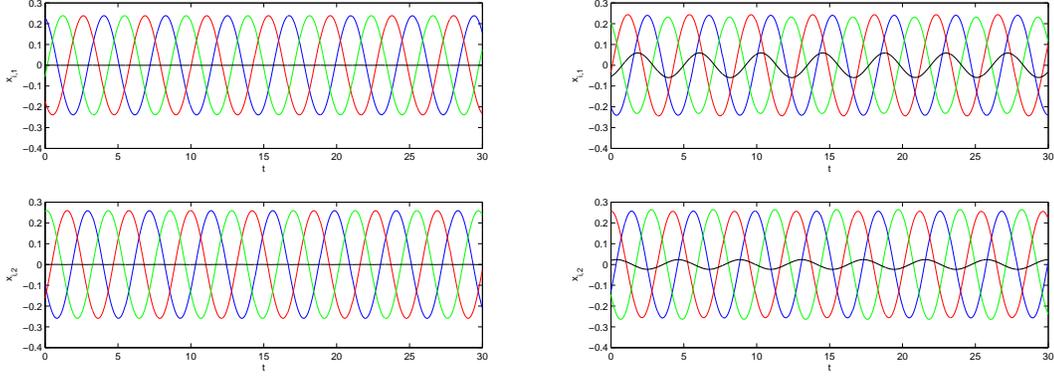


Figure 3: Solutions with  $\tilde{\mathbf{Z}}_3$  (interior) symmetry. (left) Network  $\mathcal{G}_1$  with exact  $\mathbf{S}_3$ -symmetry. (right) Network  $\mathcal{G}_2$  with  $\mathbf{S}_3$ -interior symmetry.

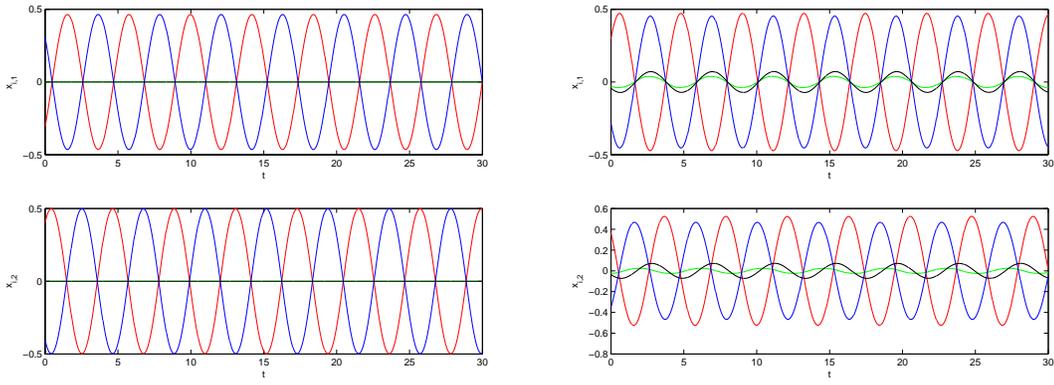


Figure 4: Solutions with  $\tilde{\mathbf{Z}}_2$  (interior) symmetry. (left) Network  $\mathcal{G}_1$  with exact  $\mathbf{S}_3$ -symmetry. (right) Network  $\mathcal{G}_2$  with  $\mathbf{S}_3$ -interior symmetry.

The upper panels show the first components and the lower panels show the second components. The left panels refer to network  $\mathcal{G}_1$  with exact  $\mathbf{S}_3$ -symmetry and the panels on the right refer to network  $\mathcal{G}_1$  with  $\mathbf{S}_3$ -interior symmetry. Figure 6 present the solution with interior symmetry  $\tilde{\mathbf{Z}}_3$  of network  $\mathcal{G}_2$ , i.e., the approximate rotating wave from Figure 3 (right), viewed in difference coordinates: blue =  $x_1 - x_2$ , green =  $x_2 - x_3$  and red =  $x_3 - x_1$ .

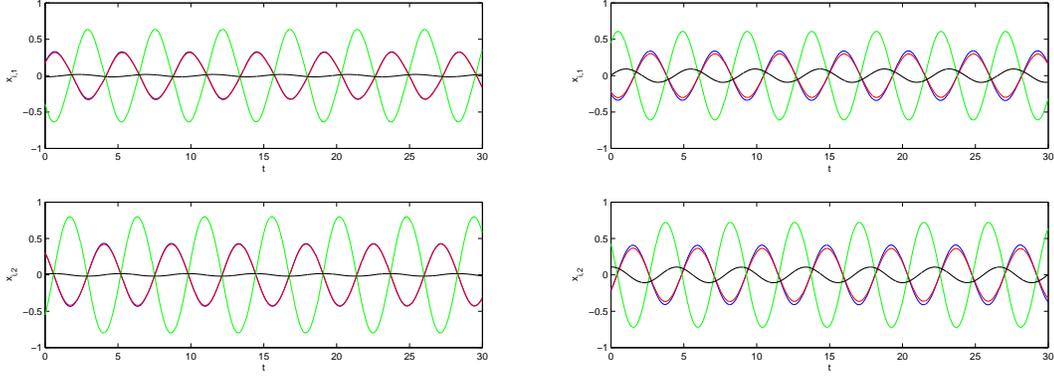


Figure 5: Solutions with  $\mathbf{Z}_2$  (interior) symmetry. (left) Network  $\mathcal{G}_1$  with exact  $\mathbf{S}_3$ -symmetry. (right) Network  $\mathcal{G}_2$  with  $\mathbf{S}_3$ -interior symmetry.

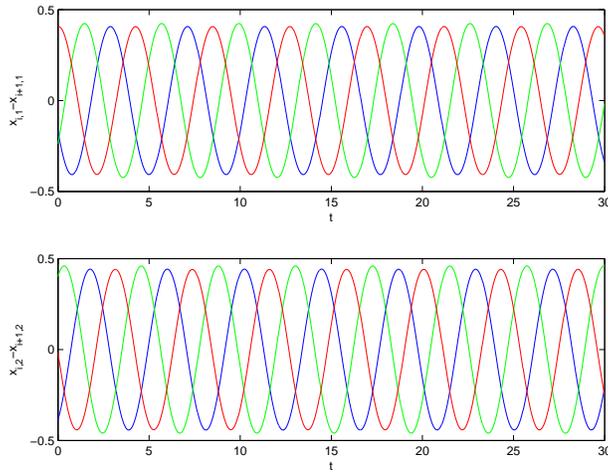


Figure 6: Approximate rotating wave in network  $\mathcal{G}_2$ , viewed in difference coordinates:  $x_1 - x_2$ ,  $x_2 - x_3$  and  $x_3 - x_1$ .

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