Hopf Bifurcation on Hemispheres

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Abstract

Field, Golubitsky and Stewart (Bifurcations on Hemispheres. J. Nonlinear Sci. 1 (1991) 201-223.) consider the steady-state bifurcations of reaction-diffusion equations defined on the hemisphere with Neumann boundary conditions on the equator. We consider Hopf bifurcations for these equations. We show the effect of the hidden symmetries on spherical domains for the type of Hopf bifurcations that can occur. We obtain periodic solutions for the hemisphere problem by extending the problem to the sphere and finding then periodic solutions with spherical spatial symmetries containing the reflection across the equator. The equations on the hemisphere have $O(2)$-symmetry and the equations on the sphere have spherical symmetry. We find orbits of periodic solutions to the sphere problem containing multiple periodic solutions that restrict to periodic solutions of the Neumann boundary value problem on the hemisphere lying on different $O(2)$-orbits.

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1 Introduction

It is known that the bifurcation behavior of reaction-diffusion equations on certain domains with certain boundary conditions can be nongeneric taking into account only the symmetries of the equations. Let $u_t + \mathcal{P}(u) = 0$ be a system of reaction-diffusion equations posed on a bounded domain $N \subset \mathbb{R}^n$ with Neumann boundary conditions (NBC) on the boundary of $N$. It follows that the group of the physical symmetries of the equations is the compact subgroup

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\( \Gamma \) of \( E(n) \) that preserves the domain \( N \) and the boundary conditions. For certain domains these problems possess more symmetry than it is immediately apparent, and these ‘hidden symmetries’ in \( E(n) \setminus \Gamma \) can be responsible for subtle changes in the generic bifurcations of such systems. These extra symmetries naturally appear when the equations may be extended to larger domains with larger symmetry groups. This was first noticed by Fujii et al. [6] and formalized by Armbruster and Dangelmayr [1] for reaction-diffusion equations on an interval with NBC. Crawford et al. [4] developed these ideas. Subsequent work on this subject has been studied by several authors. See the review paper of Gomes et al. [10] and references therein.

Field et al. [5] studied hidden symmetries on hemispherical domains, and described a general setting for hidden symmetries induced by Neumann and Dirichlet boundary conditions on a large class of spatial domains. Moreover, these authors considered steady-state bifurcations for reaction-diffusion equations defined on the hemisphere with NBC along the equator. As they pointed out, such equations have a natural \( O(2) \)-symmetry but may be extended to the full sphere where the natural symmetry group is \( O(3) \). They show that the expected bifurcations are governed not by circular symmetry but by spherical symmetry. More precisely, solutions to the Neumann boundary problem on the hemisphere can be found by first finding solutions to the extended problem on the sphere invariant by the reflection across the equator (the boundary of the hemisphere). Their results were recently applied when modelling the growth of plants, see Nagata et al. [16]. In [16], the authors referred much of the growth of plants occurs by the elongation of cylindrical stalks or roots by action mainly at a dome-shaped tip, for which a hemisphere is a reasonable approximation for modelling. Other applications include elastic buckling of hemispherical shells, see Bauer et al. [2, 3].

Hopf bifurcations of reaction-diffusion equations in \( n \)-dimensional rectangles were studied by Gomes and Stewart [11] considering the effect of hidden symetries.

In this paper we extend the results of Field et al. [5] by showing the effect of the hidden symmetries on spherical domains for the type of Hopf bifurcations that can occur. Specifically, we consider the Hopf bifurcations of reaction-diffusion equations defined on the hemisphere with NBC along the equator.

Golubitsky and Stewart [8] give a list of those conjugacy classes of isotropy subgroups of \( O(3) \times S^1 \) (action on \( V_l \oplus V_l \) for each \( l \)) that have two-dimensional fixed-point subspaces. Here, \( V_l \) denotes the space of spherical harmonics of order \( l \). See Table 2 for the natural representation of \( O(3) \). The Equivariant Hopf Theorem guarantees the existence, in generic Hopf bifurcation problems with \( O(3) \) symmetry, of a branch of periodic solutions for each isotropy subgroup in each of the conjugacy classes. Hopf bifurcation with spherical symmetry for \( l \) equals two was studied by Iooss and Rossi [14], and Haaf et al. [12].

We obtain periodic solutions for the hemisphere problem by extending the problem to the sphere and finding then periodic solutions with spherical spatial symmetries containing the reflection across the equator. Namely, we determine group orbits of periodic solutions in generic \( O(3) \)-equivariant Hopf bifurcations problems that have representative solutions with symmetry containing the above reflection. Each orbit contains multiple periodic solutions that restrict to periodic solutions of the Neumann boundary value problem on the hemisphere lying on different \( O(2) \)-orbits. See Theorems 5.1, 6.1, 6.3. In this way we find solutions to the hemisphere problem that would not have been expected if we would have considered only
We conclude the introduction by illustrating two examples contained in our results. The Equivariant Hopf Theorem guarantees the generic existence of an orbit of periodic solutions with an axis of rotation for the sphere problem. In Theorem 5.1, we prove that when \( l \) is even, this orbit of periodic solutions restricts to solutions of the hemisphere problem as follows: an isolated solution with \( O(2) \)-symmetry, and an \( O(2) \)-orbit of solutions with \( D_2 \)-symmetry. The Equivariant Hopf Theorem also guarantees the existence of an orbit of periodic solutions with twisted tetrahedral symmetry of twist type \( \mathbb{Z}_3 \), when \( l = 2, 4, 6 \). In Theorem 6.3, we show that this orbit of periodic solutions restricts to the hemisphere problem to three \( O(2) \)-orbits of solutions with spatial dihedral symmetry \( D_2 \) of trivial twist type, that is, they have no spatio-temporal symmetries. We show in Figures 1, 2 periodic one-parameter families of deformations of the sphere with twisted tetrahedral symmetry and their restriction to the hemisphere for \( l = 6 \). See Section 7 for details.

This paper is organized in the following way. In Section 2 we review the basics of Hopf bifurcation, and in Section 3 we state the main results on Hopf bifurcation with spherical symmetry. The problem of Hopf bifurcations of reaction-diffusion equations defined on the hemisphere with Neumann boundary conditions along the equator is described in Section 4. The main results of this paper are obtained in Sections 5 and Section 6. Theorem 5.1 describes solutions of the sphere problem with an axis of rotation that restrict to solutions of the hemisphere problem. In Theorems 6.1 and 6.3 we consider periodic solutions with finite spatial symmetry. Figures illustrating the results obtained in the previous theorems are presented in Section 7. A more abstract formulation of the extension problem following Field et al. [5] is presented in Section 8.
2 Symmetry-Breaking in Hopf Bifurcation

In this section we review the results of Golubitsky and Stewart [8] (see also Golubitsky et al. [9, Chapter XVI]), which relate the existence of branches of periodic solutions of \( \Gamma \)-symmetric bifurcation problems, where \( \Gamma \) is a compact Lie group, to the geometry of a \( \Gamma \times S^1 \) action on a \( \Gamma \)-simple subspace of the original total phase space.

Suppose that we have the system of ordinary differential equations

\[
\frac{dx}{dt} = F(x, \lambda)
\]  

(2.1)

where \( x \in \mathbb{R}^n, \lambda \in \mathbb{R} \) is a bifurcation parameter, and \( F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) is a smooth \((C^\infty)\) mapping commuting with the action of a compact Lie group \( \Gamma \) on \( \mathbb{R}^n \). We assume that \( F(0, \lambda) \equiv 0 \), so that there is a trivial \( \Gamma \)-invariant equilibrium solution \( x = 0 \).

Let \( (dF)_{x, \lambda} \) be the \( n \times n \) Jacobian matrix of derivatives of \( F \) with respect to the variables \( x_j \), evaluated at \( (x, \lambda) \). The most important hypothesis of the standard Hopf theorem (cf. [7, Theorem VIII 3.1]) is that \( (dF)_{0,0} \) should have a pair of simple purely imaginary eigenvalues. Under additional hypotheses of nondegeneracy, this condition implies the occurrence of a branch of periodic solutions. Generally, in the presence of a symmetry group \( \Gamma \), there are nontrivial restrictions on the imaginary eigenspace: it must contain a \( \Gamma \)-simple invariant subspace ( [9, Lemma XVI 1.2]). Recall that a representation \( W \) of \( \Gamma \) is \( \Gamma \)-simple if either: \( W \cong V \oplus V \) where \( V \) is absolutely irreducible for \( \Gamma \), or \( W \) is irreducible but not absolutely irreducible for \( \Gamma \). Note that a representation \( V \) of \( \Gamma \) is absolutely irreducible if the only linear maps on \( V \) commuting with \( \Gamma \) are real multiples of the identity. Moreover, generically the imaginary eigenspace itself is \( \Gamma \)-simple ( [9, Proposition XVI 1.4]). Rescalling time if necessary, we can assume that the imaginary eigenvalues are \( \pm i \). Suppose now that \( \mathbb{R}^n \) is
\(\Gamma\)-simple and \((dF)_{0,0}\) has \(i\) as an eigenvalue. It follows then that we may assume \((dF)_{0,0} = J\) where

\[
J = \begin{pmatrix}
0 & -I_m \\
I_m & 0
\end{pmatrix}
\]

with \(m = n/2\), and the eigenvalues of \((dF)_{0,0}\) consist of a complex conjugate pair \(\sigma(\lambda) \pm i\rho(\lambda)\), each of multiplicity \(m\), where \(\sigma\) and \(\rho\) are smooth functions of \(\lambda\) ([8, Lemma 1.2]).

