HOPF BIFURCATION WITH $S_N$-SYMMETRY

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Abstract. We study Hopf bifurcation with $S_N$-symmetry for the standard absolutely irreducible action of $S_N$ obtained from the action of $S_N$ by permutation of $N$ coordinates. Stewart (Symmetry methods in collisionless many-body problems, J. Nonlinear Sci. 6 (1996) 543-563) obtains a classification theorem for the $C$-axial subgroups of $S_N \times S^1$. We use this classification to prove the existence of branches of periodic solutions with $C$-axial symmetry in systems of ordinary differential equations with $S_N$-symmetry posed on a direct sum of two such $S_N$-absolutely irreducible representations, as a result of a Hopf bifurcation occurring as a real parameter is varied. We determine the (generic) conditions on the coefficients of the fifth order $S_N \times S^1$-equivariant vector field that describe the stability and criticality of those solution branches. We finish this paper with an application to the special case $N = 5$.

1. Introduction

The general theory of Hopf Bifurcation with symmetry was developed by Golubitsky and Stewart [8] and by Golubitsky, Stewart, and Schaeffer [11]. Golubitsky and Stewart [9] applied the theory of Hopf bifurcation with symmetry to systems of ordinary differential equations having the symmetries of a regular polygon (this is, with $D_n$-symmetry). They studied the existence and stability of symmetry-breaking branches of periodic solutions in such systems. Finally, they applied their results to a general system of $n$ nonlinear oscillators, coupled symmetrically in a ring, and describe the generic oscillation patterns. Since the development of the theory, some examples were studied with detail (see for example, [1],[4],[3],[5],[7],[12],[14],[16],[18]).

In this paper we study one of the few classic problems in the theory of Hopf bifurcation with symmetry that has not been completely investigated, we study Hopf bifurcation with $S_N$-symmetry. This problem is relevant to, for example, the behaviour of all-to-all coupled nonlinear oscillators. The basic existence theorem for Hopf bifurcation in the symmetric case is the Equivariant Hopf Theorem, which involves $C$-axial isotropy subgroups of $S_N \times S^1$ (in this case), this is, isotropy subgroups with two-dimensional fixed-point subspace. Stewart [17] obtains a classification theorem for $C$-axial subgroups of $S_N \times S^1$. We use this classification to prove the existence of branches of periodic solutions in systems of ordinary differential equations with $S_N$-symmetry taking the restriction of the standard action of $S_N$ on $C^N$ onto a $S_N$-simple
space. We use the Equivariant Hopf Theorem to prove the existence of branches of periodic solutions and we determine (generic) conditions on the coefficients of the fifth order $\text{S}_N \times \text{S}^1$-equivariant vector field that describe the stability of the different types of bifurcating periodic solutions.

Consider the natural action of $\text{S}_N$ on $\mathbb{C}^N$ where $\sigma \in \text{S}_N$ acts by permutation of coordinates:

$$\sigma(z_1, \ldots, z_N) = (z_{\sigma^{-1}(1)}, \ldots, z_{\sigma^{-1}(N)})$$

where $(z_1, \ldots, z_N) \in \mathbb{C}^N$. Observe the following decomposition of $\mathbb{C}^N$ into invariant subspaces for this action:

$$\mathbb{C}^N \cong \mathbb{C}^{N,0} \oplus V_1$$

where

$$\mathbb{C}^{N,0} = \{(z_1, \ldots, z_N) \in \mathbb{C}^N : z_1 + \cdots + z_N = 0\}$$

and

$$V_1 = \{(z, \ldots, z) : z \in \mathbb{C}\} \cong \mathbb{C}.$$ 

The action of $\text{S}_N$ on $V_1$ is trivial and the space $\mathbb{C}^{N,0}$ is $\text{S}_N$-simple:

$$\mathbb{C}^{N,0} \cong \mathbb{R}^{N,0} \oplus \mathbb{R}^{N,0}$$

where $\text{S}_N$ acts absolutely irreducibly on

$$\mathbb{R}^{N,0} = \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_1 + \cdots + x_N = 0\} \cong \mathbb{R}^{N-1}.$$ 

We say that a representation of a group $\Gamma$ on a vector space $V$ is absolutely irreducible, or the space $V$ is said to be absolutely irreducible, if the only linear mappings on $V$ that commute with $\Gamma$ are the scalar multiples of the identity.

If we have a local $\Gamma$-equivariant Hopf bifurcation problem, generically the centre subspace at the Hopf bifurcation point is $\Gamma$-simple (see [11, Proposition XVI 1.4]). We make that assumption here. Thus we consider a general $\text{S}_N$-equivariant system of ordinary differential equations (ODEs)

$$\frac{dz}{dt} = f(z, \lambda),$$

where $z \in \mathbb{C}^{N,0}, \lambda \in \mathbb{R}$ is the bifurcation parameter and $f : \mathbb{C}^{N,0} \times \mathbb{R} \to \mathbb{C}^{N,0}$ is smooth and commutes with the restriction of the natural action (1) of $\text{S}_N$ on $\mathbb{C}^N$ to the $\text{S}_N$-simple space $\mathbb{C}^{N,0}$. Observe that $f(0, \lambda) \equiv 0$ since $\text{Fix}_{\text{C}^{N,0}}(\text{S}_N) = \{0\}$.

We study Hopf bifurcation of (2) from the trivial equilibrium, say, at $\lambda = 0$, and so we assume that $(df)_{0,0}$ has purely imaginary eigenvalues $\pm i$ (after rescaling time if necessary). Thus if we denote the eigenvalues of $(df)_{0,0}$ by $\sigma(\lambda) \pm i\rho(\lambda)$ then $\sigma(0) = 0, \rho(0) = 1$ (see [11, Lemma XVI 1.5]) and we make the standard hypothesis of the Equivariant Hopf Theorem:

$$\sigma'(0) \neq 0.$$ 

Under the above hypothesis, we can assume that the action of $\text{S}^1$ on the centre space $\mathbb{C}^{N,0}$ of $(df)_{0,0}$ (that can be identified with the exponential of $(df)_{0,0}$) is given by multiplication by $e^{i\theta}$:

$$\theta(z_1, \ldots, z_N) = e^{i\theta}(z_1, \ldots, z_N)$$

for $\theta \in \text{S}^1$, $(z_1, \ldots, z_N) \in \mathbb{C}^{N,0}$. 

This paper is organized as follows. In Section 2 we recall the key points for Hopf bifurcation theory of symmetric systems. In Section 3 we recall the classification of the \(C\)-axial subgroups of \(S_N \times S^1\) acting on \(\mathbb{C}^{N,0}\) given by Stewart [17]. We use in Section 4 the Equivariant Hopf Theorem to prove the existence of branches of periodic solutions with these symmetries of (2) by Hopf bifurcation from the trivial equilibrium at \(\lambda = 0\) for a bifurcation problem with symmetry \(\Gamma = S_N\). The main result of this paper is Theorem 4.1. In this theorem we determine the directions of branching and the stability of periodic solutions guaranteed by the Equivariant Hopf Theorem. For solutions with symmetry \(\Sigma_{I}^{Iq,p}\) the terms of the degree three truncation of the vector field determines the criticality of the branches and also the stability of these solutions (near the origin). However, for solutions with symmetry \(\Sigma_{II}^{Iq}\), although the criticality of the branches is determined by the terms of degree three, the stability of solutions in some directions is not. Moreover, in one particular direction, even the degree five truncation is too degenerate (it originates a null eigenvalue which is not forced by the symmetry of the problem). In Section 5 we present the example \(N = 5\). This is the first case where the fifth degree truncation of the vector field is necessary in order to determine the branching equations and the stability of the periodic solutions guaranteed by the Equivariant Hopf Theorem. We determine explicitly the directions of branching and the stability of these solutions.

2. Background

In this section we review some key points related to Hopf bifurcation theory of symmetric systems. For the basics of equivariant bifurcation theory see, for example, Golubitsky et al. [11, Chapter XVI].

Consider a system of ODEs

\[
\frac{dx}{dt} = f(x, \lambda), \quad f(0, 0) = 0,
\]

where \(x \in V, \lambda \in \mathbb{R}\) is the bifurcation parameter, \(f : V \times \mathbb{R} \to V\) is a smooth (\(C^\infty\)) mapping and \(f(0, \lambda) \equiv 0\) for all \(\lambda \in \mathbb{R}\). We say that (4) undergoes a Hopf Bifurcation at \(\lambda = 0\) if \((df)_{0,0}\) has a pair of purely imaginary eigenvalues. Here, \((df)_{0,0}\) denotes the \(n \times n\) Jacobian matrix of the derivatives of \(f\) with respect to the variables \(x_j\), evaluated at \((x, \lambda) = (0, 0)\).

Suppose that \(\Gamma\) is a compact Lie group with a linear action on \(V = \mathbb{R}^N\) and \(f\) commutes with \(\Gamma\) (or it is \(\Gamma\)-equivariant), this is, \(f(\gamma x, \lambda) = \gamma f(x, \lambda)\) for all \(\gamma \in \Gamma, x \in V, \lambda \in \mathbb{R}\).

We are interested in branches of periodic solutions of (4) occurring by Hopf bifurcation from the trivial solution \((x, \lambda) = (0, 0)\). We say that a representation \(V\) of \(\Gamma\) is \(\Gamma\)-simple if either \(V \cong W \oplus W\), where \(W\) is absolutely irreducible for \(\Gamma\), or \(V\) is irreducible, but not absolutely irreducible for \(\Gamma\). Suppose that \((df)_{0,0}\) has purely imaginary eigenvalues \(\pm i\omega\). Then, generically, the corresponding real generalized eigenspace of \((df)_{0,0}\) is \(\Gamma\)-simple (see [11, Proposition XVI 1.4]). Assuming these conditions and supposing that \(\mathbb{R}^n\) is \(\Gamma\)-simple, after an equivariant change of coordinates and a rescaling of time if necessary, we can assume that \((df)_{0,0}\) has the
form

\[(df)_{0,0} = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} = J\]

where $I_m$ is the $m \times m$ identity matrix and $m = n/2$. This is due to the fact that if we assume that $\mathbb{R}^n$ is $\Gamma$-simple, the mapping $f$ is $\Gamma$-equivariant and $(df)_{0,0}$ has $i$ as an eigenvalue, then the eigenvalues of $(df)_{0,\lambda}$ consist of a complex conjugate pair $\sigma(\lambda) \pm i\rho(\lambda)$, each with multiplicity $m$. Moreover, $\sigma$ and $\rho$ are smooth functions of $\lambda$ and there is an invertible linear map $S : \mathbb{R}^n \to \mathbb{R}^n$, commuting with $\Gamma$, such that $(df)_{0,0} = SJS^{-1}$ (see [11, Lemma XVI 1.5]).

