Hilbert Series for Equivariant Mappings Restricted to Invariant Hyperplanes

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Abstract

In symmetric bifurcation theory it is often necessary to describe the restrictions of equivariant mappings to the fixed-point space of a subgroup. Such restrictions are equivariant under the normalizer of the subgroup, but this condition need not be the only constraint. We develop an approach to such questions in terms of Hilbert series – generating functions for the dimension of the space of equivariants of a given degree. We derive a formula for the Hilbert series of the restricted equivariants in the case when the subgroup is generated by a reflection, so the fixed-point space is a hyperplane. By comparing this Hilbert series with that of the normalizer, we can detect the occurrence of further constraints. The method is illustrated for the dihedral and symmetric groups.

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1 Introduction

A central object of study in symmetric bifurcation theory is the space $\vec{\mathcal{E}}_V(\Gamma)$ of smooth equivariant mappings $f : V \to V$ where V is a finite-dimensional representation (real or complex) of a compact Lie group Γ, see Golubitsky *et al.* [9]. This space is a module over the ring $\mathcal{E}_V(\Gamma)$ of invariant functions. For many purposes it

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is sufficient to consider the space $\vec{\mathcal{P}}_V(\Gamma)$ of equivariant polynomial mappings, which is a module over the ring $\mathcal{P}_V(\Gamma)$ of polynomial functions.

Some useful information about the structure of $\vec{\mathcal{P}}_V(\Gamma)$ can be encoded in its Hilbert series, which is a generating function for the dimensions of the polynomial equivariants of given degree. Specifically, define

$$\vec{\mathcal{P}}_V^d(\Gamma) = \{p \in \vec{\mathcal{P}}_V(\Gamma) : \partial p = d\}$$

where ∂ indicates the degree, and form the power series in an indeterminate t defined by

$$\Psi_\Gamma(t) = \sum_{d=0}^{\infty} a_d t^d,$$

where

$$a_d = \dim \vec{\mathcal{P}}_V^d(\Gamma).$$

The celebrated theorem of Molien [17] generalized to the module of equivariants [18] provides an explicit formula

$$\Psi_\Gamma(t) = \int_{\gamma\in\Gamma} \frac{\mathrm{tr}(\gamma^{-1})}{\det(\mathbf{1} - t\gamma)} d\mu_\Gamma$$

where μ_Γ is normalised Haar measure on Γ.

The problem that we discuss in this paper concerns analogous Hilbert series for the restrictions of Γ-equivariants to fixed-point spaces. Suppose that Σ is a subgroup of Γ, and let

$$\mathrm{Fix}(\Sigma) = \{x \in V : \sigma x = x \quad \forall \sigma \in \Sigma\}$$

be its fixed-point space. It is well known that $W = \mathrm{Fix}(\Sigma)$ is mapped to itself by every Γ-equivariant f, so that

$$\hat{f} = f|_W : W \to W.$$

In applications to bifurcation theory, this fact is exploited to find solutions of a Γ-equivariant differential equation that have symmetry Σ, see Golubitsky *et al.* [9]. A key question is to understand what constraints the map $\hat{f}$ must satisfy. In particular, it is useful to do so without computing the Γ-equivariants explicitly.

It is well known that every restricted mapping $\hat{f}$ possesses an equivariance property, as follows. Let

$$\overline{\Sigma} = \{\sigma \in \Gamma : \sigma x = x \quad \forall x \in W\}$$

and let $H = N_\Gamma(\overline{\Sigma})/\overline{\Sigma}$ where N_Γ indicates the normalizer. Then W is invariant under the action of H, and $\hat{f}$ is H-equivariant. Thus there is a map

$$\hat{}: \vec{\mathcal{P}}_V(\Gamma) \to \vec{\mathcal{P}}_W(H).$$

It is here that the problem becomes interesting, because in many cases the map $\hat{}$ is not surjective: for example, see Section 2, where $\Gamma = \mathbf{D}_5$ in its natural action on $\mathbf{R}^2$ and Σ is generated by a reflection. That is, in order for a map on W to be the restriction of a Γ-equivariant on V, it must satisfy non-trivial constraints in addition to H-equivariance. Another way to phrase this is that not all polynomial H-equivariants on W extend to polynomial Γ-equivariants on V. (Note that such an extension always exists in the category of continuous mappings. The obstacle to extension is to ensure a sufficient degree of smoothness.) This phenomenon has major consequences for symmetric bifurcation theory, because in many problems the 'obvious' symmetry group is H, but a larger group Γ is involved in a less obvious manner. We refer for example to [1, 2, 3, 4, 5, 6, 7, 8, 11, 16]. See Gomes *et al.* [10] for an overview of the subject.

Golubitsky, Marsden and Schaeffer [8] identified one source of extra constraints on $\hat{f}$, which they called *hidden symmetries*. Suppose that some element $\gamma \in \Gamma \setminus N_\Gamma(\overline{\Sigma})$ exists with the property that $W \cap \gamma W \neq \{0\}$. Then for all $w \in W \cap \gamma W$ we have

$$\hat{f}(\gamma w) = f(\gamma w) = \gamma f(w) = \gamma \hat{f}(w).$$

That is, $\hat{f}$ is γ-equivariant on the subspace $W \cap \gamma W$ of W.

However, hidden symmetries are not the only constraints on $\hat{f}$. Again, the example of $\mathbf{D}_5$ in Section 2 illustrates this, because W is a line, so $W \cap \gamma W = \{0\}$ for all $\gamma \in \Gamma \setminus N_\Gamma(\overline{\Sigma})$. These additional constraints arise from a combination of symmetry properties and smoothness, and are difficult to characterize in a concrete manner.

We shall say that $\Sigma \subseteq \Gamma$ is *deficient* if the map $\hat{}$ is not surjective. How can we characterize deficient subgroups?

A separate, but related, issue is whether $\hat{}$ is injective. Equivalently, if a Γ-equivariant vanishes on W, must it be zero on V?

We show here that both of these issues can usefully be studied in terms of Hilbert series, at least in the special case for which Σ is generated by a single reflection. In particular we prove an analogue of Molien's theorem for the generating function of the dimensions of the spaces of polynomials $\hat{f}$ of degree d.