In the symmetric context, periodic solutions may have not only spatial symmetry but also temporal symmetry. The temporal symmetries are phase shifts and may be thought of as elements of a circle group \(S^1\), acting on the infinite-dimensional space of \(2\pi\)-periodic functions. In order to detect those periodic solutions whose symmetries combine both space and time, we must extend the group \(\Gamma\) to \(\Gamma \times S^1\). Suppose that \(v(t)\) is a \(2\pi\)-periodic function in \(t\) (rescaling time if necessary to make the period \(2\pi\)) and identify the circle \(S^1\) with \(\mathbb{R}/2\pi\mathbb{Z}\). Then a symmetry of the periodic function \(v(t)\) is an element \((\gamma, \theta) \in \Gamma \times S^1\) such that

\[
\gamma v(t) = v(t - \theta)
\]

In this sense the symmetry \((\gamma, \theta)\) is a mixture of spatial and temporal symmetries. The term spatial refers to the vector field symmetry of the original equation rather than actual physical space, and \(S^1\) acts on the space of \(2\pi\)-periodic mappings \(v(t)\), not on \(\mathbb{R}^n\). We call this action of \(S^1\) the phase-shift action. The collection of all symmetries for \(v(t)\) forms a group

\[
\Sigma_{v(t)} = \{ (\gamma, \theta) \in \Gamma \times S^1 : \gamma v(t) = v(t - \theta) \} \subseteq \Gamma \times S^1
\]

and there is a natural action of \(\Gamma \times S^1\) on the space \(C_{2\pi}\) of \(2\pi\)-periodic mappings \(\mathbb{R} \rightarrow \mathbb{R}^n\), defined by

\[
(\gamma, \theta).v(t) = \gamma v(t + \theta)
\]

That is, the \(\Gamma\)-action is induced from its spatial action on \(\mathbb{R}^n\), and the \(S^1\) action is by phase shift. Rewriting (2.2) as \((\gamma, \theta).v(t) = v(t)\) we see that \(\Sigma_{v(t)}\) is just the isotropy subgroup of \(v(t)\) with respect to this action.

The proof of the Hopf bifurcation theorem by Liapunov-Schmidt reduction generalises to the \(\Gamma\)-equivariant context, including these phase-shift symmetries in a natural way: the Liapunov-Schmidt reduction induces a related but different action of \(S^1\) on a finite-dimensional space, which can be identified with the exponential of the linearization \(J = (dF)_{0,0}\) acting on the imaginary eigenspace (which we can assume to be \(\mathbb{R}^n\)).

**The Action of \(\Gamma \times S^1\)**

We describe now the action of \(\Gamma \times S^1\) in the case that \(\mathbb{R}^n\) is a \(\Gamma\)-simple space \(V \oplus V\), where \(V\) is an absolutely irreducible space for \(\Gamma\). We can fix an orthonormal basis of \(V\) and let the vector \((x_1, \ldots, x_m)\) denote the coordinates of the vector \(x \in V\) in this basis. The action of \(\Gamma\) allows us to identify each \(\gamma \in \Gamma\) with an \(m \times m\) matrix acting on (the coordinates of) \(V\). We denote that matrix by \(\gamma\) itself. We may identify \((x, y) \in V \oplus V\) with the \(m \times 2\) matrix with columns given by \((x_1, \ldots, x_m)^t\) and \((y_1, \ldots, y_m)^t\) and denoted by \([x|y]\). The jacobian matriz \(J\) may be put in the form

\[
\begin{bmatrix}
0 & -I_m \\
I_m & 0
\end{bmatrix}
\]
where \( m = n/2 \). Then direct computation of \( e^{-sJ} \) leads to the following description: let \((x, y) \in V \oplus V\) and \((\gamma, \theta) \in \Gamma \times S^1\). Then

\[
(\gamma, \theta)(x, y) = \gamma[x|y]R_{\theta}
\]

where

\[
R_{\theta} = \begin{bmatrix}
\cos(\theta) & -\sin(\theta) \\
\sin(\theta) & \cos(\theta)
\end{bmatrix}
\]

is the usual rotation matrix.

**Remark 2.1**

(i) The only vector of \( V \oplus V \) fixed by some \( \theta \neq 0 \) in \( S^1 \) is the zero vector. That is, the action of \( S^1 \) on \( V \oplus V \) is fixed-point-free.

(ii) The group \( S^1 \) commutes with all of \( \Gamma \times S^1 \).

\( \diamond \)

**The Equivariant Hopf Theorem**

Given an action of a compact Lie group \( \Gamma \) on \( \mathbb{R}^n \) and \( x \in \mathbb{R}^n \), the isotropy subgroup \( \Sigma_x \) of \( x \) is the subgroup of \( \Gamma \) defined by

\[
\Sigma_x = \{ \gamma \in \Gamma : \gamma \cdot x = x \}
\]

and the fixed-point subspace of \( \Sigma_x \) is the subspace of \( \mathbb{R}^n \) given by

\[
\text{Fix}(\Sigma_x) = \{ v \in \mathbb{R}^n : \sigma \cdot v = v \text{ for all } \sigma \in \Sigma_x \}
\]

If \( F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n \) is a smooth map that commutes with (or it is equivariant by) \( \Gamma \):

\[
F(\gamma \cdot x, \lambda) = \gamma \cdot F(x, \lambda)
\]  \hspace{1cm} (2.4)

for all \( \gamma \in \Gamma \), \( x \in \mathbb{R}^n \), \( \lambda \in \mathbb{R} \), then

\[
F(\text{Fix}(\Sigma_x) \times \mathbb{R}) \subseteq \text{Fix}(\Sigma_x)
\]

since if \( v \in \text{Fix}(\Sigma_x) \) and \( \sigma \in \Sigma_x \), then \( \sigma \cdot F(v) = F(\sigma \cdot v) = F(v) \) and so \( F(v) \in \text{Fix}(\Sigma_x) \). Thus \( \text{Fix}(\Sigma_x) \) is a flow-invariant subspace for (2.1).

Assuming the generic hypothesis that the eigenvalues \( \sigma(\lambda) \pm i\rho(\lambda) \) of \( (dF)_{0,\lambda} \) cross the imaginary axis with nonzero speed, that is,

\[
\sigma'(0) \neq 0
\]  \hspace{1cm} (2.5)

we have the following result:

**Theorem 2.2 (Equivariant Hopf Theorem)** Let the system of ordinary differential equations (2.1), where \( F \) commutes with a compact Lie group \( \Gamma \), satisfy

\[
(dF)_{0,0} = J = \begin{pmatrix}
0 & -I_m \\
I_m & 0
\end{pmatrix}
\]

\[ 6 \]
and (2.5) where \( m = n/2 \). Suppose that \( \Sigma \) is an isotropy subgroup of \( \Gamma \times S^1 \) such that
\[
\text{dim Fix}(\Sigma) = 2.
\]

Then there exists a unique branch of small amplitude periodic solutions to (2.1) with period near \( 2\pi \), having \( \Sigma \) as their group of symmetries.

**Proof** See Golubitsky and Stewart [8, Theorem 1.2]. \( \square \)

### Isotropy Subgroups of \( \Gamma \times S^1 \)

In order to apply the Equivariant Hopf Theorem we require information on what form isotropy subgroups \( \Sigma \) of \( \Gamma \times S^1 \) can take and which of them have two-dimensional fixed-point subspaces. We now discuss the form of the isotropy subgroups.

The isotropy subgroups of \( \Gamma \times S^1 \) can be characterized as “twisted” subgroups, as defined below:

**Definition 2.3** Let \( H \subset \Gamma \) be a subgroup and let \( \theta : H \to S^1 \) be a group homomorphism. We call
\[
H^\theta = \{(h, \theta(h)) \in \Gamma \times S^1 : h \in H\}
\]
a twisted subgroup of \( \Gamma \times S^1 \).

All proper isotropy subgroups of \( \Gamma \times S^1 \) are twisted subgroups [8, Proposition 6.2]. That is, if \( \Sigma \) is an isotropy subgroup of \( \Gamma \times S^1 \) acting on a \( \Gamma \)-simple space, with \( \Sigma \neq \Gamma \times S^1 \), and if \( H \) is the projection of \( \Sigma \) on \( \Gamma \) then there is an homomorphism \( \theta : H \to S^1 \) such that \( \Sigma = H^\theta \). In particular, it follows that \( K = \ker(\theta) \) is a normal subgroup of \( H \). Moreover, \( H/K \) is isomorphic to \( \text{Im}(\theta) \). Also \( \text{Im}(\theta) \) is a closed subgroup of \( S^1 \). Recall that the closed subgroups of \( S^1 \) are 1, \( \mathbb{Z}_n \) (\( n = 2, 3, 4, \ldots \)) and \( S^1 \). We say \( \theta \) is trivial, \( \mathbb{Z}_n \), or \( S^1 \) according as \( \text{Im}(\theta) = 1 \), \( \mathbb{Z}_n \), or \( S^1 \).

Intuitively we think of elements of \( \Gamma \) as spatial symmetries (acting on \( \mathbb{R}^n \)), and elements of \( S^1 \) as temporal symmetries, acting on periodic solutions by phase shift. In this sense, an element \( \sigma = (h, \theta(h)) \in \Gamma \times S^1 \) is a spatial symmetry if \( \theta(h) = 0 \) and a combined spatio-temporal symmetry if \( \theta(h) \neq 0 \). For a given isotropy subgroup \( H^\theta \subset \Gamma \times S^1 \), the spatial symmetries form the subgroup \( K \) of \( H \). Note that if \( H^\theta = \Sigma_{v(t)} \) then
\[
K = \{ \gamma \in \Gamma : \gamma v(t) = v(t) \ \forall t \}
\]
and
\[
H = \{ \gamma \in \Gamma : \gamma \{v(t)\} = \{v(t)\} \}.
\]

Let \( H^\theta \) and \( L^\psi \) be two conjugate isotropy subgroups of \( \Gamma \times S^1 \). Then \( H, L \) are conjugate isotropy subgroups of \( \Gamma \), and \( \ker(\theta), \ker(\psi) \) are conjugate isotropy subgroups of \( \Gamma \) [9, Lemma XVI 7.3].
3 Hopf Bifurcation with \(O(3)\)-symmetry

In this section, we study the existence of branches of periodic solutions for \(O(3)\)-equivariant bifurcation problems. We state, without proofs, the main results on Hopf bifurcation with spherical symmetry (\cite{8}, see also \cite{9, Section XVIII 5}). Namely, we give, for each irreducible representation \(V_l\) of the orthogonal group \(O(3)\), a list of isotropy subgroups \(\Sigma \subset O(3) \times S^1\) for which the Equivariant Hopf Theorem proves the existence of a branch of periodic solutions. Specifically, we give a classification of those isotropy subgroups \(\Sigma\) for which \(\dim \text{Fix}_{V_l \oplus V_l}(\Sigma) = 2\), when \(V_l\) is any irreducible representation of \(O(3)\).