We define the \textit{isotropy subgroup} of $x \in V$ in $\Gamma$ as

$$\Sigma_x = \{ \gamma \in \Gamma : \gamma x = x \} \subseteq \Gamma$$

and the \textit{fixed-point space} of a subgroup $\Sigma \subseteq \Gamma$ is the subspace of $V$ defined by

$$\text{Fix}(\Sigma) = \{ x \in V : \gamma x = x, \ \forall \ \gamma \ \in \ \Sigma \}.$$ 

For any $\Gamma$-equivariant mapping $f$ and any subgroup $\Sigma \subseteq \Gamma$ we have

$$f(\text{Fix}(\Sigma) \times \mathbb{R}) \subseteq \text{Fix}(\Sigma).$$

Identify the circle $S^1$ with $\mathbb{R}/2\pi\mathbb{Z}$ and suppose that $x(t)$ is a periodic solution of (4) in $t$ of period $2\pi$. A symmetry of $x(t)$ is an element $(\gamma, \theta) \in \Gamma \times S^1$ such that

$$\gamma x(t) = x(t - \theta).$$

The set of all symmetries of $x(t)$ forms a subgroup

$$\Sigma_{x(t)} = \{ (\gamma, \theta) \in \Gamma \times S^1 : \gamma x(t) = x(t - \theta) \}.$$ 

Take the natural action of $\Gamma \times S^1$ on the space $C_{2\pi}$ of $2\pi$-periodic functions from $\mathbb{R}$ into $V$, given by

$$(\gamma, \theta) \cdot x = \gamma \cdot x(t + \theta).$$

Thus, the action of $\Gamma$ on $C_{2\pi}$ is induced through its spatial action on $v$ and $S^1$ acts by phase shift.

This way, the initial definition of symmetry of the periodic solution $x(t)$ may be rewritten as

$$(\gamma, \theta)x(t) = x(t)$$

and with respect to this action, $\Sigma_{x(t)}$ is the isotropy subgroup of $x(t)$.

So if we assume (4) where $f$ commutes with $\Gamma$ and $(df)_{0,0} = L$ has purely imaginary eigenvalues, we can apply a Liapunov-Schmidt reduction preserving symmetries that will induce a different action of $S^1$ on a finite-dimensional space, which can be identified with the exponential of $L|_{E_i}$ acting on the imaginary eigenspace $E_i$ of $L$. The reduced function of $f$ will commute with $\Gamma \times S^1$ (see [11, Chapter XVI Section 3]).

Consider the system of ODEs (4), where $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is smooth and commutes with a compact Lie group $\Gamma$. Assume the generic hypothesis that $\mathbb{R}^n$ is $\Gamma$-simple and $(df)_{0,0}$ has $i$ as eigenvalue. Thus, after a change of coordinates, we can assume that $(df)_{0,0} = J$, where $m = n/2$. The eigenvalues of $(df)_{0,\lambda}$ are $\sigma(\lambda) \pm i\rho(\lambda)$ each with multiplicity $m$. Therefore $\sigma(0) = 0$ and $\rho(0) = 1$. Furthermore, assume that $\sigma'(0) \neq 0$, that is, the eigenvalues of $(df)_{0,\lambda}$ cross the imaginary axis with nonzero speed. Let $\Sigma \subseteq \Gamma \times S^1$ be an isotropy subgroup such that $\dim \text{Fix}(\Sigma) = 2$. Then
by the Equivariant Hopf Theorem (see [11, Theorem XVI 4.1]) there exists a unique branch of small-amplitude periodic solutions to (4) with period near $2\pi$, having $\Sigma$ as their group of symmetries.

The basic idea in the Equivariant Hopf Theorem is that small amplitude periodic solutions of (4) of period near $2\pi$ correspond to zeros of a reduced equation $\varphi(x, \lambda, \tau) = 0$ where $\tau$ is the period-perturbing parameter. To find periodic solutions of (4) with symmetries $\Sigma$ is equivalent to find zeros of the reduced equation with isotropy $\Sigma$ and they correspond to the zeros of the reduced equation restricted to $\text{Fix}(\Sigma)$.

The main tool for calculating the stabilities of the periodic solutions (including those guaranteed by the Equivariant Hopf Theorem) is to use a Birkhoff normal form of $f$: by a suitable coordinate change, up to any given order $k$, the vector field $f$ can be made to commute with $\Gamma$ and $S^1$ (in the Hopf case). This result is the equivariant version of the Poincaré-Birkhoff normal form Theorem.

Suppose that the vector field $f$ in (4) is in Birkhoff normal form. Then it is possible to perform a Liapunov-Schmidt reduction on (4) such that the reduced equation $\varphi$ has the explicit form

$$\varphi(x, \lambda, \tau) = f(x, \lambda) - (1 + \tau)Jx,$$

where $\tau$ is the period-scaling parameter (see [11, Theorem XVI 10.1]). Let $(x_0, \lambda_0, \tau_0)$ be a solution to $\varphi = 0$ with isotropy $\Sigma$, and let $x(t)$ be the corresponding solution of (4). Then $x(t)$ is orbitally stable if the $n - d_\Sigma$ (where $d_\Sigma = \dim \Gamma + 1 - \dim \Sigma$) eigenvalues of $(d\varphi)_{x_0, \lambda_0, \tau_0}$ which are not forced to zero by the group action have negative real parts (see [11, Corollary XVI 10.2]).

Thus, the assumptions of Birkhoff normal form implies that we can apply the standard Hopf Theorem to $\dot{x} = f(x, \lambda)$ restricted to $\text{Fix}(\Sigma) \times \mathbb{R}$. In this case, exchange of stability happens, so that if the trivial steady-state solution is stable subcritically, then a subcritical branch of periodic solutions with isotropy subgroup $\Sigma$ is unstable. Supercritical branches may be stable or unstable depending on the signs of the real part of the eigenvalues on the complement of $\text{Fix}(\Sigma)$.

Call the system

$$\dot{y} = Ly + g_2(y) + \cdots + g_k(y)$$

the ($k$th order) truncated Birkhoff normal form.

The dynamics of the truncated Birkhoff normal form are related to, but not identical with, the local dynamics of the system (4) around the equilibrium $x = 0$.

On the other hand, in general, it is not possible to find a single change of coordinates that puts $f$ into normal form for all orders. And if it is, then there is the problem of the first ‘tail’.

When discussing the stability of the solutions found using the Equivariant Hopf Theorem we suppose that the $k$th order truncation of $f$ commutes also with $S^1$. Thus we are ignoring terms of higher order that do not commute necessarily with $S^1$ and that can change the dynamics (and so the stability of these periodic solutions that exist even for the nontruncated system by the Equivariant Hopf Theorem).

However, in some cases, the stability results for the periodic solutions can hold even when $f$ is of the form

$$\tilde{f}(x, \lambda) + o(\|x\|^k),$$
where \( \tilde{f} \) commutes with \( \Gamma \times S^1 \) but \( o(||x||^k) \) commutes only with \( \Gamma \), provided \( k \) is large enough (see [11, Theorem XVI 11.2]). We use \( h(x) = o(||x||^k) \) to mean that
\[
h(x)/||x||^k \to 0 \text{ as } ||x|| \to 0.
\]

Suppose that \( \dim \text{Fix}(\Sigma) = 2 \). Following [11, Definition XVI 11.1] \( \Sigma \) has \( p \)-determined stability if all eigenvalues of \( (d\tilde{f})(x_0,\lambda_0) - (1 + \tau_0)J \), other than those forced to zero by \( \Sigma \), have the form
\[
\mu_j = \alpha_j a^{m_j} + o(a^{m_j})
\]
on a periodic solution \( x(s) \) of
\[
\dot{x} = \tilde{f}(x,\lambda)
\]
such that \( ||x(s)|| = a \), where \( \alpha_j \) is a \( \mathbb{C} \)-valued function of the Taylor coefficients of terms of degree lower or equal \( p \) in \( \tilde{f} \). We expect that the real parts of the \( \alpha_j \) to be generically nonzero: these are the nondegeneracy conditions on the Taylor coefficients of \( \tilde{f} \) at the origin that are obtained when computing stabilities along the branches. In this case, we say that \( \tilde{f} \) is nondegenerate for \( \Sigma \).

Suppose that the hypotheses of the Equivariant Hopf Theorem hold, and the isotropy subgroup \( \Sigma \subseteq \Gamma \times S^1 \) has \( p \)-determined stability. Let \( k \geq p \) and assume that \( f(x,\lambda) = \tilde{f}(x,\lambda) + o(||x||^k) \) where \( \tilde{f} \) commutes with \( \Gamma \times S^1 \) and is nondegenerate for \( \Sigma \). Then for \( \lambda \) sufficiently near 0, the stabilities of a periodic solution of \( \dot{x} = f(x,\lambda) \) with isotropy \( \Sigma \) are given by the same expressions in the coefficients of \( f \) as those that define the stability of a solution of the truncated Birkhoff normal form \( \dot{x} = \tilde{f}(x,\lambda) \) with isotropy subgroup \( \Sigma \) (see [11, Theorem XVI 11.2]). As it has been said there always exists a polynomial change putting \( f \) in the form \( \tilde{f}(x,\lambda) + o(||x||^k) \). Thus, if the \( p \)-determined stability condition holds, the stability analysis for \( f \) is completed.