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2 An Example

As a motivation we start with a simple example. Consider $\Gamma = \mathbf{D}_5$ acting on $V = \mathbf{R}^2 \equiv \mathbf{C}$ by the standard action generated by

$$\xi = \begin{pmatrix} \cos(\frac{2\pi}{5}) & -\sin(\frac{2\pi}{5}) \\ \sin(\frac{2\pi}{5}) & \cos(\frac{2\pi}{5}) \end{pmatrix},$$

$$\kappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is well known that the ring of the invariant polynomial functions $\mathcal{P}_V(\Gamma)$ is generated by

$$\begin{aligned} u &= x^2 + y^2 \equiv z\overline{z}, \\ v &= x^5 - 10y^2x^3 + 5y^4x \equiv \mathrm{Re}(z^5) \end{aligned}$$

which are algebraically independent, and the module of the equivariant polynomial mappings $\vec{\mathcal{P}}_V(\Gamma)$ is freely generated over $\mathcal{P}_V(\Gamma)$ by

$$U = \begin{pmatrix} x \\ y \end{pmatrix} \equiv z,$$

$$V = \begin{pmatrix} x^4 - 6x^2y^2 + y^4 \\ -4x^3y + 4xy^3 \end{pmatrix} \equiv \overline{z}^4$$

(see for example [9]). Thus if $f \in \vec{\mathcal{P}}_V(\Gamma)$, then there are polynomial functions $p_1,\ p_2$ such that

$$f(x,y) = p_1(u,v)U + p_2(u,v)V.$$

Let $\Sigma = \{1,\ \kappa\}$ and so $W = \mathrm{Fix}(\Sigma) = \{(x,0):\ x \in \mathbf{R}\}$. Note that $\overline{\Sigma} = \Sigma$ and $N_\Gamma(\Sigma) = \Sigma$. Thus $H = N_\Gamma(\overline{\Sigma})/\overline{\Sigma} = 1$ and so we have the trivial group acting on W. Consider $\hat{f}$. As

$$\begin{aligned} \hat{u} &= x^2, \\ \hat{v} &= x^5 \end{aligned}$$

and

$$\hat{U} = \begin{pmatrix} x \\ 0 \end{pmatrix},$$

$$\hat{V} = \begin{pmatrix} x^4 \\ 0 \end{pmatrix},$$

then

$$\hat{f}(x) = p_1(x^2, x^5)\hat{U} + p_2(x^2, x^5)\hat{V}.$$

Note that although

$$\begin{pmatrix} x^2 \\ 0 \end{pmatrix}$$

is a polynomial mapping on W that is H-equivariant (H is trivial), it can not appear as a restriction of a polynomial mapping f to W. Thus $\{1,\ \kappa\}$ is deficient in $\mathbf{D}_5$.

3 Preliminaries

Let V be a real vector space and let Γ be a compact Lie group acting linearly and (without loss of generality) orthogonally on V. Recall that a polynomial function $p : V \to \mathbf{R}$ is *invariant* under Γ if $p(\gamma \cdot v) = p(v)$ for all $\gamma \in \Gamma$, and a polynomial mapping $g : V \to V$ is *equivariant* under Γ if $g(\gamma \cdot v) = \gamma.g(v)$ for all $\gamma \in \Gamma$. In applications to bifurcation theory, g is a truncation of the Taylor series of a smooth equivariant vector field, but in this paper we focus only on polynomial functions and mappings in the abstract.

Denote by $\mathcal{P}_V(\Gamma)$ the ring of Γ-invariant polynomials and by $\mathcal{P}_V^d(\Gamma)$ the vector space of homogeneous Γ-invariant polynomials of degree d. Let $\vec{\mathcal{P}}_V(\Gamma)$ be the space of Γ-equivariant polynomial mappings from V to V, which is a module over $\mathcal{P}_V(\Gamma)$. Denote by $\vec{\mathcal{P}}_V^d(\Gamma)$ the vector space of homogeneous Γ-equivariant polynomial maps of degree d.

3.1 Review of Hilbert Series

We review Hilbert series for the rings of invariants and modules of equivariants for general compact Lie groups.

It will be convenient to change from a real representation to a complex representation, and we briefly explain why this useful step produces no extra complications. Let Γ be a compact Lie group acting on $V = \mathbf{R}^k$, so that $\gamma \in \Gamma$ acts as a matrix M_γ. The matrix M_γ has real entries, and we can view it as a matrix acting on $\mathbf{C}^k$. If $(x_1, \ldots, x_k)$ denote real coordinates on $\mathbf{R}^k$, $x_j \in \mathbf{R}$, then we obtain complex coordinates on $\mathbf{C}^k$ by permitting the x_j to be complex. Moreover, there is a natural inclusion $\mathbf{R}[x_1, \ldots, x_k] \subseteq \mathbf{C}[x_1, \ldots, x_k]$ where these are the rings of polynomials in the x_j with coefficients in $\mathbf{R}$, $\mathbf{C}$ respectively.

Every real-valued Γ-invariant in $\mathbf{R}[x_1, \ldots, x_k]$ is also a complex-valued Γ-invariant in $\mathbf{C}[x_1, \ldots, x_k]$. Conversely the real and imaginary parts of a complex-valued invariant are real invariants (because the matrices M_γ have real entries). Therefore a basis over $\mathbf{R}$ for the real vector space of real-valued invariants of degree d is also a basis over $\mathbf{C}$ for the complex vector space of $\mathbf{C}$-valued invariants of degree d. That is, the 'real' and 'complex' Hilbert series are the same. Similar remarks apply to the equivariants. Bearing these facts in mind, we 'complexify' the entire problem, reducing it to the following situation.