Representations of the group \(O(3)\)

The orthogonal group \(O(3)\) consists of all \(3 \times 3\) matrices \(A\) such that \(A^t = A^{-1}\), That is \(\det(A) = \pm 1\). The special orthogonal group \(SO(3)\) consists of the elements in \(O(3)\) with positive determinant. Algebraically, the orthogonal group is just a direct sum

\[
O(3) = SO(3) \oplus \mathbb{Z}_2^c
\]

where \(\mathbb{Z}_2^c = \{\pm I\}\). Each irreducible representation of \(SO(3)\) gives rise to two irreducible representations of \(O(3)\) corresponding to the possibilities of \(-I\) acting trivially or as minus the identity. In the natural action (or standard action) of \(O(3)\) the element \(-I\) acts trivially if \(l\) is even and nontrivially when \(l\) is odd.

The irreducible representations of the rotation group \(SO(3)\) have dimension \(2l + 1\), \(l = 0, 1, 2, \ldots\) Up to isomorphism, there is only one such representation, denoted by \(V_l\), the spherical harmonics of order \(l\) and the action of \(SO(3)\) on \(V_l\) is induced from the standard action on \(\mathbb{R}^3\). See for example Miller \cite{15}.

Subgroups of \(O(3)\)

The closed subgroups of \(O(3)\) fall in three classes \cite{15}:

Class I. Subgroups of \(SO(3)\).

Class II. Subgroups of \(O(3)\) that contain \(-I\).

Class III. Subgroups of \(O(3)\) that do not fall into the classes I and II.

The subgroups of class I consist of the planar subgroups \(O(2)\), \(SO(2)\), \(D_m\) \((m \geq 2)\), \(Z_m\) \((m \geq 1)\), and the exceptional subgroups \(I, O, T\). The planar subgroups are the symmetry groups of the unoriented and oriented circle and \(m\)-gon respectively. The exceptional subgroups are the rotation groups of the icosahedron, octahedron and tetrahedron.

The subgroups of class II are of the form \(\Sigma \oplus \mathbb{Z}_2^c\) where \(\Sigma\) is a subgroup of class I.

From \cite[Lemma 2.7]{13} each subgroup \(\Sigma\) of class III is determined by two subgroups \(K\) and \(L\) of \(SO(3)\) where \(K\) is isomorphic to \(\Sigma\), and \(L\) is of index 2 in \(K\). Moreover, \(K = \pi(\Sigma)\), \(L = \Sigma \cap SO(3)\), where \(\pi\) is the projection

\[
\pi: O(3) \rightarrow SO(3) \quad \pi(\pm I\gamma) = \gamma \quad \text{for all } \gamma \in SO(3)
\]

See Table 1. Note that

\[
O(2)^\perp = SO(2) \cup (-\kappa)SO(2)
\]
<table>
<thead>
<tr>
<th>$\Sigma$</th>
<th>$K$</th>
<th>$L$</th>
<th>$\Sigma$</th>
<th>$K$</th>
<th>$L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O(2)^-$</td>
<td>$O(2)$</td>
<td>$SO(2)$</td>
<td>$D^d_{2m}$ ($m \geq 2$)</td>
<td>$D_{2m}$</td>
<td>$D_m$</td>
</tr>
<tr>
<td>$O^-$</td>
<td>$O$</td>
<td>$T$</td>
<td>$D^-<em>{m} = D^z</em>{m}$ ($m \geq 2$)</td>
<td>$D_{m}$</td>
<td>$Z_m$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$Z^c_{2m}$ ($m \geq 1$)</td>
<td>$Z_{2m}$</td>
<td>$Z_m$</td>
</tr>
</tbody>
</table>

Table 1: The subgroups $\Sigma$ of $O(3)$ of class III.

where

$$-\kappa = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\cup$ denotes disjoint union.

If $\Sigma$ is a subgroup of $O(3)$, its normalizer $N_{O(3)}(\Sigma)$ is the group

$$N_{O(3)}(\Sigma) = \{ \gamma \in O(3) : \gamma^{-1} \Sigma \gamma = \Sigma \}$$

This is the largest subgroup of $O(3)$ in which $\Sigma$ is normal. Moreover,

$$N_{O(3)}(\Sigma) = N_{SO(3)}(\pi(\Sigma)) \oplus Z^c_2$$

In particular, we have

$$N_{O(3)}(O(2)^-) = O(2) \oplus Z^c_2, \quad N_{O(3)}(O(2) \oplus Z^c_2) = O(2) \oplus Z^c_2$$

**Isotropy Subgroups of $O(3) \times S^1$**

Recall that the natural representation of $O(3)$, arising in most applications, is plus when $l$ is even, and minus when $l$ is odd. (These are the natural representations induced on spherical harmonics by the standard actions of $O(3)$ on the 2-sphere in $\mathbb{R}^3$.) It is the natural representation that we shall consider in this paper.

Golubitsky and Stewart [8] give a list of those conjugacy classes of isotropy subgroups of $O(3) \times S^1$ (action on $V_l \oplus V_l'$ for each $l$) that have two-dimensional fixed-point subspaces. The Equivariant Hopf Theorem (Theorem 2.2) guarantees the existence of a branch of periodic solutions, in generic Hopf bifurcation problems with $O(3)$ symmetry, for each isotropy subgroup in each of those conjugacy classes. We reproduce their results for the natural representation in Table 2. Recall that any isotropy subgroup of $O(3) \times S^1$ is a twisted subgroup $H^\theta$ where $H$ is the projection of $\Sigma$ on $O(3)$ and $\theta : H \to S^1$ is a group homomorphism. Denote by $K = \ker(\theta)$. Suppose now that $H^\theta$ is an isotropy subgroup of $O(3) \times S^1$ with two-dimensional fixed-point space. It turns out that $H$ must be a closed subgroup of $O(3)$ of type II, so that $H = J \oplus Z^c_2$ where $J \subset SO(3)$. To see this note that for the plus representation, $(-I, 0)$ lies in every isotropy subgroup, and for the minus representation, $(-I, \pi)$ lies in every isotropy subgroup. Therefore $H$ is a closed subgroup of $O(3)$ that contains $Z^c_2$ and
so it is of type II. It also follows that for the plus representation, $K$ is of type II, and for the minus representation, $K$ is of type I or III. The strategy of Golubitsky and Stewart [8] for finding the isotropy subgroups with two-dimensional fixed-point subspaces was to classify first by twist type, and second by the type (I, II, or III) of $K$.

<table>
<thead>
<tr>
<th>$J$</th>
<th>Type of $K$</th>
<th>Type of $K$</th>
<th>Twist $\theta(H)$</th>
<th>Value of $l$</th>
<th>Value of $l$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$O(2)$</td>
<td>II</td>
<td>$O(2) \oplus \mathbb{Z}_2^c$</td>
<td>1</td>
<td>even $l$</td>
</tr>
<tr>
<td>2</td>
<td>$O(2)$</td>
<td>III</td>
<td>$O(2)^-$</td>
<td>$\mathbb{Z}_2$</td>
<td>odd $l$</td>
</tr>
<tr>
<td>3</td>
<td>$SO(2)$</td>
<td>II</td>
<td>$Z_k \oplus \mathbb{Z}_2^c$</td>
<td>$S^1$</td>
<td>even $l$</td>
</tr>
<tr>
<td>4</td>
<td>$SO(2)$</td>
<td>III</td>
<td>$Z_{2k}^-$</td>
<td>$S^1$</td>
<td>odd $l$</td>
</tr>
<tr>
<td>5</td>
<td>$I$</td>
<td>II</td>
<td>$I \oplus \mathbb{Z}_2^c$</td>
<td>1</td>
<td>6, 10, 12, 16, 18, 20, 22, 24, 26, 28, 32, 34, 38, 44</td>
</tr>
<tr>
<td>6</td>
<td>$I$</td>
<td>I</td>
<td>$I$</td>
<td>$\mathbb{Z}_2$</td>
<td>21, 25, 27, 31, 33, 35, 37, 39, 41, 43, 47, 49, 53, 59</td>
</tr>
<tr>
<td>7</td>
<td>$O$</td>
<td>II</td>
<td>$O \oplus \mathbb{Z}_2^c$</td>
<td>1</td>
<td>4, 6, 8, 10, 14</td>
</tr>
<tr>
<td>8</td>
<td>$O$</td>
<td>I</td>
<td>$O$</td>
<td>$\mathbb{Z}_2$</td>
<td>9, 13, 15, 17, 19, 23</td>
</tr>
<tr>
<td>9</td>
<td>$O$</td>
<td>II</td>
<td>$T \oplus \mathbb{Z}_2^c$</td>
<td>$\mathbb{Z}_2$</td>
<td>6, 10, 12, 14, 16, 20</td>
</tr>
<tr>
<td>10</td>
<td>$O$</td>
<td>III</td>
<td>$O^-$</td>
<td>$\mathbb{Z}_2$</td>
<td>3, 7, 9, 11, 13, 17</td>
</tr>
<tr>
<td>11</td>
<td>$T$</td>
<td>II</td>
<td>$D_2 \oplus \mathbb{Z}_2^c$</td>
<td>$\mathbb{Z}_2$</td>
<td>2, 4, 6</td>
</tr>
<tr>
<td>12</td>
<td>$T$</td>
<td>I</td>
<td>$D_2$</td>
<td>$\mathbb{Z}_0$</td>
<td>5, 7, 9</td>
</tr>
<tr>
<td>13</td>
<td>$D_n$</td>
<td>II</td>
<td>$D_n/2 \oplus \mathbb{Z}_2^c$</td>
<td>$\mathbb{Z}_2$</td>
<td>$l &lt; n \leq 2l$</td>
</tr>
<tr>
<td>(even $n$)</td>
<td></td>
<td></td>
<td></td>
<td>even $l$</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>$D_n$</td>
<td>I</td>
<td>$D_n$</td>
<td>$\mathbb{Z}_2$</td>
<td>$l/2 &lt; n \leq l$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>odd $l$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2: Isotropy subgroups $\Sigma = H^\theta$ of $O(3) \times S^1$ on $V_t \oplus V_t$, for the standard representation of $O(3)$, having two-dimensional fixed-point subspaces. Here $H = J \oplus \mathbb{Z}_2^c$ where $J \subset SO(3)$ and $K = \ker(\theta)$. For $S^1$ twists, $\theta : SO(2) \oplus \mathbb{Z}_2^c \rightarrow S^1$ is given by $\theta(\psi) = k\psi$ for $\psi \in SO(2)$ and $k = 1, \ldots, l$ occur; also $\theta(-I) = 0$ for the plus representation and $\theta(-I) = \pi$ for the minus representation.