### 3. C-Axial Subgroups of \( S_N \times S^1 \)

In order to apply the Equivariant Hopf Theorem we require information on the \( \mathbb{C} \)-axial isotropy subgroups (this is, on the isotropy subgroups with two-dimensional fixed-point subspace) of \( S_N \times S^1 \). Such subgroups are of the type \( H^\theta = \{(h, \theta(h)) : h \in H \} \) where \( H \subseteq S_N \) and \( \theta : H \to S^1 \) is a group homomorphism (see [11, Definition XVI 7.1, Proposition XVI 7.2]). Also they are maximal with respect to fixing a complex line \( \mathbb{C}z = \{\mu z : \mu \in \mathbb{C}\} \), where \( \mu \neq 0 \). A vector \( z \) such that the isotropy subgroup \( \Sigma_z \) in \( S_N \times S^1 \) fixes only \( \mathbb{C}z \) is called an axis.

**Theorem 3.1 (Stewart [17]).** Suppose that \( N \geq 2 \). Then the axes of \( S_N \times S^1 \) acting on \( \mathbb{C}^{N,0} \) have orbit representatives as follows:

**Type I**

Let \( N = qk + p \) where \( 2 \leq k \leq N \), \( q \geq 1 \), \( p \geq 0 \). Let \( \xi = e^{2\pi i/k} \) and set
\[
z = \left( \frac{1}{q}, \ldots, \frac{1}{q}; \frac{\xi}{q}, \ldots, \frac{\xi}{q}; \frac{\xi^2}{q}, \ldots, \frac{\xi^2}{q}; \ldots; \frac{\xi^{k-1}}{q}, \ldots, \frac{\xi^{k-1}}{q}; 0, \ldots, 0 \right)
\]
Type II
Let \( N = q + p, 1 \leq q < N/2 \) and set

\[
(7) \quad z = \left( \overbrace{1, \ldots, 1}^{q}, \overbrace{a, \ldots, a}^{p} \right)
\]

where \( a = -q/p \).

Proof. See Stewart [17, Theorem 7]. \( \square \)

Next we consider the corresponding isotropy subgroups as in [17]. For type I we have \( C \)-axial subgroups \( H_\theta = \Sigma_z \) where

\[
(8) \quad \Sigma_z = S_q \wr Z_k \times S_p \overset{\text{def}}{=} \Sigma_{q,p}.
\]

Here \( \wr \) denotes the wreath product (see Hall [13, p. 81]) and the tilde indicates that \( Z_k \) is twisted into \( S^1 \). Let

\[
(9) \quad K = \ker(\theta) = \mathbb{S}_1 \times \cdots \times \mathbb{S}_k^k \times \mathbb{S}_p,
\]

where \( \mathbb{S}_q^j \) is the symmetric group on \( B_j = \{(j-1)q+1, \ldots, jq\} \) and \( \mathbb{S}_p \) is the symmetric group on \( B_0 = \{kq + 1, \ldots, n\} \). Observe that if \( \mathbb{S}_r \) acts by permuting \( \{1, \ldots, r\} \) then it is generated by \((1 \ 2), (1 \ 3), \ldots, (1 \ r)\).

Let \( \alpha = (1 \ q + 1 \ 2q + 1 \ \ldots \ (k-1)q + 1) \) and \( \xi = 2\pi/k \). Then \( \Sigma_{q,p} \) is generated by \((\alpha, \xi)\) and \( K \).

For the type II, the isotropy subgroup is

\[
(10) \quad \Sigma_z = S_q \times S_p \overset{\text{def}}{=} \Sigma_{q}^{II}
\]

where the respective factors are the symmetric groups on \( \{1, \ldots, q\} \) and \( \{q+1, \ldots, N\} \). Thus we have the generators \((1 \ 2), \ldots, (1 \ q), (q + 1 \ q + 2), \ldots, (q + 1 \ N)\).

4. Periodic Solutions with Maximal Isotropy

Consider the system of ODEs

\[
(11) \quad \frac{dz}{dt} = f(z, \lambda),
\]

where \( f : \mathbb{C}^{N,0} \times \mathbb{R} \to \mathbb{C}^{N,0} \) is smooth, commutes with \( \Gamma = \mathbb{S}_N \) and \((df)_{0,\lambda} \) has eigenvalues \( \sigma(\lambda) \pm i\rho(\lambda) \) with \( \sigma(0) = 0, \rho(0) = 1 \) and \( \sigma'(0) \neq 0 \).

In order to determine the direction of branching and the stability of the bifurcating branches of periodic solutions of (11), we use the results stated at the end of Section 3 and so we must compute the general form of a \( \mathbb{S}_N \times \mathbb{S}^1 \)-equivariant bifurcation problem, up to degree 5. As we mentioned before, the coefficients of the degree 5 terms of \( f \) in (11) are necessary to describe the stability of some of the periodic solutions guaranteed by the Equivariant Hopf Theorem. See Dias et al. [2] and the details in Rodrigues [15, Chapter 4, Section 3] for the computation of the fifth order truncation of \( f \) in (11). The idea is to consider the action of \( \mathbb{S}_N \times \mathbb{S}^1 \) extended to \( \mathbb{C}^N \) given by (1) and (3) and obtain the cubic and the fifth order truncation of a general smooth vector field equivariant under \( \mathbb{S}_N \times \mathbb{S}^1 \) and defined on \( \mathbb{C}^N \). The restriction and projection onto \( \mathbb{C}^{N,0} \) of this truncation describes the lower or equal degree 5 terms of
the vector field in (11) defined on \( \mathbb{C}^{N,0} \). If we suppose that the Taylor series of degree five of \( f \) around \( z = 0 \) commutes also with \( \mathbb{S}^1 \), then we can write \( f = (f_1, f_2, \ldots, f_N) \), where

\[
f_1(z_1, \ldots, z_N, \lambda) = \mu(\lambda)z_1 + f_1^{(3)}(z_1, \ldots, z_N, \lambda) + f_1^{(5)}(z_1, \ldots, z_N, \lambda) + \cdots
\]

(12) \[
f_2(z_1, \ldots, z_N, \lambda) = f_1(z_2, z_1, \ldots, z_N, \lambda)
\]

\[
\vdots
\]

\[
f_N(z_1, \ldots, z_N, \lambda) = f_1(z_N, z_2, \ldots, z_1, \lambda)
\]

and

\[
f_1^{(3)}(z_1, \ldots, z_N, \lambda) = A_1 \left[ |z_1|^2 z_1 - \frac{1}{N} \sum_{k=1}^{N} |z_k|^4 z_k \right] + A_2 \sum_{k=1}^{N} z_k^2 + A_3 \sum_{k=1}^{N} |z_k|^2
\]

\[
f_1^{(5)}(z_1, \ldots, z_N, \lambda) = A_4 \left[ |z_1|^4 z_1 - \frac{1}{N} \sum_{k=1}^{N} |z_k|^4 z_k \right] + A_5 \sum_{i=1}^{N} |z_i|^4 + A_6 \sum_{i=1}^{N} z_i^2 \sum_{j=1}^{N} \bar{z}_j^2 + A_7 \sum_{i=1}^{N} |z_i|^2 \sum_{j=1}^{N} \bar{z}_j^2 + A_8 \left[ |z_1|^2 \sum_{j=1}^{N} \bar{z}_j^2 - \frac{1}{N} \sum_{k=1}^{N} z_k^2 \sum_{j=1}^{N} \bar{z}_j^2 \right] + A_9 \left[ |z_1|^2 \sum_{j=1}^{N} \bar{z}_j^2 - \frac{1}{N} \sum_{k=1}^{N} z_k^2 \sum_{j=1}^{N} \bar{z}_j^2 \right] + A_{10} \left[ |z_1|^2 \sum_{i=1}^{N} |z_i|^2 \sum_{j=1}^{N} \bar{z}_j^2 \right] + A_{11} \sum_{i=1}^{N} |z_i|^4 + A_{12} \sum_{j=1}^{N} \bar{z}_j^2 - \frac{1}{N} \sum_{k=1}^{N} \bar{z}_j^2 \sum_{j=1}^{N} \bar{z}_j^2 + A_{13} \left[ |z_1|^2 \sum_{k=1}^{N} |z_k|^2 z_k - \frac{1}{N} \sum_{i=1}^{N} |z_i|^2 \sum_{j=1}^{N} |z_j|^2 z_j \right] + A_{14} \left[ |z_1|^2 \sum_{k=1}^{N} |z_k|^2 z_k - \frac{1}{N} \sum_{i=1}^{N} |z_i|^2 \sum_{j=1}^{N} |z_j|^2 \right] + A_{15} \left[ |z_1|^2 \sum_{k=1}^{N} z_k^2 - \frac{1}{N} \sum_{i=1}^{N} |z_i|^2 \sum_{j=1}^{N} z_j^2 \right]
\]

with \( z_N = -z_1 - \cdots - z_{N-1} \). The coefficients \( A_i \), for \( i = 1, \ldots, 15 \) are complex smooth functions of \( \lambda \), \( \mu(0) = i \) and \( \text{Re}(\mu'(0)) \neq 0 \). Suppose that \( \text{Re}(\mu'(0)) > 0 \). Rescaling \( \lambda \) if necessary we can suppose that

\[
\text{Re}(\mu(\lambda)) = \lambda + \cdots
\]

where \( + \cdots \) stands for higher order terms in \( \lambda \). Thus the trivial solution of (11) is stable for \( \lambda \) negative and unstable for \( \lambda \) positive (near zero).

Throughout, subscripts \( r \) and \( i \) on the coefficients \( A_1, \ldots, A_{15} \) refer to real and imaginary parts.

**Theorem 4.1.** Consider the system (11) where \( f \) is as in (12). Assume that \( \text{Re}(\mu'(0)) > 0 \), such that the trivial equilibrium is stable if \( \lambda < 0 \) and it is unstable if \( \lambda > 0 \) (near the origin). For each type of the isotropy subgroups of the form \( \Sigma^I_{q,p} \) and \( \Sigma_q^I \) listed in Table 1, let \( \Delta_0, \ldots, \Delta_r \) be the functions of \( A_1, \ldots, A_{15} \) listed in Tables 3, 4 and 5 evaluated at \( \lambda = 0 \). Then:

1. For each \( \Sigma_i \) the corresponding branch of periodic solutions is supercritical if \( \Delta_0 < 0 \) and subcritical if \( \Delta_0 > 0 \). Table 2 lists the branching equations.
HOPF BIFURCATION WITH $S_N$-SYMMETRY

Isotropy Subgroup Fixed-Point Subspace

$\Sigma^I_{q,p} = \tilde{S}_q \wr Z_k \times S_p$

$$\left\{ \left( \frac{z_1, \ldots, z_{q-1}, z_q; \xi^{k-1}z_1, \ldots, 0, \ldots, 0}{q} \right) : z_1 \in \mathbb{C} \right\}$$

$N = kq + p$, $2 \leq k \leq N$, $q \geq 1, p \geq 0$

$\Sigma^{II}_{q} = S_q \times S_p$

$$\left\{ \left( \frac{z_1, \ldots, z_q; -\frac{q}{p} z_1, \ldots, -\frac{q}{p} z_q}{q} \right) : z_1 \in \mathbb{C} \right\}$$

$N = q + p$, $1 \leq q < \frac{N}{2}$

Table 1. $\mathbb{C}$-axial isotropy subgroups of $S_N \times S^1$ acting on $\mathbb{C}^{N,0}$ and fixed-point subspaces. Here $\xi = e^{2\pi i/k}$.