Let V be a k-dimensional vector space over $\mathbf{C}$, and let $x_1, \ldots, x_k$ denote coordinates relative to a basis for V. Let $\Gamma \subseteq \mathbf{GL}(V)$ be a compact Lie group. Let $\mathbf{C}[x_1, \ldots, x_k]$ denote the ring of polynomials over $\mathbf{C}$ in $x_1, \ldots, x_k$. Consider an action of Γ on V. Note that $\mathbf{C}[x_1, \ldots, x_k]$ is graded:

$$\mathbf{C}[x_1, \ldots, x_k] = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$$

where R_i consists of all homogeneous polynomials of degree i. If $f(x) \in R_i$ for some i, then $f(\gamma x) \in R_i$ for all $\gamma \in \Gamma$. Therefore for any subgroup Γ of $\mathbf{GL}(V)$ the space $\mathcal{P}_V(\Gamma)$ has the structure

$$\mathcal{P}_V(\Gamma) = \mathcal{P}_V^0(\Gamma) \oplus \mathcal{P}_V^1(\Gamma) \oplus \mathcal{P}_V^2(\Gamma) \oplus \cdots$$

of an N-graded $\mathbf{C}$-algebra given by $\mathcal{P}_V^i(\Gamma) = \mathcal{P}_V(\Gamma) \cap R_i$. Similarly $\vec{\mathcal{P}}_V(\Gamma)$ is a graded module over the ring $\mathcal{P}_V(\Gamma)$:

$$\vec{\mathcal{P}}_V(\Gamma) = \vec{\mathcal{P}}_V^0(\Gamma) \oplus \vec{\mathcal{P}}_V^1(\Gamma) \oplus \vec{\mathcal{P}}_V^2(\Gamma) \oplus \cdots.$$

Since Γ is compact, consider the normalized Haar measure μ_Γ defined on Γ [9] and denote the integral with respect to μ_Γ of a continuous function f defined on Γ by

$$\int_\Gamma f d\mu_\Gamma.$$

Recall that if Γ is finite, then the normalized Haar integral on Γ is

$$\int_\Gamma f d\mu_\Gamma \equiv \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma),$$

where $|\Gamma|$ denotes the order of Γ [9].

The *Hilbert series* or *Poincaré series* of the graded algebra $\mathcal{P}_V(\Gamma)$ is a generating function for the dimension of the vector space of invariants at each degree and is defined to be

$$\Phi_\Gamma(t) = \sum_{d=0}^{\infty} \dim(\mathcal{P}_V^d(\Gamma)) t^d. \tag{3.1}$$

There is a famous explicit formula:

Theorem 3.1 (Molien's Theorem) *Let Γ be a compact Lie group. Then the Hilbert series of $\mathcal{P}_V(\Gamma)$ is*

$$\Phi_\Gamma(t) = \int_\Gamma \frac{1}{\det(\mathbf{1} - t\gamma)} d\mu_\Gamma.$$

Proof. See [17] (or [20]) for the original proof of the finite case, and [18] for the extension to a compact group. Both proofs make use of the complex representation. ∎

The *Hilbert series* of the graded module $\vec{\mathcal{P}}_V(\Gamma)$ over the ring $\mathcal{P}_V(\Gamma)$ is the generating function

$$\Psi_\Gamma(t) = \sum_{d=0}^{\infty} \dim(\vec{\mathcal{P}}_V^d(\Gamma))\ t^d.$$

There is an explicit formula that generalizes the Molien Theorem for the equivariants:

Theorem 3.2 (Equivariant Molien Theorem) *Let Γ be a compact Lie group. Then the module $\vec{\mathcal{P}}_V(\Gamma)$ over the ring $\mathcal{P}_V(\Gamma)$ has a Hilbert series given by*

$$\Psi_\Gamma(t) = \int_\Gamma \frac{\operatorname{tr}(\gamma^{-1})}{\det(\mathbf{1} - t\gamma)} d\mu_\Gamma.$$

Proof. See [18]. Again the proof involves complexifying the representation. ∎

Remark 3.3

Theorem 3.2 also holds for the module of Γ-equivariant polynomial mappings $f : V \to W$ where the actions of Γ on V and W may be different (non-isomorphic). In this case the expression is

$$\Psi_\Gamma(t) = \int_\Gamma \frac{\mathrm{tr}(\gamma_W^{-1})}{\det(\mathbf{1} - t\gamma_V)} d\mu_\Gamma,$$

where γ_V and γ_W represent the matrices γ corresponding to the actions of Γ on V and on W respectively. See [21].

Note that for orthogonal group representations $\mathrm{tr}(\gamma^{-1}) = \mathrm{tr}(\gamma)$. Molien's Theorem has considerable theoretical interest, but is not always a practical way to compute Molien series because of difficulties in evaluating the integral. Two alternatives to the Molien formula, which can be more suitable for computations, can be found in Jarić and Birman [13]. An example of their application to a crystallographic space group is given in [14].

4 Hilbert Series and Deficient Subgroups

As before, we consider a compact Lie group Γ acting orthogonally on $V = \mathbf{R}^k$. Denote the coordinates for a given basis of V by $x_1, \ldots, x_k$. Recall the beginning of section 3.1, where it is shown that the 'real' and 'complex' Hilbert series are the same for Γ. That is, the (real) dimension of the space of homogeneous equivariants $\vec{\mathcal{P}}_V^d(\Gamma)$ (for each d) is equal to the (complex) dimension of the space $\vec{\mathcal{P}}_{V_\mathbf{C}}^d(\Gamma)$, where the action of Γ on $V_\mathbf{C}$ is given by the same (real) matrices.

Suppose that Σ is a subgroup of Γ and let $W = \mathrm{Fix}(\Sigma)$ be the fixed-point space of Σ on V. Recall that $\mathrm{Fix}(\Sigma)$ is given by the vectors on V that are fixed by the group Σ. Note that the space W, complexified, is the space $\mathrm{Fix}(\Sigma)$ in the complexified space $V_\mathbf{C}$. If f is Γ-equivariant, then

$$f(\mathrm{Fix}(\Sigma)) \subseteq \mathrm{Fix}(\Sigma).$$

Denote by $\hat{f}$ the restriction of f to W:

$$\hat{f} = f\,|_W \colon W \to W.$$

Call

$$\vec{Q}_W = \{\hat{p} \ : \ p \in \vec{\mathcal{P}}_V(\Gamma)\}$$

and

$$\vec{Q}_W^d = \{\hat{p} \ : \ p \in \vec{\mathcal{P}}_V^d(\Gamma)\}.$$

Since $\vec{\mathcal{P}}_V(\Gamma)$ is a graded module, we have

$$\vec{Q}_W = \vec{Q}_W^0 \oplus \vec{Q}_W^1 \oplus \vec{Q}_W^2 \oplus \cdots.$$

Consider the generating function for the dimensions of the $\vec{Q}^d_W$:

$$\Psi_\Sigma(t) = \sum_{d=0}^{\infty} \dim(\vec{Q}^d_W)\ t^d.$$

The question we address at this point is: using the Hilbert series Ψ_Γ of the module $\vec{\mathcal{P}}_V(\Gamma)$, how can we compute the Hilbert series Ψ_Σ of $\vec{Q}_W$? If this is possible, then we can compare the result with the Hilbert series Ψ_H for the H-equivariants on W. The group Σ is deficient if and only if $\Psi_\Sigma(t) \neq \Psi_H(t)$.