4 Hopf Bifurcation on Hemispheres

In this paper we study Hopf bifurcations of reaction-diffusion equations defined on the hemisphere with Neumann boundary conditions along the equator. In [5] the authors studied steady-state bifurcations for the same class of equations. As they pointed out, such equations have a natural $O(2)^-$-symmetry but may be extended to the full sphere where the
natural symmetry group is $O(3)$. Field et al. [5] show that the expected bifurcations are governed not by circular symmetry but by spherical symmetry – subject to a final restriction back to the hemispherical domain. We show that the same is true for Hopf bifurcations.

Denote the coordinates on $\mathbb{R}^3$ by $(x_1, x_2, x_3)$, the unit sphere in $\mathbb{R}^3$ by $S$ and the upper hemisphere of $S$, $\{x \in S : x_3 \geq 0\}$ by $H$. Let $\Delta$ denote the Laplacian on $S$ and $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth map. Consider the reaction-diffusion equation defined on $H$ by

$$\frac{\partial u}{\partial t} + \Delta u + f(u, \lambda) = 0 \quad (4.6)$$

where $u : H \times [0, +\infty[ \to \mathbb{R}$. Assume (4.6) satisfies Neumann boundary conditions on $\partial H = \{(x_1, x_2, x_3) \in H : x_3 = 0\}$:

$$\frac{\partial u}{\partial x_3}(x_1, x_2, 0, t) = 0 \quad \text{on} \quad \partial H \times \mathbb{R}_0^+ \quad (4.7)$$

Solutions of (4.6) on $H$ that satisfy the boundary condition (4.7) can be extended to solutions of (4.6) on $S$ by defining $u$ on the lower hemisphere by reflection. Namely, if $\tau : S \to S$ denotes the reflection across $\partial H$ defined by

$$\tau(x_1, x_2, x_3) = (x_1, x_2, -x_3) \quad (4.8)$$

then we can define $u$ on the lower hemisphere by

$$u(\tau(x), t) \equiv u(x, t) \quad \forall x \in H, \forall t \in \mathbb{R}_0^+$$

A function $u$ on $S$ is $\tau$-invariant if and only if

$$u(\tau(x), t) = u(x, t) \quad \forall x \in S, \forall t \in \mathbb{R}_0^+$$

The extension $u$ defined on $S$ is $\tau$-invariant. See Theorem 8.5 for the regularity of the extended solution along $\partial H$. Conversely, suppose that $u$ is a $\tau$-invariant solution for the reaction-diffusion equation (4.6) on $S$. Then $u|_H$ is a solution for the Neumann boundary value problem (4.6) on $H$. We follow the approach of Field et al. [5] for finding solutions to the Neumann problem on the hemisphere by first finding solutions to the extended problem on $S$ that are $\tau$-invariant.

As observed in [5], the equation (4.6) defined on the hemisphere $H$ and satisfying boundary conditions (4.7) has symmetry group $O(2)$, whereas (4.6) defined on the sphere $S$ has symmetry group $O(3)$.

Suppose that

$$f(0, \lambda) \equiv 0$$

Thus, equation (4.6) has the trivial group-invariant steady-state solution $u = 0$. If a Hopf bifurcation occurs then let $V$ be the imaginary eigenspace of the linearization of (4.6) about $u = 0$ at $\lambda = 0$. It follows then that the group of symmetries of the equation leaves the space $V$ invariant. Moreover, generically the action of the symmetry group $O(2)$ on $V$ is $O(2)$-simple. See [9, Proposition XVI 1.4]. Let $V_S$ be the imaginary eigenspace of the linearization of (4.6) on the sphere (about $u = 0$ at $\lambda = 0$). Then $V$ consists of those eigenfunctions in $V_S$ that are $\tau$-invariant. The irreducible representations of $O(2)$ have dimension either
one or two. A direct application of the general $O(2)$-symmetric Hopf theory would imply that generically we should expect the dimension of $V$ to be two or four. However, from the general $O(3)$-symmetric Hopf theory for the reaction-diffusion equation on the full sphere, we expect the action of $O(3)$ on $V_S$ to be $O(3)$-simple. That is, the direct sum of two isomorphic absolutely $O(3)$-irreducible spaces. The irreducible representations of $O(3)$ correspond to the action of $O(3)$ on the spherical harmonics of order $l$ which have dimension $2l + 1$. Moreover, the vectors in $V_S$ that are $\tau$-invariant form a subspace of dimension approximately $(1/2) \dim V_S$. Thus the space $V$ may be of higher dimension than would have been expected from the $O(2)$-symmetric Hopf bifurcation problem. We show in the next sections that periodic solutions to the hemisphere problem that would not be expected if the extension property was not valid can exist.

$O(3)$-symmetric Hopf bifurcation problems have been studied [8, 9]. In particular, subgroups of $O(3) \times S^1$ that are known to support branches of periodic solutions. Recall Table 2 and Section 3. Using that, we can determine group orbits of periodic solutions in generic $O(3)$-equivariant Hopf bifurcations problems that have representative solutions with isotropy containing $\tau$. Observe that a solution to the $O(3)$-symmetric equation (4.6) on $S$ restricts to the hemisphere if and only if it is invariant under the reflection $\tau$. Specifically, we determine the subgroups in the conjugacy classes of the groups $H^g$ in Table 2 that support periodic solutions on the hemisphere problem. That is, those subgroups $\Sigma$ of $O(3) \times S^1$ with two-dimensional fixed-point subspace such that $\tau \in \Sigma$. We classify such solutions up to $O(2)$-symmetry, the symmetry group of the hemisphere. We find that some group orbits of periodic solutions contain multiple periodic solutions with symmetry $\tau$. Each of these periodic solutions then restrict to solutions of the Neumann boundary value problem on the hemisphere lying on different $O(2)$-orbits.

**Remark 4.1** Let $\Sigma = H^g$ be an isotropy subgroup of $O(3) \times S^1$. Then $\tau \in \Sigma$ if and only if $\tau \in K = \ker(\theta)$. Moreover, we have that $\tau$ belongs to an isotropy subgroup of $O(3) \times S^1$ in the conjugacy class of $H^g$ if and only if $\tau$ belongs to a subgroup of $O(3)$ in the conjugacy class of $K = \ker(\theta)$. Note that if $\tau \in \gamma K \gamma^{-1}$ for some $\gamma \in O(3)$, and $K = \ker(\theta)$ for some homomorphism $\theta : H \rightarrow S^1$ such that $H^g$ is a two-dimensional isotropy subgroup of $O(3) \times S^1$, then $\tau \in \gamma H^g \gamma^{-1}$ and $\gamma H^g \gamma^{-1}$ is conjugate to $H^g$.

**Definition 4.2** Denote the identity map on $R^3$ by $I_{R^3}$. Given a map $f : R^3 \rightarrow R^3$, let $\text{Fix}(f) = \{x \in R^3 : f(x) = x\}$ denote the fixed-point set of $f$. We define:

(i) A linear map $\sigma : R^3 \rightarrow R^3$ is an involution if $\sigma \neq I_{R^3}$ and $\sigma^2 = I_{R^3}$.

(ii) An involution $\sigma \in O(3)$ is a reflection if the fixed-point set of $\sigma$ is two-dimensional.

**Remark 4.3** If $\sigma \in O(3)$ is an involution which is not a reflection and not equal to $I_{R^3}$, then $-\sigma$ is a reflection.

Note that $\tau$ is a reflection in $O(3)$. Moreover, we have:

**Lemma 4.4** An involution $\sigma \in O(3)$ is a reflection if and only if $\sigma$ is conjugate to $\tau$. That is, if and only if there exists $\gamma \in O(3)$ such that $\sigma = \gamma \tau \gamma^{-1}$.
Lemma 4.5 The periodic solutions to $\mathbf{O}(3)$-equivariant Hopf bifurcation problems with isotropy conjugate to the twisted groups with spatial group $\mathbf{I}$ or $\mathbf{O}$ or $\mathbf{D}_n$ (see Table 3) cannot restrict to solutions of the Neumann problem on the hemisphere.

Proof The groups $\mathbf{I}$, $\mathbf{O}$, $\mathbf{D}_2$ and $\mathbf{D}_n$ contain no orientation-reversing elements. Therefore, they do not contain reflections.