<table>
<thead>
<tr>
<th>Isotropy Subgroup</th>
<th>Branching Equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma^I_{q,p}$, $2 &lt; k \leq N$</td>
<td>$\lambda = - (A_{1r} + kq A_{3r})</td>
</tr>
<tr>
<td>$N = kq + p$, $q \geq 1, p \geq 0$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma^I_{q,p}$, $k = 2$</td>
<td>$\lambda = - [A_{1r} + 2q(2r + 3r)]</td>
</tr>
<tr>
<td>$N = 2q + p$, $q \geq 1, p \geq 0$</td>
<td></td>
</tr>
<tr>
<td>$\Sigma^{II}_{q}$</td>
<td>$\lambda = - A_{1r} \left[ 1 - \frac{q}{N} \left( 1 - \frac{q^2}{p^2} \right) \right]</td>
</tr>
<tr>
<td>$N = q + p$, $1 \leq q &lt; \frac{N}{2}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2. Branching equations for $S_N \times S^1$ Hopf bifurcation. Subscript $r$ on the coefficients refer to the real part and $+\cdots$ stands for higher order terms.

(2) For each $\Sigma_i$, if $\Delta_j > 0$ for some $j = 0, \ldots, r$, then the corresponding branch of periodic solutions is unstable. If $\Delta_j < 0$ for all $j$, then the branch of periodic solutions is stable near $\lambda = 0$ and $z = 0$. 
Isotropy Subgroup \[ \Delta_0 \]

\[ \begin{align*}
\Sigma^I_{q,p}, & \quad 2 < k \leq N \\
N = kq + p, & \quad q \geq 1, p \geq 0 \\
\Sigma^I_{q,N-2q}, & \quad k = 2 \\
N = 2q + p, & \quad q \geq 1, p \geq 0 \\
\Sigma^{II}_q, & \quad N = q + p, \quad 1 \leq q < \frac{N}{2}
\end{align*} \]

Table 3. Stability for \( S_N \) Hopf bifurcation in the direction of \( W_0 = \text{Fix}(\Sigma) \).

**Proof.** Our aim is to study periodic solutions of (11) obtained by Hopf bifurcation from the trivial equilibrium. Note that we are assuming that \( f \) satisfies the conditions of the Equivariant Hopf Theorem.

From Theorem 3.1 we have (up to conjugacy) the \( C \)-axial subgroups of \( S_N \times S^1 \). See Table 1. Therefore, we can use the Equivariant Hopf Theorem to prove the existence of periodic solutions with these symmetries for a bifurcation problem with symmetry \( \Gamma = S_N \).

As stated in Section 2, periodic solutions of (11) of period \( 2\pi/(1 + \tau) \) are in one-to-one correspondence with the zeros of \( g(z, \lambda, \tau) \), the reduced function obtained by the Lyapunov-Schmidt procedure where \( \tau \) is the period-perturbing parameter. Assuming that \( f \) commutes with \( \Gamma \times S^1 \), \( g(z, \lambda, \tau) \) has the explicit form

\[
(13) \quad g(z, \lambda, \tau) = f(z, \lambda) - (1 + \tau)i z.
\]

(see [11, Theorem XVI 10.1]). Throughout denote by \( \nu(\lambda) = \mu(\lambda) - (1 + \tau)i \). If \( z(t) \) is a periodic solution of (11) with \( \lambda = \lambda_0 \) and \( \tau = \tau_0 \), and \( (z_0, \lambda_0, \tau_0) \) is the corresponding solution of (13), then there is a correspondence between the Floquet multipliers of \( z(t) \) and the eigenvalues of \( (dg)_{(z_0, \lambda_0, \tau_0)} \) such that a multiplier lies inside (respectively outside) the unit circle if and only if the corresponding eigenvalue has negative (respectively positive) real part (see [11, Corollary XVI 10.2]). So, we determine the stability of each type of bifurcating periodic orbit by calculating the eigenvalues of \( (dg)_{(z_0, \lambda_0, \tau_0)} \) (to the lowest order in \( z \)).

Recall Table 1. As \( g \) commutes with \( \Gamma \times S^1 \), it maps \( \text{Fix}(\Sigma) \) into itself (where \( \Sigma \) is either of type \( \Sigma^I_{q,p} \) or \( \Sigma^{II}_q \)). By the Equivariant Hopf Theorem, for each of the conjugacy classes \( \Sigma^I_{q,p} \) and \( \Sigma^{II}_q \), we have a distinct branch of periodic solutions of (11) that are in correspondence with the zeros of \( g \) with isotropy \( \Sigma^I_{q,p} \) and \( \Sigma^{II}_q \). These zeros are found by solving \( g|_{\text{Fix}(\Sigma_{q,p})} = 0 \) and \( g|_{\text{Fix}(\Sigma^{II}_q)} = 0 \) (and \( \text{Fix}(\Sigma^I_{q,p}), \text{Fix}(\Sigma^{II}_q) \) are two-dimensional). Note that to find the zeros of \( g \), it suffices to look at representative points on \( \Gamma \times S^1 \) orbits. See Table 2.
\[
\Delta_1, \ldots, \Delta_r
\]

<table>
<thead>
<tr>
<th>Isotropy Subgroup</th>
<th>(\Delta_1, \ldots, \Delta_r)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\Sigma^I_{p,q}, k = 2)</td>
<td>((1 - \frac{2q}{N}) A_{1r} - 2q A_{2r}, \text{ if } p \geq 1)</td>
</tr>
<tr>
<td>(N = 2q + p)</td>
<td>(-</td>
</tr>
<tr>
<td>(q \geq 1, p \geq 0)</td>
<td>(-A_{1r} - 2q A_{2r}, \text{ if } p &gt; 1)</td>
</tr>
<tr>
<td>(- (</td>
<td>A_1 + 2q A_2</td>
</tr>
</tbody>
</table>

\[
\Sigma^I_{p,q}, \quad k = 3
\]

| \(N = 3q + p, q \geq 1, p \geq 0\) | \((1 - \frac{6q}{N}) A_{1r}, \text{ if } p \geq 1\) |
| \(- |\left(1 - \frac{6q}{N}\right) A_1|^2 + |A_2|^2, \text{ if } p \geq 1\) | \(-|A_1|^2, \text{ if } p > 1\) |
| \(-A_{1r}, \text{ if } p > 1\) | \(-(-3q + \frac{6q}{N}) \text{ Re}(A_1 A_{12}), \text{ if } q \geq 2\) |
| \(- \left(0 - 2q A_{2r}\right) A_{1r} + 6A_{2r}\) | \(- (|A_1 + 6A_2|^2 - (1 - \frac{3}{N}) A_1|^2)\) |

\[
\Sigma^I_{q}^{II}
\]

| \(N = q + p, 1 \leq q \leq \frac{N}{2}\) | \((1 + \frac{q}{N} - \frac{q^3}{N^2 p^2}) A_{1r} - q\left(1 + \frac{q}{p}\right) A_{2r}\) |
| \(- \left|\left(1 + \frac{q}{N} - \frac{q^3}{N^2 p^2}\right) A_1 - q\left(1 + \frac{q}{p}\right) A_2\right|^2 - |A_1 + q\left(1 + \frac{q}{p}\right) A_2|^2\) | \(-(-1 + \frac{q}{N} - \frac{q^3}{N^2 p^2} + \frac{2q^2}{p^2}) A_{1r} - q\left(1 + \frac{1}{p}\right) A_{2r}\) |
| \(-\left|\left(-1 + \frac{q}{N} - \frac{q^3}{N^2 p^2} + \frac{2q^2}{p^2}\right) A_1 - q\left(1 + \frac{1}{p}\right) A_2\right|^2 - |\frac{q^2}{p^2} A_1 + q\left(1 + \frac{q}{p}\right) A_2|^2\) | \(-\left|\left(1 + \frac{1}{p}\right) A_1 - q\left(1 + \frac{1}{p}\right) A_2\right|^2\) |

**Table 4.** Stability for \(S_N\) Hopf bifurcation.

Let \(\Sigma_{z_0} \subset \Gamma\) be the isotropy subgroup of \(z_0\). Then, for \(\sigma \in \Sigma_{z_0}\) we have

\[(dg)_{z_0}\sigma = \sigma (dg)_{z_0}\]

That is, \((dg)_{z_0}\) commutes with the isotropy subgroup \(\Sigma\) of \(z_0\).

For the two types of isotropy subgroups \(\Sigma_{q,p}^I\) and \(\Sigma_{q}^{II}\), it is possible to put the Jacobian matrix \((dg)_{z_0}\) into block diagonal form. We do this by decomposing \(C^{N,0}\) into subspaces, each of which is invariant under a different representation of the corresponding isotropy subgroup. The isotypic components for the action of \(\Sigma_{q,p}^I\) and \(\Sigma_{q}^{II}\) on \(C^{N,0}\) are listed in Table 6.

Specifically, for \(\Sigma_{q,p}^I = \widehat{S_q} \times \mathbb{Z}_k \times S_p\) we form the isotypic decomposition

\[C^{N,0} = W_0 \oplus W_1 \oplus W_2 \oplus W_3 \oplus \sum_{j=2}^{k-1} P_j\]
where $W_0 = \text{Fix}(\Sigma_{q,p}^I), W_1$ and the $k - 2$ subspaces $P_j, j = 2, \cdots, k - 1$ are complex one-dimensional subspaces, invariant under $\Sigma_{q,p}^I$. Moreover, $W_2$ and $W_3$ are complex invariant subspaces of dimension respectively $p - 1$ and $k(q - 1)$.