Remark 4.1

Recall that for any subgroup Σ of Γ, if we define

$$\overline{\Sigma} = \{\sigma \in \Gamma\ :\ \sigma \cdot w = w \quad \forall w \in W\},$$

then the normalizer $N_\Gamma(\overline{\Sigma})$ is the biggest subgroup of Γ that leaves W invariant, and $\overline{\Sigma}$ acts trivially on W. If we let $H = N_\Gamma(\overline{\Sigma})/\overline{\Sigma}$ then H acts on W.

Lemma 4.2 *Let Γ be a compact Lie group acting orthogonally on $V = \mathbf{R}^k$. Suppose that Σ is a subgroup of Γ and let $W = \mathrm{Fix}(\Sigma)$ be the fixed-point space of Σ on V. Let d be a nonnegative integer, define the map*

$$\begin{array}{rcl} R^d : \vec{\mathcal{P}}^d_V(\Gamma) & \to & \vec{Q}^d_W \\ p & \mapsto & \hat{p} = p\,|_W \end{array}$$

and call $\vec{K}^d_V = \ker(R^d)$. Then

$$\vec{Q}^d_W \cong_{\mathbf{R}} \vec{\mathcal{P}}^d_V(\Gamma)/\vec{K}^d_V.$$

Proof. Note that for $p \in \vec{\mathcal{P}}^d_V(\Gamma)$, since $\hat{p}$ is the restriction of p to W, then $\hat{p}$ belongs to $\vec{Q}^d_W$. Thus R^d is well defined. Moreover R^d is surjective by definition of $\vec{Q}^d_W$. Since R^d is $\mathbf{R}$-linear, the result follows. ∎

Lemma 4.2 implies that

$$\vec{K}_V = \vec{K}^0_V \oplus \vec{K}^1_V \oplus \vec{K}^2_V \oplus \cdots$$

and

$$\dim_{\mathbf{R}}(\vec{\mathcal{P}}^d_V(\Gamma)) = \dim_{\mathbf{R}}(\vec{K}^d_V) + \dim_{\mathbf{R}}(\vec{Q}^d_W)$$

for each nonnegative integer d. Moreover, if we complexify V and W and consider the action of Γ on $V_{\mathbf{C}}$ by the same matrices, we get

$$\dim_{\mathbf{C}}(\vec{\mathcal{P}}^d_{V_{\mathbf{C}}}(\Gamma)) = \dim_{\mathbf{C}}(\vec{K}^d_{V_{\mathbf{C}}}) + \dim_{\mathbf{C}}(\vec{Q}^d_{W_{\mathbf{C}}}).$$

Recall that $\dim_{\mathbf{C}}(\vec{\mathcal{P}}^d_{V_{\mathbf{C}}}(\Gamma)) = \dim_{\mathbf{R}}(\vec{\mathcal{P}}^d_V(\Gamma))$ and $\dim_{\mathbf{C}}(\vec{Q}^d_{W_{\mathbf{C}}}) = \dim_{\mathbf{R}}(\vec{Q}^d_W)$. Therefore if we define

$$\Psi^0_\Sigma(t) = \sum_{d=0}^{\infty} \dim(\vec{K}^d_V)\ t^d$$

we obtain:

Lemma 4.3 *The Hilbert series Ψ_Σ, Ψ_Γ and Ψ^0_Σ are related by:*

$$\Psi_\Sigma(t) = \Psi_\Gamma(t) - \Psi^0_\Sigma(t).$$

5 Reflection Subgroups

In this section we obtain a formula for the series Ψ^0_Σ in the case where Σ is a subgroup of Γ generated by a single geometric reflection σ.

5.1 Semi-invariance

As before, Γ is a compact Lie group, so we may assume without loss of generality that Γ acts orthogonally on $V = \mathbf{R}^k$. Suppose that there is a decomposition of the space V as $W \oplus W^\perp$, where W has dimension $k-1$ and there exists $\sigma \in \Gamma$ that acts as the identity on W and as minus the identity on $W^\perp$; namely σ is a *reflection.* Orthogonality is defined with respect to a Γ-invariant inner product $<,>_\Gamma$ defined on V. Throughout, let Σ be the subgroup of Γ generated by σ, so $W = \mathrm{Fix}(\Sigma)$.

Remark 5.1

Note that $x \in W$ if and only if $L_W(x) = 0$ for some linear polynomial function $L_W(x)$. That is $L_W(x) = 0$ is an equation defining the hyperplane W. Take $L_W(x) =< n_\Sigma, x >_\Gamma$ with n_Σ any nonnull vector in $W^\perp$. Observe that $W^\perp$ is one-dimensional.

Lemma 5.2 *Let $f \in \vec{\mathcal{P}}_V(\Gamma)$. Then f vanishes on W if and only if f is of the form*

$$f(x) = L_W(x)g(x),$$

for some polynomial mapping $g : V \to V$.