<table>
<thead>
<tr>
<th>$J$</th>
<th>$K$</th>
<th>$K$</th>
<th>Value of $l$ Plus</th>
<th>Value of $l$ Minus</th>
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</thead>
<tbody>
<tr>
<td>$\mathbf{I}$</td>
<td>$\mathbf{I}$</td>
<td>$\mathbf{I}$</td>
<td>$\mathbf{Z}_2$</td>
<td>21, 25, 27, 31, 33, 35, 37, 39, 41, 43, 47, 49, 53, 59</td>
</tr>
<tr>
<td>$\mathbf{O}$</td>
<td>$\mathbf{I}$</td>
<td>$\mathbf{O}$</td>
<td>$\mathbf{Z}_2$</td>
<td>9, 13, 15, 17, 19, 23</td>
</tr>
<tr>
<td>$\mathbf{T}$</td>
<td>$\mathbf{I}$</td>
<td>$\mathbf{D}_2$</td>
<td>$\mathbf{Z}_6$</td>
<td>5, 7, 9</td>
</tr>
<tr>
<td>$\mathbf{D}_n$</td>
<td>$\mathbf{I}$</td>
<td>$\mathbf{D}_n$</td>
<td>$\mathbf{Z}_2$</td>
<td>$l/2 &lt; n \leq l$ odd</td>
</tr>
</tbody>
</table>

Table 3: Entries 6, 8, 12 and 14 of Table 2 with isotropy subgroups $H^\theta$ of $\mathbf{O}(3) \times \mathbf{S}^1$ on $V_1 \oplus V_1$, for the standard representation of $\mathbf{O}(3)$, having two-dimensional fixed-point subspaces. Here $H = J \oplus \mathbf{Z}_2^c$ where $J \subset \mathbf{SO}(3)$ and $K = \ker(\theta)$.

5 Axisymmetric Solutions

There are two types of twisted isotropy subgroups of $\mathbf{O}(3) \times \mathbf{S}^1$ that contain $\mathbf{SO}(2)$: those with spatial group $\mathbf{O}(2) \oplus \mathbf{Z}_2^c$ (when $l$ is even) and $\mathbf{O}(2)^-$ (when $l$ is odd). See Table 4. We call those solutions axisymmetric since they have an axis of rotation.

Theorem 5.1 Each orbit of axisymmetric periodic solutions to the sphere problem restricts to solutions of the hemisphere problem as follows:

(a) When $l$ is even, an isolated axisymmetric solution with the $x_3$-axis as axis of symmetry. The symmetry of the solution is $\mathbf{O}(2)^-$. 

(b) For all $l$, a unique orbit of solutions:

(b.1) with spatial symmetry group $\mathbf{Z}_2^-$ (inside $\mathbf{O}(2)^-$) and spatio-temporal symmetry $(\mathbf{D}_2)^\theta$ (inside $\mathbf{O}(2)^- \times \mathbf{S}^1$) where $\theta$ has twist type $\mathbf{Z}_2$, when $l$ is odd;

(b.2) with spatial symmetry group $\mathbf{D}_2^-$ (inside $\mathbf{O}(2)^-$) when $l$ is even.
| $J$ | Type of | $K$ | Twist $\theta(H)$ | Value of $l$ | Value of $l$
|---|---|---|---|---|---|
| $\mathbf{O}(2)$ | II | $\mathbf{O}(2) \oplus \mathbb{Z}_2^c$ | 1 | even $l$ |  \\
| $\mathbf{O}(2)$ | III | $\mathbf{O}(2)^-$ | $\mathbb{Z}_2$ | odd $l$ |

Table 4: Entries 1 and 2 of Table 2 with isotropy subgroups $H^\theta$ of $\mathbf{O}(3) \times S^1$ on $V_l \oplus V_l$, for the standard representation of $\mathbf{O}(3)$, having two-dimensional fixed-point subspaces. Here $H = J \oplus \mathbb{Z}_2^c$ where $J \subset \mathbf{SO}(3)$ and $K = \ker(\theta)$.

**Proof** See [5, Theorem 3.1]. We include a more detailed proof. We begin by determining those subgroups of $\mathbf{O}(3)$ conjugate to $\mathbf{O}(2) \oplus \mathbb{Z}_2^c$ (when $l$ is even) that also contain $\tau$. An element in the conjugacy class of $\mathbf{O}(2) \oplus \mathbb{Z}_2^c$ in $\mathbf{O}(3)$ is $\mathbf{O}(2) \oplus \mathbb{Z}_2^c$, where $\mathbf{O}(2) = \mathbf{SO}(2) \cup (-\tilde{\sigma})\mathbf{SO}(2)$ and $\mathbf{SO}(2)$ contains all rotations of the plane orthogonal to the axis of symmetry that fix this axis. Here we can take $\tilde{\sigma}$ any reflection with fixed point set containing the axis of symmetry of $\mathbf{SO}(2)$. Therefore $\mathbf{O}(2)$ is generated by $\mathbf{SO}(2)$ and $-\tilde{\sigma}$.

Let $\Sigma = \mathbf{O}(2) \oplus \mathbb{Z}_2^c$ such that $\tau \in \Sigma$. Since $\tau$ reverses orientation ($\det \tau = -1$), then $\tau \not\in \mathbf{SO}(2)$. Since $\tau \in \Sigma$, then $\tau$ leaves the axis of rotation of $\mathbf{SO}(2)$ invariant (all the elements of $\Sigma$ fix or transform in $-v$ an element $v$ in the axis of rotation of $\mathbf{SO}(2)$). That is, $\tau$ is in the normalizer of $\mathbf{SO}(2)$. As $\tau(x_1, x_2, x_3) = (x_1, x_2, -x_3)$ and $\tau$ maps the axis of rotation of $\mathbf{SO}(2)$ into itself, then this axis is either: (a) the $x_3$-axis or (b) perpendicular to the $x_3$-axis.

**Case (a)** In this case $\Sigma = \mathbf{O}(2) \oplus \mathbb{Z}_2^c$, where $\mathbf{O}(2) = \mathbf{SO}(2) \cup \kappa \mathbf{SO}(2)$. The elements of $\Sigma$ are of one of the following types: $R_\theta \in \mathbf{SO}(2) \subset \mathbf{SO}(3)$, $\kappa R_\theta \in \kappa \mathbf{SO}(2) \subset \mathbf{SO}(3)$, $-R_\theta \in -\mathbf{ISO}(2)$, $-\kappa R_\theta \in -\kappa \mathbf{SO}(2)$. Note that $\tau \not\in \mathbf{SO}(3)$ since $\det \tau = -1$. Moreover $\tau = -R_\tau$ (where $R_\tau \in \mathbf{SO}(2)$) (and $-\tau = R_\tau$). Thus $\tau \in \Sigma$. Moreover,

$$\mathbf{(O(2) \oplus Z_2^c)} \cap \mathbf{O(2)^-} = \mathbf{O(2)^-}$$

since $\mathbf{O(2)^-} = \mathbf{SO}(2) \cup (-\kappa)\mathbf{SO}(2)$.

**Case (b)** Suppose the axis of rotation is $\mathbf{R}\{a_1\}$ where $a_1 = (a, b, 0)$ (and $a^2 + b^2 \neq 0$). Now $\tau a_1 = a_1$. Thus $\tau$ is a reflection fixing the axis of rotation and so $-\tau \in \mathbf{O(2)}$. We conclude that $\tau \in \Sigma = \mathbf{O(2)} \oplus \mathbb{Z}_2^c$.

We show that

$$\mathbf{(O(2) \oplus Z_2^c)} \cap \mathbf{O(2)^-} = \mathbf{D_2^-}$$

To see this, note that the elements of $\Sigma$ are of one of the following four types:

**b.1** $\tilde{\tau}_\theta \in \mathbf{SO(2)} \subset \mathbf{SO(3)}$ with positive determinant;

**b.2** $-\tilde{\sigma} \tilde{\tau}_\theta \in \mathbf{O(2)} \setminus \mathbf{SO(2)} \subset \mathbf{SO(3)}$ with positive determinant;
(b.3) \( \tilde{R}_\theta \in O(3) \) with negative determinant;

(b.4) \( \tilde{\sigma}R_\theta \in O(3) \) with negative determinant.

Recall that \( \tilde{\sigma} \) fixes the rotation axis \( a_1 \) and \( O(2)^- = SO(2) \cup (-\kappa)SO(2) \), where \( \det(R_\theta) = 1 \) and \( \det(-\kappa R_\theta) = -1 \), for \( R_\theta \in SO(2) \). We consider the four cases, each corresponding to the intersection of \( O(2)^- \) with the set of elements of one of the four types that we have in \( \Sigma \).

**Case (b.1):** the intersection of \( O(2)^- \) with the set of elements of type \( \tilde{R}_\theta \) is formed by the elements of the type \( \tilde{R}_\theta \in SO(2) \subset SO(3) \) that are also of type \( R_\theta \in SO(2) \subset SO(3) \). One element in these conditions would have to fix the \( a_1 \) axis and the \( x_3 \) axis of rotation of \( SO(2) \), so it can only be the identity.

**Case (b.2):** the intersection of \( O(2)^- \) with the set of elements of type \( -\tilde{\sigma}R_\theta \) is given by the elements of type \( -\tilde{\sigma}R_\theta \in O(2) \subset SO(3) \) that are of type \( R_\theta \in SO(2) \subset SO(3) \) since the elements \( -\tilde{\sigma}R_\theta \) have positive determinant. The second ones fix the \( x_3 \) axis, so they have 1 as an eigenvalue. The first ones are such that \( -\tilde{\sigma}R_\theta(a_1) = -a_1 \) because \( -\tilde{\sigma}R_\theta \in O(2) \setminus SO(2) \). They have \(-1\) as an eigenvalue. The third eigenvalue must be \(-1\) because these elements are in \( SO(3) \). Therefore \( \theta = \pi \) and \( R_\theta = R_\pi \). Moreover, taking the orthogonal basis \((a_1, b_1, (0,0,1))\), where \( b_1 = (-b, a, 0) \), we can take \( \tilde{\sigma} \) such that \( -\tilde{\sigma} = R_\pi \) in this basis (and in the canonical basis).