Note that if $p = 0$ we have $\Sigma_{q,p}^I = \mathbf{S}_q / \mathbf{Z}_k$ and then $W_1$ does not occur in the isotypic decomposition of $\mathbb{C}^{N,0}$ for the action of $\Sigma_{q,p}^I$. Moreover, we only have the occurrence of $W_2$ in the isotypic decomposition if $p \geq 2$. Furthermore, we only have the isotypic component $W_3$ if $q \geq 2$ and $P_j$ if $k > 2$.

For $\Sigma_{q}^{II} = \mathbf{S}_q \times \mathbf{S}_p$ we form the isotypic decomposition

\[
\mathbb{C}^{N,0} = W_0 \oplus W_1 \oplus W_2
\]

where $W_0 = \text{Fix}(\Sigma_{q}^{II})$ and $W_1, W_2$ are complex invariant subspaces of dimension respectively $q - 1$ and $p - 1$ that are the sum of two isomorphic real absolutely irreducible representations of dimension respectively $q - 1$ and $p - 1$ of $\Sigma_{q}^{II}$.

Note that as the group action forces some of the Floquet multipliers to be equal to one, it also forces the corresponding eigenvalues of $(dg)_{(z_0, \lambda_0, \tau_0)}$ to be equal to zero. (Recall [11, Theorem XVI 6.2].) The eigenvectors associated with these eigenvalues are the tangent vectors to the orbit of $\mathbf{S}_N \times \mathbf{S}^1$ through $z_0$. If the solution $z_0$ has symmetry $\Sigma_{z_0}$, then the group orbit has the dimension of $(\Gamma \times \mathbf{S}^1) / \Sigma_{z_0}$ and so the number of zero eigenvalues of $(dg)_{(z_0, \lambda_0, \tau_0)}$ forced by the group action is

\[
d_{\Sigma_{z_0}} = 1 - \dim(\Sigma_{z_0})
\]

since $\dim(\mathbf{S}_N \times \mathbf{S}^1) = 1$. The groups $\Sigma_{z_0} = \Sigma_{q,p}^I$ and $\Sigma_{z_0} = \Sigma_{q}^{II}$ are discrete, then there is one eigenvalue forced by the symmetry to be zero (this is, we get $d_{\Sigma_{z_0}} = 1$).

<table>
<thead>
<tr>
<th>Isotropy Subgroup</th>
<th>$\Delta_1, \ldots, \Delta_r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma_{p,q}^I$, $3 &lt; k \leq N$</td>
<td>$(1 - \frac{2kq}{N}) A_{1r}$, if $p \geq 1$</td>
</tr>
<tr>
<td>$N = kq + p$</td>
<td>$-</td>
</tr>
<tr>
<td>$q \geq 1$, $p \geq 0$</td>
<td>$A_{1r}$, if $p &gt; 1$</td>
</tr>
<tr>
<td></td>
<td>$-</td>
</tr>
<tr>
<td></td>
<td>If $k &gt; 3, q \geq 2$ then the fifth degree truncation is too degenerate to determine the stability in the directions in $W_3$</td>
</tr>
<tr>
<td></td>
<td>$A_{1r}$</td>
</tr>
<tr>
<td></td>
<td>$- (</td>
</tr>
<tr>
<td></td>
<td>$A_{1r} + 2kqA_{2r}$, if $k \geq 4$</td>
</tr>
<tr>
<td></td>
<td>$- (</td>
</tr>
<tr>
<td></td>
<td>$- \text{Re}(A_1 \overline{\xi_1}) + \text{Re}(2A_1 \overline{\xi_4} + kqA_1 \overline{\xi_4})$, if $k \geq 5$</td>
</tr>
<tr>
<td></td>
<td>$- \text{Re}(A_1 \overline{\xi_2}) + \text{Re}(2A_1 \overline{\xi_4} + kqA_1 \overline{\xi_4})$, if $k \geq 6$</td>
</tr>
</tbody>
</table>

Table 5. Stability for $\mathbf{S}_N$ Hopf bifurcation. Here $\xi_1 = 2A_4 + 3kqA_{12} + q(kq - 1) (2 - \frac{2kq}{N}) A_{13} + kqA_{14} + q(kq - 1) (1 - \frac{2kq}{N}) A_{14} + 2q(kq - 1) A_{15}$ and $\xi_2 = \xi_1 - 3kqA_{12} - kqA_{14}$. 

\[
\text{Table 5. Stability for } \mathbf{S}_N \text{ Hopf bifurcation. Here } \xi_1 = 2A_4 + 3kqA_{12} + q(kq - 1) (2 - \frac{2kq}{N}) A_{13} + kqA_{14} + q(kq - 1) (1 - \frac{2kq}{N}) A_{14} + 2q(kq - 1) A_{15} \\
\text{and } \xi_2 = \xi_1 - 3kqA_{12} - kqA_{14}.
\]
Type of Isotropy Isotypic components
Subgroup

$\Sigma^I_{q,p}$ $W_0 = \{ (z_1, \ldots, z_1; \xi z_1, \ldots, \xi z_1; \xi^{k-1} z_1, \ldots, \xi^{k-1} z_1; 0, \ldots, 0) : z_1 \in \mathbb{C} \}$

$N = kq + p,$

$2 \leq k \leq N$ $W_1 = \{ (z_1, \ldots, z_1; -\frac{pq}{kq} z_1, \ldots, -\frac{pq}{kq} z_1) : z_1 \in \mathbb{C} \}$ if $p \geq 1$

$q \geq 1, p \geq 0$ $W_2 = \{ (0, \ldots, 0; z_1, \ldots, z_{p-1}, -z_1 - \cdots - z_{p-1}) : z_1, \ldots, z_{p-1} \in \mathbb{C} \}$ if $p \geq 2$

$W_3 = \{ (z_1, \ldots, z_{q-1}, z_q; \ldots; z_q, \ldots, z_{q(k-1)+1}, \ldots, z_{kq-1}, z_{kq}; 0, \ldots, 0) \}$ if $q \geq 2$

$P_j = \{ (z_1, \ldots, z_j z_1, \ldots, z_j; \ldots; z_1; 0, \ldots, 0) : z_1 \in \mathbb{C} \}$

and $j = 2, \ldots, k - 1$

$\Sigma^I_q$ $W_0 = \text{Fix}(\Sigma^I_q) = \left\{ \left( z_1, \ldots, \frac{-q}{p} z_1, \ldots, \frac{-q}{p} z_1 \right) : z_1 \in \mathbb{C} \right\}$

$N = q + p$ $W_1 = \{ (z_1, \ldots, z_{q-1}, -z_1 - \cdots - z_{q-1}, 0, \ldots, 0) : z_1, \ldots, z_{q-1} \in \mathbb{C} \}$ if $q > 1$

$W_2 = \{ (0, \ldots, 0, z_q+1, \ldots, z_{N-1}, -z_q-1 - \cdots - z_{N-1}) : z_q+1, \ldots, z_{N-1} \in \mathbb{C} \}$ if $p > 1$

**Table 6.** Isotypic components of $\mathbb{C}_{N,0}$ for the action of $\Sigma^I_{q,p}$ and $\Sigma^I_q$.

Here, in $W_3$ we have $z_q = -z_1 - \cdots - z_{q-1}, z_{kq} = -z_{q(k-1)+1} - \cdots - z_{kq-1}$ and $z_1, \ldots, z_{q-1}, z_{q(k-1)+1}, \ldots, z_{kq-1} \in \mathbb{C}$.

To compute the eigenvalues it is convenient to use the complex coordinates. We take co-ordinate functions on $\mathbb{C}^N$: $z_1, \overline{z}_1, z_2, \overline{z}_2, \ldots, z_N, \overline{z}_N$. These correspond to a basis $B$ for $\mathbb{C}^N$ with elements denoted by $b_1, \overline{b}_1, b_2, \overline{b}_2, \ldots, b_N, \overline{b}_N$.

Recall that an $\mathbb{R}$-linear mapping on $\mathbb{C} \equiv \mathbb{R}^2$ has the form

\begin{equation}
\omega \mapsto \alpha \omega + \beta \overline{\omega}
\end{equation}

where $\alpha, \beta \in \mathbb{C}$. The matrix of this mapping in these coordinates,

\begin{equation}
M = \begin{pmatrix}
\alpha & \beta \\
\beta & \alpha
\end{pmatrix}
\end{equation}
has

$$\text{tr}(M) = 2\Re(\alpha), \quad \text{det}(M) = |\alpha|^2 - |\beta|^2.$$  

The eigenvalues of this matrix are

$$\frac{\text{tr}(M)}{2} \pm \sqrt{\left(\frac{\text{tr}(M)}{2}\right)^2 - \text{det}(M)}.$$  

If one eigenvalue is zero, then $\text{det}(M) = 0$ and the sign of the other eigenvalue (if it is not zero) is given by the sign of the real part of $\alpha$. If $M$ has no zero eigenvalues, then the eigenvalues have negative real part if and only if the determinant is positive and the trace is negative.

$$\left(\Sigma^I_{q,p} = \widehat{S}_q \mathcal{Z}_k \times S_p, \text{where } N = qk + p, \ 2 \leq k \leq N, \ q \geq 1, \ p \geq 0\right)$$

The fixed-point subspace of $\Sigma^I_{q,p} = \widehat{S}_q \mathcal{Z}_k \times S_p$ is

$$\text{Fix}(\Sigma^I_{q,p}) = \{(z, \ldots, z; z, \xi z, \ldots, \xi^{k-1} z; \ldots; \xi^k z; 0, \ldots, 0) : z \in \mathbb{C}\}$$

where $\xi = e^{2\pi i/k}$. Using the equation (13) where $f$ is as in (12), after dividing by $z$ we have if $k \neq 2$

$$\nu(\lambda) + (A_1 + kqA_3)|z|^2 + \cdots = 0$$

where $+ \cdots$ denotes terms of higher order in $z$ and $\overline{z}$, and taking the real part of this equation, we obtain,

$$\lambda = -(A_{1r} + kqA_{3r})|z|^2 + \cdots$$

It follows that if $A_1 + kqA_3 < 0$, then the branch bifurcates supercritically.