Proof. This is an immediate consequence of the Remainder Theorem for Multivariable Polynomials [15], but it is easy to give a direct proof. Choose a basis for V such that x_1 is the coordinate corresponding to $W^\perp$, so that $L_W(x) = x_1$ is a linear polynomial that vanishes on W. Write f as $(f_1, \ldots, f_n)$. Then

$$f(x) = 0 \text{ for } x_1 = 0 \Longleftrightarrow f_1(x) = \cdots = f_n(x) = 0 \text{ for } x_1 = 0,$$

thus

$$f_j(x) = x_1 g_j(x),$$

for some $g_j : V \to \mathbf{R}$, and so $f(x) = x_1(g_1(x), \ldots, g_n(x))$. ∎

Remarks 5.3

(a) If f is Γ-equivariant and $f(x) = 0$ for $x \in W$, it follows that $f(\gamma \cdot x) = \gamma \cdot f(x) = \gamma \cdot 0 = 0$ for all $x \in W$. Therefore, if we define

$$\Gamma W = \{\gamma \cdot w \ : \ w \in W, \ \gamma \in \Gamma\},$$

then

$$f \mid_W \equiv 0 \Longleftrightarrow f \mid_{\Gamma W} \equiv 0.$$

The set ΓW is a union of hyperplanes γW.

(b) The linear function L_W in Lemma 5.2 is unique up to a nonzero real constant multiple.

Recall Remark 4.1 and let $N = N_\Gamma(\overline{\Sigma})$ (the normalizer of $\overline{\Sigma}$ in Γ). Suppose now that N is of finite index in Γ. Let $\{\gamma_1 = 1, \ \gamma_2, \ \ldots, \ \gamma_s\}$ be a set of coset representatives for N, where s is the index of N, so that

$$\Gamma = \gamma_1 N \ \dot{\cup} \ \gamma_2 N \ \dot{\cup} \ \cdots \ \dot{\cup} \ \gamma_s N.$$

Choose $n_\Sigma \neq 0$ in $W^\perp$. We have:

Corollary 5.4 *(a) If N is of finite index in Γ, then a polynomial mapping $f \in \vec{\mathcal{P}}_V(\Gamma)$ vanishes on W if and only if there exists a polynomial mapping $g : V \to V$ such that*

$$f(x) = (L_{\gamma_1}(x) L_{\gamma_2}(x) \cdots L_{\gamma_s}(x)) \ \ g(x)$$

where

$$L_{\gamma_i}(x) = < \gamma_i \cdot n_\Sigma, x >_\Gamma, \ \ i = 1, \ldots, s.$$

(b) If N is of infinite index in Γ, then the map $\hat{}$ is injective.

Proof. (a) Note that $L_{\gamma_1}(x) = < n_\Sigma, x >_\Gamma$ and so $L_{\gamma_1}(x) = 0$ if and only if $x \in W$. Since f vanishes on W, it follows that $L_{\gamma_1}(x)$ divides each component $f_j(x)$ of $f(x)$ (Lemma 5.2). Moreover,

$$L_{\gamma_i}(\gamma_i \cdot x) = < \gamma_i \cdot n_\Sigma, \gamma_i \cdot x >_\Gamma = < n_\Sigma, x >_\Gamma = 0$$

for $x \in W$ and $i = 2, \ldots, n$. That is, each $L_{\gamma_i}(x)$ divides f_j since it represents a linear polynomial that vanishes on $\gamma_i W = \{\gamma_i \cdot w \ : \ w \in W\}$ (recall Remark 5.3). Note that $\gamma_i W \cap \gamma_j W = \{0\}$ for $i \neq j$.

It follows easily that the polynomials $L_{\gamma_i}(x)$ are coprime in the ring of k-variable polynomials. Since this is a unique factorization domain [15], it follows that we can choose $g = (g_1, \ldots, g_n)$ with $g_i : V \to \mathbf{R}$ such that $f(x) = L_{\gamma_1}(x) L_{\gamma_2}(x) \cdots L_{\gamma_s}(x) \ g(x)$.

(b) The same argument applies for an unbounded set of coset representatives $\{\gamma_1, \ \ldots, \ \gamma_s\}$, $s \to \infty$. Therefore $f = 0$. ∎

Remark 5.5

When the index of N is finite, define

$$\Delta(x) = L_{\gamma_1}(x)L_{\gamma_2}(x)\cdots L_{\gamma_s}(x) \tag{5.2}$$

as in Corollary 5.4. Then

$$\Delta(\gamma \cdot x) = L_{\gamma^t\gamma_1}(x)L_{\gamma^t\gamma_2}(x)\cdots L_{\gamma^t\gamma_s}(x).$$

Note that for each $i = 1, \ldots, n$, we have

$$\gamma^t\gamma_i \cdot n_\Sigma = \gamma_p\sigma^* \cdot n_\Sigma$$

for some p and some $\sigma^* \in N$. Moreover $\sigma^* \cdot n_\Sigma$ belongs to $W^\perp$ since $< \sigma^* \cdot n_\Sigma, w >_\Gamma = < n_\Sigma, {\sigma^*}^t \cdot w >_\Gamma = 0$ for all $w \in W$. As $W^\perp$ is an one-dimensional real vector space and $\sigma^* \cdot n_\Sigma$ belongs to $W^\perp$, it follows that $\sigma^* \cdot n_\Sigma = c_i(\gamma)n_\Sigma$ for $c_i(\gamma) = 1$ or $c_i(\gamma) = -1$, and so

$$L_{\gamma^t\gamma_i}(x) = c_i(\gamma)L_{\gamma_p}(x).$$

Moreover, for a fixed $\gamma \in \Gamma$, if $i \neq j$, then $\gamma^t\gamma_i n_\Sigma \neq \mp\gamma^t\gamma_j n_\Sigma$. Thus

$$\Delta(\gamma \cdot x) = c_1(\gamma)c_2(\gamma)\cdots c_s(\gamma)\Delta(x), \tag{5.3}$$

where each constant $c_i(\gamma)$ is 1 or -1.

In the next proposition we use the notion of *semi-invariant*.

Definition 5.6

Following [19], we say that a polynomial function $f : V \to \mathbf{R}$ is a *semi-invariant* for Γ if there exists a homomorphism $C : \Gamma \to \{+1, -1\}$ such that

$$f(\gamma \cdot x) = C(\gamma)f(x)$$

for all $\gamma \in \Gamma$.