**Case (b.3):** the intersection of \( O(2)^- \) with elements of type \( \tilde{\sigma}R_{\theta_1} \in O(3) \). In this case we must find \( \theta \) such that \( \gamma = -\kappa R_{\theta_1} = -\tilde{\sigma}R_{\theta_1} \). Since \( \gamma \) fixes the \( x_3 \)-axis and maps \( a_1 \) into \(-a_1 \), where \( a_1 \) is the rotation axis of \( O(2) \), then as \( \det(\gamma) = -1 \), it follows that \( -\tilde{\sigma}R_{\theta_1} = -\tilde{\sigma}R_\pi \) (and so \( \theta_1 = \pi \)). Moreover, \( \gamma \) has eigenvalues \(-1,1,1\) and

\[
-\kappa R_\theta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\cos \theta & \sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

in the canonical coordinates \( x_1, x_2, x_3 \). Recall that this is a reflection on the line \( x_2 = \tan(\pi/2 - \theta/2) x_1 \) if \( \theta \neq 0 \) and \( x_1 = 0 \) otherwise (in the \( x_1 x_2 \)-plane). On the other hand \( -\tilde{\sigma}R_\pi \) is a reflection along the axis \((-b, a, 0)\). This axis must coincide with the line \( x_2 = \tan(\pi/2 - \theta/2) x_1 \) if \( \theta \neq 0 \), otherwise with \( x_1 = 0 \). If \( b = 0 \) we take \( \theta = 0 \) and \( \gamma = -\kappa R_0 = -\kappa = -\tilde{R}_\pi \). If not, then \( \theta \) is the angle such that

\[
\tan\left(\frac{\pi - \theta}{2}\right) = \frac{a}{b}
\]

In any case,

\[
\gamma = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

in the \((a_1, b_1, (0,0,1))\)-basis.

**Case (b.4):** the intersection of \( O(2)^- \) with elements of type \( \tilde{\sigma}R_{\theta_1} \in O(3) \). We must find \( \theta \) such that \( \gamma = -\kappa R_\theta = \tilde{\sigma}R_{\theta_1} \). As \( \gamma = -\kappa R_\theta \) fixes the \( x_3 \)-axis and \( \gamma = \tilde{\sigma}R_{\theta_1} \) fixes the \( a_1 \)-axis,
and \( \det(\gamma) = -1 \), it follows that the eigenvalues of \( \gamma \) are 1, 1, -1. In the basis \((a_1, b_1, (0, 0, 1))\) we have
\[
\widetilde{\sigma} R_\pi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

Recall that in case (b.2) we chose
\[
\widetilde{\sigma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}
\]

Note that \( \widetilde{\sigma} R_\pi \) is a reflection on the \( a_1 \)-axis (in the \( x_1 x_2 \)-plane). Also \(-\kappa R_\theta\) is a reflection on the line \( x_2 = \tan(\pi/2 - \theta/2)x_1 \) if \( \theta \neq 0 \) and \( x_1 = 0 \) otherwise (in the same plane). If \( a = 0 \) we take \( \theta = 0 \) and \( \gamma = -\kappa R_0 = -\kappa = \widetilde{\sigma} R_\pi \). If \( a \neq 0 \) we choose \( \theta \) such that
\[
\tan\left(\frac{\pi - \theta}{2}\right) = \frac{b}{a}
\]

In both cases
\[
\gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]
in the \((a_1, b_1, (0, 0, 1))\)-basis. Moreover
\[
(\widehat{O(2)} \oplus \mathbb{Z}_2^c) \cap O(2)^- = D_2^-
\]
where
\[
D_2^- = \left\{ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}
\]
in the \(((a, b, 0), (-b, a, 0), (0, 0, 1))\) basis.

Note that for odd \( l \) now \( \Sigma = \widehat{O(2)^-} = \widehat{\text{SO}(2)} \cup \widehat{\sigma \text{SO}(2)} \) and, following the same lines as before, we also have to consider cases (a) and (b). In case (a) \( \tau \notin \widehat{O(2)^-} \) so these solutions do not occur when \( l \) is odd. In case (b) we have
\[
(\widehat{O(2)^-}) \cap O(2)^- = \mathbb{Z}_2^-.
\]

To see this, note that the elements of \( \Sigma \) are of one of the following two types:

- \( \widetilde{R}_\theta \in \widehat{\text{SO}(2)} \subset \text{SO}(3) \),
- \( \widetilde{\sigma} \widetilde{R}_\theta \in \text{O}(3) \).
Following cases (b.1) and (b.4), we obtain

\[ (\tilde{O}(2)^-) \cap O(2)^- = Z_2^- = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\} \]

in the \=((a, b, 0), (-b, a, 0), (0, 0, 1)) basis.

For \(H^\theta \cap (O(2)^- \times S^1)\), when \(l\) is odd, we need to compute \((\tilde{O}(2) \oplus Z_2^c) \cap O(2)^-\). The above computations, done for even \(l\), are also valid for odd \(l\), so we have

\[ (\tilde{O}(2) \oplus Z_2^c) \cap O(2)^- = D_2^- \]

Note that the axis of symmetry is perpendicular to \(x_3\) (case (b) in the computations for even \(l\)).

6 Solutions with Finite Spatial Symmetry

We consider now periodic solutions with twisted symmetry groups and finite spatial symmetry groups for the sphere problem that restrict to solutions of the hemisphere problem.

<table>
<thead>
<tr>
<th>Type of</th>
<th>Value of (l) Plus</th>
<th>Value of (l) Minus</th>
</tr>
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<td>(J) (K) (K)</td>
<td>(\theta(H)) Representation</td>
<td>(\theta(H)) Representation</td>
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<tr>
<td>I II I (\oplus Z_2^c) 1</td>
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<tr>
<td>O II O (\oplus Z_2^c) 1</td>
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<tr>
<td>O III O(^-) (Z_2)</td>
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</tbody>
</table>

Table 5: Entries 5, 7 and 10 of Table 2 with isotropy subgroups \(H^\theta\) of \(O(3) \times S^1\) on \(V_i \oplus V_i\), for the standard representation of \(O(3)\), having two-dimensional fixed-point subspaces. Here \(H = J \oplus Z_2^c\) where \(J \subset SO(3)\) and \(K = \ker(\theta)\).

**Theorem 6.1** Suppose that we have an \(O(3) \times S^1\)-orbit of periodic solutions to the sphere problem with finite group of spatial symmetries given in Table 5. On restriction, we obtain the following \(O(2)^- \times S^1\) orbits of solutions to the hemisphere problem:

(a) **Spatial group** \(K = I \oplus Z_2^c\) (Icosahedral solutions)
    
    There are 15 orbits of solutions with (spatial) symmetry group \(Z_2^-\).

(b) **Spatial group** \(K = O \oplus Z_2^c\) (Octahedral solutions)
    
    There are 3 orbits of solutions with (spatial) symmetry group \(D_4^-\).
    
    There are 6 orbits of solutions with (spatial) symmetry group \(D_2^-\).
(c) Spatial group $K = \mathbf{O}^{-}$ (Octahedral solutions)

There are 6 orbits of solutions with spatial symmetry group $\mathbf{Z}_2^-$, and spatio-temporal symmetry $(\mathbf{D}_2)\theta$ where $\theta$ has twist type $\mathbf{Z}_2$.

Proof  See [5, Theorem 4.3].

We recall that a group $G$ is the disjoint union of subgroups $G_i$, $i \in I$, if $G = \bigcup_{i \in I} G_i$ and for all $i, j \in I$, $i \neq j$, $G_i \cap G_j$ is the identity element of $G$. We write $G = \bigcup_{i \in I} G_i$.

Lemma 6.2 The subgroup $T$ of $\mathbf{O}(3)$ has the following disjoint union decomposition:

$$T = \bigcup^4 \mathbf{Z}_3 \cup^3 \mathbf{Z}_2$$

Here $\bigcup^k\mathbf{Z}_l$ denotes a disjoint decomposition of $k$ copies of subgroups all conjugate (in $\mathbf{SO}(3)$) to $\mathbf{Z}_l$.

Proof  See [13] or [9, pp. 105].

<table>
<thead>
<tr>
<th>$J$</th>
<th>Type of $K$</th>
<th>$K$</th>
<th>Twist $\theta(H)$</th>
<th>Value of $l$</th>
<th>Value of $l$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>Value of $l$</td>
<td>Value of $l$</td>
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<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Plus Representation</td>
<td>Minus Representation</td>
</tr>
<tr>
<td>SO(2)</td>
<td>II</td>
<td>$\mathbf{Z}_k \oplus \mathbf{Z}_2^c$</td>
<td>$\mathbf{S}^1$ [1, \ldots, $l$]</td>
<td>even $l$</td>
<td></td>
</tr>
<tr>
<td>SO(2)</td>
<td>III</td>
<td>$\mathbf{Z}_2^{-k}$</td>
<td>$\mathbf{S}^1$ [1, \ldots, $l$]</td>
<td>odd $l$</td>
<td></td>
</tr>
<tr>
<td>$\mathbf{O}$</td>
<td>II</td>
<td>$T \oplus \mathbf{Z}_2^c$</td>
<td>$\mathbf{Z}_2$</td>
<td>6, 10, 12, 14, 16, 20</td>
<td></td>
</tr>
<tr>
<td>$T$</td>
<td>II</td>
<td>$\mathbf{D}_2 \oplus \mathbf{Z}_2^c$</td>
<td>$\mathbf{Z}_3$</td>
<td>2, 4, 6</td>
<td></td>
</tr>
<tr>
<td>$\mathbf{D}_n$</td>
<td>II</td>
<td>$\mathbf{D}_{n/2} \oplus \mathbf{Z}_2^c$</td>
<td>$\mathbf{Z}_2$</td>
<td>$l &lt; n \leq 2l$</td>
<td></td>
</tr>
<tr>
<td>(even $n$)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6: Entries 3, 4, 9, 11 and 13 of Table 2 with isotropy subgroups $H^\theta$ of $\mathbf{O}(3) \times \mathbf{S}^1$ on $V_2 \oplus V_1$, for the standard representation of $\mathbf{O}(3)$, having two-dimensional fixed-point subspaces. Here $H = J \oplus \mathbf{Z}_2^c$ where $J \subset \mathbf{SO}(3)$ and $K = \ker(\theta)$. For $\mathbf{S}^1$ twists, $\theta : \mathbf{SO}(2) \oplus \mathbf{Z}_2^c \to \mathbf{S}^1$ is given by $\theta(\psi) = k\psi$ for $\psi \in \mathbf{SO}(2)$ and $k = 1, \ldots, l$ occur. Also $\theta(-I) = 0$ for the plus representation and $\theta(-I) = \pi$ for the minus representation.