In the particular case $k = 2$ we have

$$\nu(\lambda) + [A_1 + 2q(A_2 + A_3)]|z|^2 + \cdots = 0$$

and taking the real part of this equation,

$$\lambda = -[A_{1r} + 2q(A_{2r} + A_{3r})]|z|^2 + \cdots$$

where the functions $A_{ir}$ for $i = 1, 2, 3$ are evaluated at $\lambda = 0$. It follows in this case that if $A_{1r} + 2q(A_{2r} + A_{3r}) < 0$, then the branch bifurcates supercritically.

Throughout we denote by $(z_0, \lambda_0, \tau_0)$ a zero of $g(z, \lambda, \tau) = 0$ with $z_0 \in \text{Fix}(\Sigma)$. Specifically, we wish to calculate $(dg)_{(z_0, \lambda_0, \tau_0)}$.

Recall the generators for $\Sigma^I_{q,p}$ given in Section 3. With respect to the basis $B$, any “real” matrix commuting with $\Sigma^I_{q,p} = \widehat{S}_q \mathcal{Z}_k \times S_p$ has the form

$$(dg)_{(z_0, \lambda_0, \tau_0)} = \begin{pmatrix}
M_1 & M_3 & M_4 & \cdots & M_{k+1} & M_{k+2} \\
M_{k+1}^{\xi^2} & M_1^{\xi^2} & M_3^{\xi^2} & \cdots & M_k^{\xi^2} & M_{k+2}^{\xi^2} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
M_3^{2(k-1)} & \cdots & M_k^{2(k-1)} & \cdots & M_{k+3}^{2(k-1)} & M_{k+2}^{2(k-1)} \\
M_{k+3} & M_{k+3}^{\xi^2} & M_{k+3}^{\xi^4} & \cdots & M_{k+3}^{2(k-1)} & M_{k+4}
\end{pmatrix}$$
where \( M_1 \) commutes with \( S_q \), \( M_{k+4} \) commutes with \( S_p \) and the other matrices are defined below.

Suppose \( M \) is a square matrix of order \( a \) with rows \( l_1, \ldots, l_a \) and commuting with \( S_a \). It follows then that \( M = (l_1, (12) \cdot l_1, \ldots, (1a) \cdot l_1)^T \), where if \( l_1 = (m_1, \ldots, m_a) \) then \( (1i) \cdot l_1 = (m_i, m_2, \ldots, m_{i-1}, m_1, m_{i+1}, \ldots, m_a) \). Moreover, \( l_1 \) is invariant under \( S_{a-1} \) in the last \( a-1 \) entries and so it has the following form: \( (m_1, m_2, \ldots, m_2) \).

Applying this to \( M_1 \) and \( M_{k+4} \) we get

\[
M_1 = \begin{pmatrix}
C_1 & C_2 & \cdots & C_2 \\
C_2 & C_1 & \cdots & C_2 \\
\vdots & \ddots & \ddots & \vdots \\
C_2 & C_2 & \cdots & C_1
\end{pmatrix},
M_{k+4} = \begin{pmatrix}
C_{k+4} & C_{k+5} & \cdots & C_{k+5} \\
C_{k+5} & C_{k+4} & \cdots & C_{k+5} \\
\vdots & \ddots & \ddots & \vdots \\
C_{k+5} & C_{k+5} & \cdots & C_{k+4}
\end{pmatrix},
\]

where \( M_1 \) is a \( 2q \times 2q \) matrix and \( M_{k+4} \) is a \( 2p \times 2p \) matrix.

The other symmetry restrictions on the \( M_i \), for \( i = 3, \ldots, k + 3 \), imply that each have one identical entry,

\[
M_i = \begin{pmatrix}
C_i & \cdots & C_i \\
& \ddots & \\
C_i & \cdots & C_i
\end{pmatrix}.
\]

Note that each \( M_i \) for \( i = 1, \ldots, k + 1 \) is a \( 2q \times 2q \) matrix and \( M_{k+2}, M_{k+3} \) are, respectively, \( 2q \times 2p \) and \( 2p \times 2q \) matrices. Furthermore, we have

\[
M_1^{\xi_j} = \begin{pmatrix}
C_1^{\xi_j} & C_2^{\xi_j} & \cdots & C_2^{\xi_j} \\
C_2^{\xi_j} & C_1^{\xi_j} & \cdots & C_2^{\xi_j} \\
\vdots & \ddots & \ddots & \vdots \\
C_2^{\xi_j} & C_2^{\xi_j} & \cdots & C_1^{\xi_j}
\end{pmatrix}
\]

for \( j = 2, \ldots, 2(k - 1) \) and

\[
M_1^{\xi_j} = \begin{pmatrix}
C_1^{\xi_j} & C_1^{\xi_j} & \cdots & C_1^{\xi_j} \\
C_1^{\xi_j} & C_1^{\xi_j} & \cdots & C_1^{\xi_j} \\
\vdots & \ddots & \ddots & \vdots \\
C_1^{\xi_j} & C_1^{\xi_j} & \cdots & C_1^{\xi_j}
\end{pmatrix}
\]

for \( l = 3, \ldots, k + 3 \) and \( j = 2, \ldots, 2(k - 1) \).

Now, each \( C_i \) is of the type

\[
C_i = \left( \begin{array}{cc}
c_i & c_i' \\
\xi_i & \xi_i'
\end{array} \right), \quad C_i^{\xi_j} = \left( \begin{array}{cc}
c_i & \xi_i \\
\xi_i' & c_i'
\end{array} \right)
\]

for \( i = 1, \ldots, k + 3, j = 2, \ldots, 2(k - 1) \) and

\[
C_{k+2} = \left( \begin{array}{cc}
c_{k+2} & c_{k+2}' \\
\xi_{k+2} & \xi_{k+2}'
\end{array} \right), \quad C_{k+4} = \left( \begin{array}{cc}
c_{k+4} & c_{k+4}' \\
\xi_{k+4} & \xi_{k+4}'
\end{array} \right), \quad C_{k+5} = \left( \begin{array}{cc}
c_{k+5} & c_{k+5}' \\
\xi_{k+5} & \xi_{k+5}'
\end{array} \right),
\]

where
\[ c_1 = \frac{\partial g_1}{\partial z_1}, \quad c_1' = \frac{\partial g_1}{\partial z_1}, \quad c_2 = \frac{\partial g_1}{\partial z_2}, \quad c_2' = \frac{\partial g_1}{\partial z_2}, \]

\[ c_3 = \frac{\partial g_1}{\partial z_{q+1}}, \quad c_3' = \frac{\partial g_1}{\partial z_{q+1}}, \quad \ldots \quad c_{k+1} = \frac{\partial g_1}{\partial z_{(k-1)+1}}, \quad c_{k+1}' = \frac{\partial g_1}{\partial z_{(k-1)+1}}, \]

\[ c_{k+2} = \frac{\partial g_1}{\partial z_N}, \quad c_{k+2}' = \frac{\partial g_1}{\partial z_N}, \quad c_{k+3} = \frac{\partial g_{k+1}}{\partial z_1}, \quad c_{k+3}' = \frac{\partial g_{k+1}}{\partial z_1}, \]

\[ c_{k+4} = \frac{\partial g_N}{\partial z_N}, \quad c_{k+4}' = \frac{\partial g_N}{\partial z_N}, \quad c_{k+5} = \frac{\partial g_N}{\partial z_{N-1}}, \quad c_{k+5}' = \frac{\partial g_N}{\partial z_{N-1}}, \]

calculated at \((z_0, \lambda_0, \tau_0)\).

Throughout we denote by \((dg)_{(z_0,\lambda_0,\tau_0)}W_k\) the restriction of \((dg)_{(z_0,\lambda_0,\tau_0)}\) to the subspace \(W_k\).

We begin by computing \((dg)_{(z_0,\lambda_0,\tau_0)}|W_0\) where

\[ W_0 = \{ (z_1, \ldots, z_j; \xi_1 z_1, \ldots, \xi_j z_j; \ldots; \xi^{k-1}_j z_1, \ldots, \xi^{k-1}_j z_j; 0, \ldots, 0) : z_1 \in \mathbb{C} \}. \]

The tangent vector to the orbit of \(\Gamma \times S^1\) through \(z_0\) is the eigenvector

\[ (iz, \ldots, iz; i\xi z, \ldots, i\xi z; \ldots; i\xi^{k-1}_j z, \ldots, i\xi^{k-1}_j z; 0, \ldots, 0). \]

Note that

\[ \frac{d}{dt}(e^{it} z, \ldots, e^{it} z, \ldots, e^{it} \xi^{k-1}_j z, \ldots, e^{it} \xi^{k-1}_j z) \bigg|_{t=0} = (iz, \ldots, iz, \ldots, i\xi^{k-1}_j z, \ldots, i\xi^{k-1}_j z). \]

Now since \(g(\text{Fix}(\Sigma^I_{q,p})) \subseteq \text{Fix}(\Sigma^I_{q,p})\) we have that \(g(\text{Fix}(\Sigma^I_{q,p}))\) is two-dimensional. Thus, \((dg)_{(z_0,\lambda_0,\tau_0)}|W_0\) is as in (14) and the matrix of this mapping has the form (15).

The matrix \((dg)_{(z_0,\lambda_0,\tau_0)}|W_0\) has a single eigenvalue equal to zero and the other is given by

\[ 2\text{Re}(\alpha) = 2\text{Re}(A_1 + kqA_3)|z|^2 + \cdots \]

if \(k \geq 3\), whose sign is determined by \(A_{1r} + kqA_{3r}\) if it is assumed nonzero (where \(A_{1r} + kqA_{3r}\) is calculated at zero). In the particular case \(k = 2\), the nonzero eigenvalue is given by

\[ 2\text{Re}(\alpha) = 2\text{Re}[A_1 + 2q(A_2 + A_3)]|z|^2 + \cdots \]

whose sign is determined by \(A_{1r} + 2q(A_{2r} + A_{3r})\) if it is assumed nonzero (where \(A_{1r} + 2q(A_{2r} + A_{3r})\) is calculated at zero).