Proposition 5.7 *With the conditions of Corollary 5.4 the polynomial function $\Delta(x)$ (as defined in (5.2)) is a semi-invariant of Γ.*

Proof. From Remark 5.5, $\Delta(\gamma\cdot x) = C(\gamma)\Delta(x)$ where $C(\gamma) = c_1(\gamma)c_2(\gamma)\cdots c_s(\gamma)$. We prove now that $C : \Gamma \to \{1, -1\}$ is a group homomorphism. Since

$$\Delta((\gamma_1\gamma_2) \cdot x) = C(\gamma_1\gamma_2)\Delta(x)$$

and

$$\Delta((\gamma_1\gamma_2) \cdot x) = \Delta(\gamma_1 \cdot (\gamma_2 \cdot x)) = C(\gamma_1)\Delta(\gamma_2 \cdot x) = C(\gamma_1)C(\gamma_2)\Delta(x)$$

for all $x \in V$, then

$$C(\gamma_1\gamma_2) = C(\gamma_1)C(\gamma_2)$$

for all $\gamma_1,\ \gamma_2 \in \Gamma$. ∎

Definition 5.8

Let Γ be a compact Lie group with an action defined on V, and let $C : \Gamma \to \{1, -1\}$ be a homomorphism. A polynomial mapping $g : V \to V$ is (Γ, C)-*equivariant* if

$$g(\gamma \cdot x) = C(\gamma)\gamma \cdot g(x)$$

for all $x \in V$ and $\gamma \in \Gamma$.

Theorem 5.9 *A polynomial mapping $f \in \vec{\mathcal{P}}_V(\Gamma)$ vanishes on W if and only if there exists a (Γ, C)-equivariant $g : V \to V$ such that*

$$f(x) = \Delta(x)g(x),$$

where $\Delta : V \to \mathbf{R}$ is a semi-invariant for Γ as obtained in (5.2) and $C : \Gamma \to \{1, -1\}$ is the corresponding homomorphism determined by $\Delta(x)$.

Proof. Since $C(\gamma) = \mp 1$, then $C(\gamma)^{-1} = C(\gamma)$. The result follows from Corollary 5.4, Remark 5.5, and Proposition 5.7. ∎

5.2 Hilbert Series

Recall Section 4 where we use the notation $f \in \vec{K}_V$ for those $f \in \vec{\mathcal{P}}_V(\Gamma)$ that vanish on W. We can now combine Lemma 4.3 and Theorem 5.9, and obtain a formula for Ψ_Σ for the cases where Σ is a subgroup of Γ generated by a geometric reflection (as defined at beginning of Section 5.1).

Theorem 5.10 *Let Γ be a compact Lie group acting orthogonally on $V = \mathbf{R}^k$, and let Σ be a subgroup of Γ generated by a reflection σ that fixes W (a $k-1$ dimensional subspace of V). With the conditions of Theorem 5.9, if $N = N_\Gamma(\overline{\Sigma})$ and the index s of N in Γ is finite, then*

$$\Psi_\Sigma(t) = \Psi_\Gamma(t) - t^s \Psi_{(\Gamma,C)}(t),$$

where

$$\Psi_{(\Gamma,C)}(t) = \int_{\gamma\in\Gamma} \frac{C(\gamma)\mathrm{tr}(\gamma)}{\det(\mathbf{1} - t\gamma)} d\mu_\Gamma.$$

Proof. From Theorem 5.9, Ψ_Σ^0 in Lemma 4.3 becomes

$$\Psi_\Sigma^0(t) = t^s \Psi_{(\Gamma,C)}(t).$$

The conclusion of the theorem follows from a direct application of Remark 3.3. Note that $\mathrm{tr}((C(\gamma)\gamma)^{-1}) = C(\gamma)\mathrm{tr}(\gamma)$ since $C(\gamma) = \mp 1$ and Γ acts orthogonally. ∎

Remark 5.11

Applying Lemma 4.3 and the ideas involved in the proof of Theorem 5.9, it follows that for the cases where the index of N in Γ is infinite, and again Σ is a subgroup of Γ generated by a reflection,

$$\Psi_\Sigma(t) = \Psi_\Gamma(t).$$

So $\hat{}$ is injective.

Theorem 5.12 *With the conditions of Theorem 5.10,*

$$\Psi_\Sigma(t) = (1+t^s)\Psi_\Gamma(t) - t^s\Psi_{\Gamma_0}(t)$$

where $\Gamma_0 = \ker(C)$.

Proof. Let

$$A(t) = \Psi_{(\Gamma,C)}(t) = \int_{\gamma\in\Gamma} \frac{C(\gamma)\mathrm{tr}(\gamma)}{\det(\mathbf{1}-t\gamma)} d\mu_\Gamma.$$

If $\Gamma_1 = \Gamma \setminus \Gamma_0$, then

$$\int_\Gamma f d\mu_\Gamma = \int_{\Gamma_0} f d\mu_\Gamma + \int_{\Gamma_1} f d\mu_\Gamma$$

for any continuous function f on Γ. Thus

$$A(t) = B_0(t) - B_1(t), \tag{5.4}$$

where

$$B_i(t) = \int_{\gamma\in\Gamma_i} \frac{\mathrm{tr}(\gamma)}{\det(\mathbf{1}-t\gamma)} d\mu_\Gamma$$

for $i = 0, 1$. Also

$$\Psi_\Gamma(t) = B_0(t) + B_1(t). \tag{5.5}$$

Moreover, since $|\Gamma/\Gamma_0| = 2$

$$B_0(t) = \frac{1}{2}\Psi_{\Gamma_0}(t). \tag{5.6}$$

From (5.5) and (5.6),

$$B_1(t) = \Psi_\Gamma(t) - \frac{1}{2}\Psi_{\Gamma_0}(t). \tag{5.7}$$

From (5.4), (5.6) and (5.7),

$$A(t) = \Psi_{\Gamma_0}(t) - \Psi_\Gamma(t). \tag{5.8}$$

Therefore

$$\Psi_\Sigma(t) = \Psi_\Gamma(t) - t^s(\Psi_{\Gamma_0}(t) - \Psi_\Gamma(t)) = (1+t^s)\Psi_\Gamma(t) - t^s\Psi_{\Gamma_0}(t). \qquad \blacksquare$$

6 Examples

We illustrate our methods by applying them to the dihedral groups $\mathbf{D}_n$ and symmetric groups $\mathbf{S}_n$.