Theorem 6.3 Suppose that we have an $\mathbf{O}(3) \times \mathbf{S}^1$-orbit of periodic solutions to the sphere problem with finite group of spatial symmetries given in Table 6. On restriction, we obtain the following $\mathbf{O}(2)^- \times \mathbf{S}^1$ orbits of solutions to the hemisphere problem:

(a) Spatial group $K = \mathbf{Z}_k \oplus \mathbf{Z}_2^c$

For each even $k$ (and $2 \leq k \leq l$) there is one orbit of solutions with spatial symmetry group $\mathbf{Z}_k$, and spatio-temporal symmetry $(\mathbf{SO}(2))\theta$ where $\theta(\psi) = k\psi$ for $\psi \in \mathbf{SO}(2)$.

(b) Spatial group $K = \mathbf{Z}_{2k}^-$

For each odd $k$ (and $1 \leq k \leq l$) there is one orbit of solutions with spatial symmetry group $\mathbf{Z}_k$, and spatio-temporal symmetry $(\mathbf{SO}(2))\theta$ where $\theta(\psi) = k\psi$ for $\psi \in \mathbf{SO}(2)$. 

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(c) **Spatial group** $K = T \oplus Z_2^c$

There are 3 orbits of solutions with spatial symmetry group $D_2^-$, and spatio-temporal symmetry $(D_3^\theta)^\theta$ where $\theta$ has twist type $Z_2$.

(d) **Spatial group** $K = D_2 \oplus Z_2^c$

There are 3 orbits of solutions with spatial symmetry group $D_2^-$.

(e) **Spatial group** $K = D_{n/2} \oplus Z_2^c$

For each even $n/2$ (and $l < n \leq 2l$) there are three orbits of solutions and for each odd $n/2$ (and $l < n \leq 2l$) there is one orbit of solutions. The solutions have spatial symmetry group $D_{n/2}^-$, and spatio-temporal symmetry $(D_n^-)^\theta$ where $\theta$ has twist type $Z_2$.

**Proof**

(a) Suppose $K = Z_k \oplus Z_2^c$. When $k$ is odd, $K$ does not contain nontrivial involutions. When $k$ is even $R_\pi \in SO(2)$ is the only nontrivial involution of $Z_k$. Moreover, $\tau = -R_\pi \in K$. Thus by Lemma 4.4 for each even $k$ there is one orbit of solutions.

Let $k$ be even. For the spatial intersection, $K \cap O(2)^-$, where $K = Z_k \oplus Z_2^c$, note that $O(2)^- = SO(2) \cup (-\kappa)SO(2)$ and $Z_k \oplus Z_2^c = Z_k \cup (-I)Z_k$. The intersection $SO(2) \cap Z_k$ is $Z_k$ and the intersection $[( -\kappa)SO(2)] \cap [(-I)Z_k]$ is empty because every element of $(-\kappa)SO(2)$ fixes the third coordinate and no element of $(-I)Z_k$ does so. We conclude that the intersection $(Z_k \oplus Z_2^c) \cap O(2)^-$ is $Z_k$.

In order to calculate $H^\theta \cap (O(2)^- \times S^1)$ where $H = SO(2) \oplus Z_2^c$, we calculate now $(SO(2) \oplus Z_2^c) \cap O(2)^-$ which is $SO(2)$. Therefore we have that $H^\theta \cap (O(2)^- \times S^1) = (SO(2))^\theta$ where $\theta(\psi) = k\psi$, for $\psi \in SO(2)$.

(b) Note that the group $Z_{2k}^-$ is generated by $-R_{\pi/k}$. If $k$ is even then the nontrivial involution of $Z_{2k}^-$ is $(-R_{\pi/k}) = R_{\pi} = \tau \in Z_{2k}^-$. Let $k$ be odd. The elements of $Z_{2k}^-$ with determinant one form the group $Z_k$, which is contained in $O(2)^-$. By arguments similar to the last item, there are no more elements in the intersection between $Z_{2k}^-$ and $O(2)^-$. Thus $Z_{2k}^- \cap O(2)^- = Z_k$. Moreover, $(SO(2) \oplus Z_2^c) \cap O(2)^- = SO(2)$.

(c) By Lemma 6.2, in $T$ there are three nontrivial involutions in $Z_2$. Composing these involutions with $-I$ it follows that $T \oplus Z_2^c$ has three distinct reflections (recall Remark 4.3). Moreover, by Lemma 4.4, these reflections are conjugate to $\tau$.

For the intersection $K \cap O(2)^-$, note that the group $T \oplus Z_2^c$ can be realized as $Z_3 + Z_2^3$. The elements of $Z_3 + Z_2^3$ that fix the third coordinate are two groups isomorphic to $Z_2$, forming the group $D_3^- = <R_\pi, -\kappa>$.

For the intersection $H^\theta \cap (O(2)^- \times S^1)$, note that the group $H = O \oplus Z_2^c$ can be realized as $S_3 + Z_2^3$. Moreover, $H \cap O(2)^-$ is the group $D_4^- = <R_{\pi/2}, -\kappa>$.

(d) We have that $D_2 = <R_\pi, \kappa>$ and so $D_2 \oplus Z_2^c$ contains three reflections, $\tau = -R_\pi$, $-\kappa$ and $-\kappa R_\pi$, which are conjugate to $\tau$ by Lemma 4.4.
The intersection $K \cap O(2)^- \text{ where } K$ is (conjugate to) $D_2 \oplus Z_2^c$ is the group $D_2^-$ (generated by $R_\pi$ and $-\kappa$).

From the previous item we have that $H^\theta \cap (O(2)^- \times S^1) = (D_2^-)^\theta$.

(e) When $n/2$ is even, $R_\pi \in SO(2)$ is the only nontrivial involution of $Z_{n/2}$. Moreover, $\tau = -R_\pi \in K$. We also have another nontrivial involution in $D_{n/2}$, which is $\kappa$. Now $(-I)\kappa$ is a reflection in $K$, so it is conjugate to $\tau$. Also $\kappa R_\pi$ is a nontrivial involution in $D_{n/2}$.

When $n/2$ is odd, there is only one nontrivial involution in $D_{n/2}$, which is $\kappa$. Now $(-I)\kappa$ is a reflection in $K$, so it is conjugate to $\tau$. There are no more reflections.

We conclude that when $n/2$ is even we have three orbits of solutions containing $\tau$ for the sphere problem. When $n/2$ is odd we have one orbit of solutions.

For the intersection $K \cap O(2)^- \text{ where } K = D_{n/2} \oplus Z_2^c$, note that

$$D_{n/2} \oplus Z_2^c = Z_{n/2} \cup \kappa Z_{n/2} \cup (-I)Z_{n/2} \cup (-\kappa)Z_{n/2}.$$ 

The elements of $K$ that fix the $x_3$-axis belong to $Z_{n/2} \cup (-\kappa)Z_{n/2}$. This is the group $D_{n/2}^-$ (generated by $R_{3\pi/n}$ and $-\kappa$) and it is contained in $O(2)^-$. It follows then that $K \cap O(2)^-$ is $D_{n/2}^-$.

Now, $H = D_n \oplus Z_2^c$. The elements of $H$ that fix the $x_3$-axis form the group $D_n^-$ (generated by $R_{2\pi/n}$ and $-\kappa$) which is contained in $O(2)^-$. It follows then that the intersection $H^\theta \cap (O(2)^- \times S^1)$ is $(D_n^-)^\theta$.

\[\square\]

7 Figures

Following Field et al. [5], given $l \geq 2$, we identify the spherical harmonics of order $2l + 1$ (in $V_l$) with the deformations of a sphere in the following way: since a spherical harmonic is a real-valued function on the sphere, we can picture it by deforming the sphere in the radial direction by an amount equal to the value of that spherical harmonic.

In this section we show periodic one-parameter families of deformations of the sphere, the parameter being time, to illustrate the symmetries of the periodic solutions predicted by Theorem 5.1 and Theorem 6.3 for the sphere problem and for its restriction to the hemisphere.