We compute \((dg)_{(z_0,\lambda_0,\tau_0)}|W_1\) where

\[ W_1 = \{ (z_1, \ldots, z_j; -\frac{kq}{p} z_1, \ldots, -\frac{kq}{p} z_1) : z_1 \in \mathbb{C} \}. \]
We have \(((dg)_{(z_0,\lambda_0,\tau_0)}|W_1)z \to \alpha z + \beta \bar{z}\) where
\[
\alpha = c_1 + (q - 1)c_2 + qc_3 - 2qc_4,
\beta = c'_1 + (q - 1)c'_2 + qc'_3 - 2qc'_4,
\]
for \(k = 2\). Recall that this case is special case since the branching equation is different from the one we obtain for \(k \geq 3\). Thus, we study this case separately. We get for \(k = 2\) (see [15, Chapter 4, Section 4, p.71] for the explicit expressions for \(c_1, \ldots, c_4, c'_1, \ldots, c'_4\)) that

\[
\text{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_1) = 2\text{Re} \left[ \left(1 - \frac{q}{N} \right) A_1 - 2qA_2 \right] |z|^2 + \cdots,
\]

\[
\text{det}((dg)_{(z_0,\lambda_0,\tau_0)}|W_1) = \left( \left| \left(1 - \frac{q}{N} \right) A_1 - 2qA_2 \right|^2 - \left| \left(1 - \frac{q}{N} \right) A_1 + 2qA_2 \right|^2 \right) |z|^4 + \cdots.
\]

If \(k \geq 3\) we have
\[
\alpha = c_1 + (q - 1)c_2 + qc_3 + \cdots + qc_{k+1} - kqc_{k+2},
\beta = c'_1 + (q - 1)c'_2 + qc'_3 + \cdots + qc'_k + 1 - kqc'_k + 2.
\]

and we get

\[
\text{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_1) = 2\text{Re} \left[ \left(1 - \frac{2q}{N} \right) A_1 \right] |z|^2 + \cdots,
\]

\[
\text{det}((dg)_{(z_0,\lambda_0,\tau_0)}|W_1) = \left( \left| \left(1 - \frac{2q}{N} \right) A_1 \right|^2 - |A_1|^2 \right) |z|^4 + \cdots.
\]

(see [15, Chapter 4, Section 4, p.72] for the explicit expressions for \(c_1, \ldots, c_{k+2}, c'_1, \ldots, c'_{k+2}\). We compute now \((dg)_{(z_0,\lambda_0,\tau_0)}|W_2\) where

\[
W_2 = \{(0, \ldots, 0; z_1, \ldots, z_p - 1, -z_1 - \cdots - z_{p-1}) : z_1, \ldots, z_{p-1} \in \mathbb{C} \}.
\]

Recall that we only have this isotypic component in the decomposition of \(\mathbb{C}^{N,0}\) for the action of \(\Sigma^I_{q,p}\) when \(p > 1\). Recall (9). The action of \(K \subset \Sigma^I_{q,p}\) on \(W_2\) decomposes in the following way:

\[
W_2 = W^1_2 \oplus W^2_2
\]

where

\[
W^1_2 = \{(0, \ldots, 0; x_1, \ldots, x_{p-1}, -x_1 - \cdots - x_{p-1}) : x_1, \ldots, x_{p-1} \in \mathbb{R} \},
\]

\[
W^2_2 = \{(0, \ldots, 0; ix_1, \ldots, ix_{p-1}, -ix_1 - \cdots - ix_{p-1}) : x_1, \ldots, x_{p-1} \in \mathbb{R} \}.
\]

Moreover, the actions of \(K\) on \(W^1_2\) and on \(W^2_2\) are \(K\)-isomorphic and \(K\)-absolutely irreducible. Thus, it is possible to choose a basis of \(W_2\) such that \((dg)_{(z_0,\lambda_0,\tau_0)}|W_2\) in the new coordinates has the form

\[
(16) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]

where \(\text{Id}_{(p-1)\times(p-1)}\) is the \((p-1)\times(p-1)\) identity matrix. Furthermore, the eigenvalues of (16) are the eigenvalues of \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) each with multiplicity \(p - 1\).
With respect to the basis $B'$ of $W_2$ given by
\[ b_{kq+1} - b_N, \bar{b}_{kq+1} - \bar{b}_N, b_{kq+2} - b_N, \bar{b}_{kq+2} - \bar{b}_N, \ldots, b_{N-1} - b_N, \bar{b}_{N-1} - \bar{b}_N, \]
we can write $(dg)_{(z_0, \lambda_0, \tau_0)}|W_2$ in the following block diagonal form
\[ (dg)_{(z_0, \lambda_0, \tau_0)}|W_2 = \text{diag}(C_{k+4} - C_{k+5}, \ldots, C_{k+4} - C_{k+5}). \]
The eigenvalues of $(dg)_{(z_0, \lambda_0, \tau_0)}|W_2$ are the eigenvalues of $C_{k+4} - C_{k+5}$, each with multiplicity $p - 1$. The eigenvalues of $C_{k+4} - C_{k+5}$ have negative real part if and only if
\[ \text{tr}(C_{k+4} - C_{k+5}) < 0 \text{ and } \det(C_{k+4} - C_{k+5}) > 0. \]
If $k = 2$ then
\[ \text{tr}((dg)_{(z_0, \lambda_0, \tau_0)}|W_2) = 2\text{Re}(-A_1 - 2qA_2)|z|^2 + \cdots, \]
\[ \det((dg)_{(z_0, \lambda_0, \tau_0)}|W_2) = (|A_1 + 2qA_2|^2 - |2qA_2|^2)|z|^4 + \cdots. \]
Moreover, if $k \geq 3$ we have
\[ \text{tr}((dg)_{(z_0, \lambda_0, \tau_0)}|W_2) = 2\text{Re}(A_1)|z|^2 + \cdots, \]
\[ \det((dg)_{(z_0, \lambda_0, \tau_0)}|W_2) = |A_1|^2|z|^4 + \cdots. \]
(see [15, Chapter 4, Section 4, p.73] for the explicit expressions for $c_{k+4}, c_{k+5}, c'_{k+4}, c'_{k+5}$ in both cases).

We compute now $(dg)_{(z_0, \lambda_0, \tau_0)}|W_3$ where
\[ W_3 = \{(z_1, \ldots, z_q-1, z_q; \ldots; z_{q(k-1)+1}, \ldots, z_{kq-1}, z_{kq}; 0, \ldots, 0) : z_1, \ldots, z_{kq} \in \mathbb{C}\} \]
with $z_q = -z_1 - \cdots - z_{q-1}, \ldots, z_{kq} = -z_{q(k-1)+1} - \cdots - z_{kq-1}$. Recall that we only have this isotypic component in the decomposition of $\mathbb{C}^{N,0}_q$ for the action of $\Sigma_{q,p}$ when $q \geq 2$.

With respect to the basis $B'$ of $W_3$ given by
\[ b_1 - b_q, \bar{b}_1 - \bar{b}_q, \ldots, b_{q-1} - b_q, \bar{b}_{q-1} - \bar{b}_q, \]
\[ b_{q+1} - b_{2q}, \bar{b}_{q+1} - \bar{b}_{2q}, \ldots, b_{2q-1} - b_{2q}, \bar{b}_{2q-1} - \bar{b}_{2q}, \]
\[ \ldots, \]
\[ b_{q(k-1)+1} - b_{kq}, \bar{b}_{q(k-1)+1} - \bar{b}_{kq}, \ldots, b_{kq-1} - b_{kq}, \bar{b}_{kq-1} - \bar{b}_{kq}, \]
we can write $(dg)_{(z_0, \lambda_0, \tau_0)}|W_3$ in the following block diagonal form:
\[ (dg)_{(z_0, \lambda_0, \tau_0)}|W_3 = \text{diag}(C_{1} - C_{2}, \ldots; C_{1}^{q} - C_{2}^{q}, \ldots; C_{1}^{q(k-1)} - C_{2}^{q(k-1)}, \ldots). \]
We have thus, we get different expressions for the derivatives. We study the case but it the particular case [4, p.76 to p.78] for the explicit computations) we get that the case which is not forced by the symmetry of the problem. We consider now the degree j case. If c

system. See [15, Chapter 4, Section 4, p.75 and p.76] for the explicit computation of the five degree truncation is too degenerate in order to determine the stability of the this component appears in the isotypic decomposition of C^{N,0} for the action of \Sigma^f_{q,p}; the five degree truncation is too degenerate in order to determine the stability of the system. See [15, Chapter 4, Section 4, p.75 and p.76] for the explicit computation of c_1, c_2, c_1', c_2'.

We compute now \((dg)_{(z_0,\lambda_0,\tau_0)}|P_j\) where

\[ P_j = \{(z_1, \ldots, z_j; \xi^j_1 z_1, \ldots, \xi^j_{j-1} z_1; \ldots; \xi^{j(k-1)}_1 z_1; 0, \ldots, 0) : z_1 \in C\} \]

and 2 ≤ j ≤ k - 1. We have \((dg)_{(z_0,\lambda_0,\tau_0)}|P_j\) z → az + \beta z where

\[ \alpha = c_1 + (q - 1)c_2 + q\xi^j_1 c_3 + \cdots + q^{\xi^{j(k-1)}_1} c_{k+1}, \]

\[ \beta = c_1' + (q - 1)c_2' + q\xi^j_1 c_3' + \cdots + q^{\xi^{j(k-1)}_1} c_{k+1}'. \]