Example 1. $\mathbf{D}_n$

Let $\Gamma = \mathbf{D}_n$ in its standard action on $V = \mathbf{R}^2 \equiv \mathbf{C}$, generated by

$$\kappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and

$$\xi = \begin{pmatrix} \cos(\frac{2\pi}{n}) & -\sin(\frac{2\pi}{n}) \\ \sin(\frac{2\pi}{n}) & \cos(\frac{2\pi}{n}) \end{pmatrix}.$$

The invariants are generated by $z\overline{z}$ and $\mathrm{Re}(z^n)$, which are algebraically independent [9]. The equivariants are generated by z and $\overline{z}^{n-1}$, and form a free module. It follows that

$$\Phi_{\mathbf{D}_n}(t) = \frac{1}{(1-t^2)(1-t^n)}$$

and

$$\begin{aligned} \Psi_{\mathbf{D}_n}(t) &= (t + t^{n-1})\Phi_{\mathbf{D}_n}(t) \\ &= \frac{t(1+t^{n-2})}{(1-t^2)(1-t^n)}. \end{aligned}$$

(For small n these formulas are easy consequence of Molien's theorem, but for general n the combinatorics becomes less tractable.)

Case (a) n odd.

When n is odd, every reflection in $\mathbf{D}_n$ is conjugate to κ. We therefore let $\Sigma =< \kappa >$. Since $N(\Sigma) = \Sigma$, so $H = 1$, it follows that

$$\Psi_H(t) = \frac{1}{1-t}.$$

The polynomial Δ has degree n, and determines a homomorphism C with kernel $\Gamma_0 = \mathbf{Z}_n$. Therefore $\Psi_{\Gamma_0}(t) = 2\Psi_{\mathbf{D}_n}(t)$. Theorem 5.12 implies that

$$\begin{aligned}\Psi_\Sigma(t) &= (1-t^n)\Psi_{\mathbf{D}_n}(t) \\ &= \frac{t}{1-t} - (t^2+t^4+\cdots+t^{n-3}).\end{aligned}$$

Thus Σ is deficient in $\mathbf{D}_n$ for all odd $n \geq 5$. In particular we recover the results of the example in Section 2.

Case (b) n even.

When n is even, every reflection in $\mathbf{D}_n$ is conjugate either to κ or to $\kappa\xi$, so we define

$$\Sigma_1 =< \kappa >$$

$$\Sigma_2 =< \kappa\xi > .$$

Since n is even, $\mathbf{D}_n$ contains $\xi^{n/2} = -id$, which commutes with every element of $\mathbf{D}_n$ and hence normalizes both Σ_1 and Σ_2. Therefore

$$\Psi_H(t) = \frac{t}{1-t^2}$$

in both cases; that is, the H-equivariants are the odd functions. Since $(z\overline{z})^m z$ restricts to W to give x^{2m+1}, we deduce immediately that neither Σ_1 nor Σ_2 is deficient. We now check that this result is compatible with our method.

Let $\Sigma_1 =< \kappa >$. Then $H \cong \mathbf{Z}_2$ and

$$\Psi_H(t) = \frac{t}{1-t^2}$$

as expected. The polynomial Δ_1 has degree $n/2$, and determines a homomorphism C_1 with kernel $\Gamma_0^1 =< \xi^2, \ \kappa\xi >$. Therefore $\Psi_{\Gamma_0^1}(t) = \Psi_{\mathbf{D}_{n/2}}(t)$. Theorem 5.12 implies that

$$\Psi_\Sigma(t) = (1+t^{n/2})\Psi_{\mathbf{D}_n}(t) - t^{n/2}\Psi_{\mathbf{D}_{n/2}}(t) = \frac{t}{1-t^2}.$$

Thus Σ_1 is not deficient, as claimed.

Now let $\Sigma_2 =< \kappa\xi >$. Again $H \cong \mathbf{Z}_2$ and

$$\Psi_H(t) = \frac{t}{1-t^2}.$$

The polynomial Δ_2 has degree $n/2$ and determines a homomorphism C_2 with kernel

$\Gamma_0^2 =< \xi^2, \ \kappa >$. Therefore $\Psi_{\Gamma_0^2}(t) = \Psi_{\mathbf{D}_{n/2}}(t)$. From Theorem 5.12

$$\Psi_{\Sigma_2}(t) = \Psi_{\Sigma_1}(t)$$

and Σ_2 is not deficient.

Example 2. $\mathbf{S}_4$

Consider $\Gamma = \mathbf{S}_4$ acting on $\mathbf{R}^4$ by permutation of a basis, and let $\Sigma =< \sigma >$ where σ is a 2-cycle. Say $\sigma = (12)$. Thus

$$W = \mathrm{Fix}(\Sigma) = \{(x, x, z, w) \ : \ x, \ z, \ w \in \mathbf{R}\}.$$

Note that $\overline{\Sigma} = \Sigma$ and $N_\Gamma(\overline{\Sigma}) = \mathbf{S}_2 \times \mathbf{S}_2$, and so $H = N_\Gamma(\overline{\Sigma})/\overline{\Sigma} \cong \mathbf{S}_2$.

By Molien's formula, the Hilbert series for equivariants of $\mathbf{S}_4$ is given by

$$\begin{aligned} \Psi_{\mathbf{S}_4}(t) &= \frac{1}{24}\left[\frac{8}{(1-t)(1-t^3)} + \frac{6 \times 2}{(1-t)^2(1-t^2)} + \frac{4}{(1-t)^4}\right] \\ &= \frac{1}{(1-t)^4(1+t)(1+t+t^2)} \\ &= 1 + 2t + 4t^2 + 7t^3 + 11t^4 + 16t^5 + 23t^6 + 31t^7 + \cdots. \end{aligned}$$

We have

$$\Delta(x, y, z, w) = (y-x)(z-x)(w-x)(z-y)(w-y)(w-z).$$

This determines a homomorphism C with kernel the alternating group $\mathbf{A}_4$, that is

$$\begin{aligned} \Gamma_0 = \{&1, \ (123), \ (132), \ (124), \ (142), \ (134), \ (143), \ (234), \ (243), \\ &(12)(34), \ (13)(24), \ (14)(23)\}. \end{aligned}$$