In Figure 3 (a) we assume $l = 3$ and we picture a standing wave (of periodic oscillations) of deformations of the sphere with spatial symmetry $O(2)^-$, twisted symmetry $(O(2) \oplus Z_2^c)^\theta$ (and twist type $Z_2$). Thus the deformations maintain an $O(2)^-$-symmetric shape and oscillate with twist type $Z_2$. In Theorem 5.1 (b) we show that this solution may be sliced in one way (up to $O(2)^-$-symmetry) to obtain a solution to the equation posed on the hemisphere, having spatial symmetry $Z_2^-$ and spatio-temporal symmetry $(D_2^-)^\theta$ (having twist type $Z_2$). We show that in Figure 3 (b).
Figure 3: (a) Oscillations of a sphere deformed by spherical harmonics of order $l = 3$: spatial symmetry $\mathbf{O}(2)^-$ and twisted symmetry $(\mathbf{O}(2) \oplus \mathbb{Z}_2)^\theta$ (twist type $\mathbb{Z}_2$). (b) Restriction to hemisphere: spatial symmetry $\mathbb{Z}_2^-$, twisted symmetry $(\mathbf{D}(2^-))^\theta$, twist type $\mathbb{Z}_2$.

Figure 4: (a) Oscillations of a sphere deformed by spherical harmonics of order $l = 3$: spatial symmetry $\mathbb{Z}_6^-$ and twisted symmetry $(\mathbf{SO}(2))^\theta$. (b) Restriction to hemisphere: spatial symmetry $\mathbb{Z}_3$ and twisted symmetry $(\mathbf{SO}(2))^\theta$. (c) Restriction to hemisphere: topview where the spatial symmetry $\mathbb{Z}_3$ is well visualized.

In Figure 4 (a) we assume $l = 3$ and we picture a rotating wave (of periodic oscillations) of deformations of the sphere with spatial symmetry $\mathbb{Z}_6^-$, twisted symmetry $\mathbf{SO}(2)^\theta$ (and twist type $\mathbb{S}^1$). Thus the deformations maintain a $\mathbb{Z}_6^-$-symmetric constant shape and rotate about the $x_3$-axis. In Figure 4 (b-c) we picture the restriction to hemisphere with $\mathbb{Z}_3$-symmetric constant shape and rotating about the $x_3$-axis. Recall Theorem 6.3 (b) for $l = 3$ and $k = 3$.

In Figure 5 (a) we assume $l = 6$ and we picture a standing wave (of periodic oscillations) of deformations of the sphere with spatial symmetry $\mathbf{T} \oplus \mathbb{Z}_2^c$, twisted symmetry $(\mathbf{O} \oplus \mathbb{Z}_2)^\theta$. Thus the deformations are $\mathbf{T} \oplus \mathbb{Z}_2^c$-symmetric and oscillate with twist type $\mathbb{Z}_2$. In Theorem 6.3 (c) we show that this solution may be sliced in three ways to obtain solutions to the equation posed on the hemisphere, all of them have spatial symmetry $\mathbf{D}_2^-$ and spatio-temporal symmetry $(\mathbf{D}_4^-)^\theta$ (having twist type $\mathbb{Z}_2$). We show that in Figure 5 (b-d).

In Figure 1 (Section 1) we assume $l = 6$ and we picture a standing wave (of periodic oscillations) of deformations of the sphere with spatial symmetry $\mathbf{D}_2 \oplus \mathbb{Z}_2^c$, twisted symmetry
Figure 5: (a) Oscillations of a sphere deformed by spherical harmonics of order $l = 6$: spatial symmetry $\mathbf{T} \oplus \mathbb{Z}_2^c$ and spatio-temporal symmetry $(\mathbf{O} \oplus \mathbb{Z}_2^c)^\theta$. (b-d) (Three orbits) Restriction to hemisphere: spatial symmetry $\mathbf{D}_2^-$ and spatio-temporal symmetry $(\mathbf{D}_4^-)^\theta$.

Figure 6: (Orbit 2) Restriction of solution of Figure 1 to hemisphere with spatial symmetry $\mathbf{D}_2^-$. 

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Figure 7: (Orbit 3) Restriction of solution of Figure 1 to hemisphere: spatial symmetry $D_2^-$. 

Thus the deformations are $T \oplus Z_2^0$-symmetric and oscillate with twist type $Z_3$. In Theorem 6.3 (d) we show that this solution may be sliced in three ways to obtain solutions to the equation posed on the hemisphere, all of them have spatial symmetry $D_2^-$ and trivial twist type. We show that in Figures 2 (Section 1), 6, 7.

In Figure 8 (a) we assume $l = 2$ and we picture a standing wave (of periodic oscillations) of deformations of the sphere with spatial symmetry $D_2 \oplus Z^0_2$, twisted symmetry $(D_4 \oplus Z^0_2)$. Thus the deformations are $D_2 \oplus Z^0_2$-symmetric and oscillate with twist type $Z_2$. In Theorem 6.3 (e) for $l = 2$ and $n = 4$, we show that this solution may be sliced in three ways to obtain solutions to the equation posed on the hemisphere, all of them have spatial symmetry $D_2^-$ and twisted symmetry $(D_4^-)^0$ (twist type $Z_2$). We show that in Figure 8 (b-d).

8 Smoothness of Extended Solutions

In Section 4 we state that solutions of (4.6) on $H$ that satisfy the boundary condition (4.7) can be extended to solutions of (4.6) on $S$ by defining $u$ on the lower hemisphere by the reflection $\tau: S \to S$ across $\partial H$. Field et al. [5, Theorem 5.18] prove the regularity of the steady-state extended solutions along $\partial H$ obtained by this method. In fact, they prove similar conclusions for a wide class of domains (for steady-state solutions). A similar result is valid for periodic solutions of (4.6). Before stating this result, we briefly describe the abstract setting assumed by [5] and where the results hold.
Assumptions 8.1 Following [5, Section 5] we make the assumptions below:

- $M$ is a smooth, compact, connected, Riemannian $n$-dimensional manifold without boundary.

- $K$ is a finite group of transformations of $M$ generated by an admissible set $\mathcal{R}$ of $p$ reflections of $M$. Therefore $K$ is isomorphic to $\mathbb{Z}_2^p$. Specifically, averaging any Riemannian metric on $M$ over $K$, we may assume that $K$ is a group of isometries of $M$.

- $N$ is a connected component of $M_K = \{ x \in M : (k \in K \land kx = x) \Rightarrow k = I_M \}$ where $I_M$ denotes the identity map of $M$. Moreover, we can assume that every isometry on $N$ (for the Riemannian structure induced from $M$) extends uniquely to an isometry on $M$. Denote by $\partial N$ the boundary of $N$.

- If $ISO(M)$ denotes the group of isometries of $M$ and $C^\infty(M)$ the space of smooth real-valued functions on $M$, we consider the natural action of $ISO(M)$ on $C^\infty(M)$.
defined by $u \rightarrow g(u)$ where for $u \in C^\infty(M)$ and $g \in ISO(M)$

$$g(u)(x) = u(g^{-1}x)$$

- $\mathcal{P}$ is a semi-linear elliptic operator on $C^\infty(M)$ defined by

$$\mathcal{P}(u) = \Delta u + f(u), \quad (8.9)$$

where $\Delta$ is the Laplace operator associated to the Riemannian structure on $M$ and $f : \mathbb{R} \rightarrow \mathbb{R}$ is smooth.

\[ \diamond \]

**Remark 8.2** The operator $\mathcal{P}$ on $C^\infty(M)$ defined by (8.9) respects the symmetries on $M$ and $N$. That is, $\mathcal{P}$ satisfies

$$\mathcal{P}(g(u)) = g(\mathcal{P}(u))$$

for all $u \in C^\infty(M)$ and $g \in ISO(M)$ ($\mathcal{P}$ is $ISO(M)$-invariant).

\[ \diamond \]

Let $C^1(M)$ (respectively, $C^1(N)$) denote the space of $C^1$ real-valued functions on $M$ (respectively, $N$), and $C([0,T], C^1(M))$ the space of continuous mappings $u : [0,T] \rightarrow C^1(M)$.

**Proposition 8.3** [17] Consider the equation

$$\frac{\partial u}{\partial t} = \Delta u + F(x,u,\nabla u, t), \quad u(x,0) = g \quad (8.10)$$

for $u(x,t)$ a function on $M \times [0,T]$, where $M$ is a compact manifold without boundary, and $F$ is $C^\infty$ in its arguments. Given $g \in C^1(M)$, the above equation has, for some $T > 0$, a unique solution

$$u \in C([0,T], C^1(M)) \cap C^\infty(M \times [0,T]). \quad (8.11)$$

**Proof** See Taylor [17, Chapter 15, Proposition 1.2].

Given $f : M \rightarrow M$, let $\text{Fix}(f) = \{ x \in M : f(x) = x \}$ denote the fixed-point set of $f$.

**Definition 8.4** Suppose that $u$ satisfying $u(x,0) \in C^1(N)$ is a solution of

$$\frac{\partial u}{\partial t} + \mathcal{P}(u) = 0$$

on $N$. We say that $u$ satisfies Neumann boundary conditions (NBC) on $N$ if for every $\tau \in \mathcal{R}$ and all $x \in \partial N \cap \text{Fix}(\tau)$, we have

$$\frac{\partial u}{\partial n}(x) = 0, \quad (8.12)$$

where $n$ is the normal direction to $\text{Fix}(\tau)$ at $x$.

\[ \diamond \]

A solution $u$ (satisfying (8.11)) of the equation (8.10) will be called smooth.

We can now state the extension theorem:
Theorem 8.5 Let $P$ be the $K$-invariant operator defined by (8.9). Then the following hold:

1. Every smooth $K$-invariant solution $u$ of

$$\frac{\partial u}{\partial t} + P(u) = 0$$ (8.13)

on $M$ restricts to a smooth solution of the Neumann problem for (8.13) on $N$.

2. Let $u$ be a solution to the Neumann problem with $u(x,0) \in C^1(N)$ for (8.13) on $N$. Then:

(a) $u$ is smooth.
(b) $u$ extends uniquely to a smooth $K$-invariant solution of (8.13) on $M$.

Proof The proof follows the same lines as [5, Theorem 5.18], where now we consider the parabolic equation (8.13). In proving item (b) we use Proposition 8.3 above instead of [5, Lemma 5.15].

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References