Now

$$\begin{aligned} \Psi_{\Gamma_0^1}(t) &= \frac{1}{12}\left[\frac{4}{(1-t)^4} + \frac{8}{(1-t)(1-t^3)}\right] \\ &= 2\Psi_{\mathbf{S}_4}(t) - \frac{1}{(1-t)^2(1-t^2)} \\ &= \frac{1-t+t^2}{(1-t)^4(1+t+t^2)} \\ &= 1 + 2t + 4t^2 + 8t^3 + 13t^4 + 20t^5 + 30t^6 + 42t^7 + \cdots, \end{aligned}$$

and Theorem 5.12 yields

$$\begin{aligned}
\Psi_\Sigma(t) &= (1+t^6)\Psi_{\mathbf{S}_4}(t) - t^6\Psi_{\Gamma_0^1}(t) \\
&= (1-t^6)\Psi_{\mathbf{S}_4}(t) + \frac{t^6}{(1-t)^2(1-t^2)} \\
&= \frac{1+t^3+t^6}{(1-t)^3(1+t)} \\
&= 1+2t+4t^2+7t^3+11t^4+16t^5+23t^6+31t^7+\cdots.
\end{aligned}$$

Comparing this series with

$$\begin{aligned}
\Psi_H(t) &= \frac{1}{2}\left[\frac{3}{(1-t)^3} + \frac{1}{(1-t)(1-t^2)}\right] \\
&= \frac{2+t}{(1-t)^3(1+t)} \\
&= 2+5t+10t^2+16t^3+24t^4+33t^5+44t^6+56t^7+\cdots
\end{aligned}$$

we conclude that Σ is deficient.

Example 3. $\mathbf{S}_n$ for $n \geq 4$.

More generally, consider $\Gamma = \mathbf{S}_n$ acting on $\mathbf{R}^n$ by permutation of a basis, and as before let $\Sigma =< (12) >$ and

$$W = \mathrm{Fix}(\Sigma) = \{(x, x, x_3, \ldots, x_n) \;:\; x,\ x_3,\ \ldots, x_n \in \mathbf{R}\}.$$

Since $N_\Gamma(\overline{\Sigma}) = \mathbf{S}_2 \times \mathbf{S}_{n-2}$ it follows that $H \cong \mathbf{S}_{n-2}$. Moreover,

$$\Psi_H(t) = \frac{2-t-t^{n-2}}{(1-t)^3(1-t^2)\cdots(1-t^{n-2})}.$$

The Hilbert series for equivariants of $\mathbf{S}_n$ is

$$\Psi_{\mathbf{S}_n}(t) = \frac{1}{(1-t)^2(1-t^2)\cdots(1-t^{n-1})}.$$

This formula is obtained from the formula for $\Phi_{\mathbf{S}_n}$. As it is well known the ring $\mathcal{P}_V(\Gamma)$ is generated by n algebraically independent homogeneous polynomials of degrees 1, ..., n: for example the symmetric functions. Again, these degrees determine $\Phi_{\mathbf{S}_n}$. That is,

$$\Phi_{\mathbf{S}_n}(t) = \frac{1}{(1-t)\cdots(1-t^n)}$$

(see [12] for details). Since $\vec{\mathcal{P}}_V(\Gamma)$ is a free module over the ring $\mathcal{P}_V(\Gamma)$ and $f = (f_1, \ldots, f_n)$ is $\mathbf{S}_n$-equivariant if and only if the first component f_1 is $\mathbf{S}_{n-1}$-invariant in the last $n-1$ variables, it follows straightforwardly that

$$\Psi_{\mathbf{S}_n}(t) = \frac{1}{1-t}\Phi_{\mathbf{S}_{n-1}}(t),$$

and we get the above formula for $\Psi_{\mathbf{S}_n}$.

We can choose

$$\Delta(x_1, \ldots, x_n) = \prod_{i<j}(x_i - x_j).$$

This determines a homomorphism C with kernel the alternating group $\mathbf{A}_n$, see [15]. The degree of Δ is $n(n-1)/2$. From Theorem 5.12 we get

$$\Psi_\Sigma(t) \;= \left(1 + t^{\frac{n(n-1)}{2}}\right)\Psi_{\mathbf{S}_n}(t) - t^{\frac{n(n-1)}{2}}\Psi_{\mathbf{A}_n}(t).$$

Since Δ is invariant under $\mathbf{A}_n$ but changes sign under even permutations, every $\mathbf{A}_n$-invariant has a unique expression of the form $p(x) + q(x)\Delta(x)$ where p and q are $\mathbf{S}_n$-invariant. Therefore

$$\Phi_{\mathbf{A}_n}(t) = (1 + t^{\frac{n(n-1)}{2}})\Phi_{\mathbf{S}_n}(t).$$

Since $\mathbf{A}_n$ acts transitively on $\{1, \ldots, n\}$, every $\mathbf{A}_n$-equivariant $f \;:\; \mathbf{R}^n \to \mathbf{R}^n$ is uniquely determined by its first component f_1. It is straightforward to show that f_1 has the unique expression

$$f_1(x) = \sum_{r\geq 0} x_1^r p_r(x_2, \ldots, x_n)$$

where the p_r are $\mathbf{A}_{n-1}$-invariant, and there are no further restrictions on f_1. Therefore

$$\begin{aligned}\Psi_{\mathbf{A}_n}(t) &= \frac{1}{1-t}\Phi_{\mathbf{A}_{n-1}}(t)\\ &= \frac{1 + t^{\frac{(n-1)(n-2)}{2}}}{(1-t)^2(1-t^2)\cdots(1-t^{n-1})},\end{aligned}$$

and it follows that

$$\begin{aligned}\Psi_\Sigma(t) &= \frac{1-t^{(n-1)^2}}{(1-t)^2(1-t^2)\cdots(1-t^{n-1})} \\ &= \frac{1+t+t^2\cdots+t^{(n-1)^2-1}}{(1-t)\cdots(1-t^{n-1})}.\end{aligned}$$

Thus Σ is deficient.

The natural common generalization for all of the above examples is when Γ is a finite Coxeter group – a group generated by reflections, see [12]. However, we shall not investigate this generalization here.

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