When we substitute the expressions for the derivatives (see [15, Chapter 4, Section 4, p.76 to p.78] for the explicit computations) we get that the case k = 3 is a particular case. If j = 2 and k ≥ 4 we have

\[ \text{tr} ((df)_{(z_0,\lambda_0,\tau_0)}|P_2) = 2 \text{Re} (A_1) |z|^2 + \cdots, \]

\[ \text{det} ((df)_{(z_0,\lambda_0,\tau_0)}|P_2) = \left( |A_1|^2 - \left| 1 - \frac{q}{N} \right| A_1 \right)^{2} |z|^4 + \cdots, \]

but the particular case k = 3 it follows that

\[ \text{tr} ((df)_{(z_0,\lambda_0,\tau_0)}|P_2) = 2 \text{Re} (A_1 + 6A_2) |z|^2 + \cdots, \]

\[ \text{det} ((df)_{(z_0,\lambda_0,\tau_0)}|P_2) = \left( |A_1 + 6A_2|^2 - \left| 1 - \frac{3}{N} \right| A_1 \right)^{2} |z|^4 + \cdots. \]
Consider now $j = k - 1$. It follows that
\[ \text{tr} \left( (df)_{(z_0, \lambda_0, \tau_0)} | P_{k-1} \right) = 2 \text{Re} \left( A_1 + 2kqA_2 \right) |z|^2 + \cdots , \]
\[ \text{det} \left( (df)_{(z_0, \lambda_0, \tau_0)} | P_{k-1} \right) = (|A_1 + 2kqA_2|^2 - |A_1|^2) |z|^4 + \cdots . \]
Moreover, if we consider $2 < j \leq k - 2$, then we obtain that
\[ \text{det} \left( (df)_{(z_0, \lambda_0, \tau_0)} | P_j \right) = 0. \]
Thus, we need to consider the five degree truncation of (12). We have for $j = k - 2$ (note that we only have this isotypic component when $k \geq 5$) that
\[ \text{tr} \left( (df)_{(z_0, \lambda_0, \tau_0)} | P_{k-2} \right) = 2 \text{Re} \left( A_1 \right) |z|^2 + \cdots , \]
\[ \text{det} \left( (df)_{(z_0, \lambda_0, \tau_0)} | P_{k-2} \right) = (|A_1 + \xi_1|z|^2|^2 - |A_1 + (2A_4 + kqA_14)|z|^2|^2) |z|^4 + \cdots \]
\[ = \left[ 2 \text{Re}(A_1 \xi_1) - 2 \text{Re}(2A_4 \xi_1 + kqA_14 \xi_1) \right] |z|^6 + \cdots , \]
where
\[ \xi_1 = 2A_4 + 3kqA_12 + q(kq - 1) (2 - \frac{2kq}{N}) A_{13} + kqA_{14} + \]
\[ + q(kq - 1) (1 - \frac{2kq}{N}) A_{14} + 2q(kq - 1)A_{15} . \]
Furthermore, for $3 \leq j \leq k - 3$ (note that we only have this isotypic component when $k \geq 6$) we get
\[ \text{tr} \left( (df)_{(z_0, \lambda_0, \tau_0)} | P_j \right) = 2 \text{Re} \left( A_1 \right) |z|^2 + \cdots , \]
\[ \text{det} \left( (df)_{(z_0, \lambda_0, \tau_0)} | P_j \right) = (|A_1 + \xi_2|z|^2|^2 - |A_1 + (2A_4 + kqA_14)|z|^2|^2) |z|^4 + \cdots \]
\[ = \left[ 2 \text{Re}(A_1 \xi_2) - 2 \text{Re}(2A_4 \xi_2 + kqA_14 \xi_2) \right] |z|^6 + \cdots , \]
with
\[ \xi_2 = \xi_1 - 3kqA_{12} - kqA_{14} . \]
\[ (\Sigma^I_q = \mathbb{S}_q \times \mathbb{S}_p, \text{ where } N = q + p, \text{ } 1 \leq q < \frac{N}{2}) \]
The fixed-point subspace of $\Sigma^I_q = \mathbb{S}_q \times \mathbb{S}_p$ is
\[ \text{Fix} \left( \Sigma^I_{q,p} \right) = \left\{ \left( z, \ldots , z; \frac{q}{p} z, \ldots , \frac{q}{p} z \right) : z \in \mathbb{C} \right\} . \]
Using the equation (13) where $f$ is as in (12), after dividing by $z$ we have
\[ \lambda = - A_1 \left[ 1 - \frac{q}{N} \left( 1 - \frac{q^2}{p^2} \right) \right] |z|^2 + (A_2 + A_3)q \left( 1 + \frac{q}{p} \right) |z|^2 + \cdots = 0 \]
where $+ \cdots$ denotes terms of higher order in $z$ and $\overline{z}$, and taking the real part of this equation, we obtain,
\[ \lambda = - A_1 r \left[ 1 - \frac{q}{N} \left( 1 - \frac{q^2}{p^2} \right) \right] |z|^2 + (A_2 + A_3)q \left( 1 + \frac{q}{p} \right) |z|^2 + \cdots . \]
It follows that if $A_1 r \left[ 1 - \frac{q}{N} \left( 1 - \frac{q^2}{p^2} \right) \right] |z|^2 + (A_2 + A_3)q \left( 1 + \frac{q}{p} \right) < 0$, then the branch bifurcates supercritically.
Let $\Sigma_q^H = S_q \times S_p$ be the isotropy subgroup of $z_0 = (z, \ldots, z; -\frac{q}{p}z, \ldots, -\frac{q}{p}z)$.

Recall the generators for $\Sigma_q^H$ given in Section 3.

Suppose $M$ is a square $(q + p) \times (q + p)$ matrix with rows $l_1, \ldots, l_q, l_{q+1}, \ldots, l_{q+p}$ and commuting with $S_q \times S_p$. Then

$$M = (l_1, (12) \cdot l_1, \ldots, (1q) \cdot l_1, (q + 1q + 2) \cdot l_{q+1}, \ldots, (q + 1q + p) \cdot l_{q+1})$$

where if $l_1 = (m_1, \ldots, m_{q+p})$ then

$$(1i) \cdot l_1 = (m_i, m_2, \ldots, m_{i-1}, m_1, m_{i+1}, \ldots, m_{q+p}).$$

Moreover, $l_1$ is $S_{q-1} \times S_p$-invariant and $l_{q+1}$ is $S_q \times S_{p-1}$-invariant. Applying this to $(dg)_{(z_0, \lambda_0, \tau_0)}$ we have

$$(dg)_{(z_0, \lambda_0, \tau_0)} = \begin{pmatrix}
C_1 & C_6 & C_2 & C_2 \\
\cdot & C_6 & C_2 & C_2 \\
C_3 & C_3 & C_4 & C_5 \\
\cdot & C_3 & C_5 & C_4
\end{pmatrix}$$

where $C_i$ for $i = 1, \ldots, 5$ are the $2 \times 2$ matrices

$$C_i = \begin{pmatrix}
c_i & c'_i \\
c_i' & c_i
\end{pmatrix}$$

and

$$c_1 = \frac{\partial g_1}{\partial z_1}, \quad c'_1 = \frac{\partial g_1}{\partial z_1}, \quad c_6 = \frac{\partial g_1}{\partial z_2}, \quad c'_6 = \frac{\partial g_1}{\partial z_2}, \quad c_2 = \frac{\partial g_1}{\partial z_{q+1}}, \quad c'_2 = \frac{\partial g_1}{\partial z_{q+1}}.$$

$$c_3 = \frac{\partial g_{q+1}}{\partial z_1}, \quad c'_3 = \frac{\partial g_{q+1}}{\partial z_1}, \quad c_4 = \frac{\partial g_{q+1}}{\partial z_{q+1}}, \quad c'_4 = \frac{\partial g_{q+1}}{\partial z_{q+1}}, \quad c_5 = \frac{\partial g_{q+1}}{\partial z_{q+2}}, \quad c'_5 = \frac{\partial g_{q+1}}{\partial z_{q+2}}.$$

calculated at $(z_0, \lambda_0, \tau_0)$.

We begin by computing $(dg)_{(z_0, \lambda_0, \tau_0)}|W_0$. In coordinates $z, \bar{z}$ we have $((dg)_{(z_0, \lambda_0, \tau_0)}|W_0)z = \alpha z + \beta \bar{z}$ where

$$\alpha = c_1 + (q - 1)c_6 - [q(N - q)/p]c_2,$$

$$\beta = c'_1 + (q - 1)c'_6 - [q(N - q)/p]c'_2.$$

The tangent vector to the orbit of $\Gamma \times S^1$ through $z_0$ is the eigenvector

$$\left(iz, \ldots, iz, -\frac{q}{p}z, \ldots, -\frac{q}{p}z\right).$$

Note that

$$\frac{d}{dt}\left(e^{it}z, \ldots, e^{it}z, -e^{it}\frac{q}{p}z, \ldots, -e^{it}\frac{q}{p}z\right)|_{t=0} = \left(iz, \ldots, iz, -\frac{q}{p}z, \ldots, -\frac{q}{p}z\right).$$

The matrix $(dg)_{(z_0, \lambda_0, \tau_0)}|W_0$ has a single eigenvalue equal to zero and the other is

$$2\text{Re}(\alpha) = 2\text{Re} \left[ A_1 \left(1 - \frac{q}{N} + \frac{q^3}{Np^2}\right) + (A_2 + A_3)q \left(1 + \frac{q}{p}\right) \right] |z|^2 + \cdots$$

whose sign is determined by

$$A_{1r} \left[1 - \frac{q}{N} \left(1 - \frac{q^2}{p^2}\right)\right] + (A_{2r} + A_{3r})q \left(1 + \frac{q}{p}\right).$$
if it is assumed nonzero (where \( A_{1r}, A_{2r}, A_{3r} \) are calculated at zero).

We compute now \((dg)_{(z_0, \lambda_0, \tau_0)}|W_1\) where

\[
W_1 \equiv \left\{ \begin{pmatrix} z_1, \ldots, z_{q-1}, -z_1 - \cdots - z_{q-1}; 0, \ldots, 0 \end{pmatrix} : z_1, \ldots, z_{q-1} \in \mathbb{C} \right\}.
\]

The action of \( \Sigma_q^{II} \) on \( W_1 \) decomposes in the following way

\[
W_1 = W_1^1 \oplus W_1^2
\]

where

\[
W_1^1 = \left\{ \begin{pmatrix} x_1, \ldots, x_{q-1}, -x_1 - \cdots - x_{q-1}; 0, \ldots, 0 \end{pmatrix} : x_1, \ldots, x_{q-1} \in \mathbb{R} \right\},
\]

\[
W_1^2 = \left\{ \begin{pmatrix} ix_1, \ldots, ix_{q-1}, -ix_1 - \cdots - ix_{q-1}; 0, \ldots, 0 \end{pmatrix} : x_1, \ldots, x_{q-1} \in \mathbb{R} \right\}.
\]

Moreover, the actions of \( \Sigma_q^{II} \) on \( W_1^1 \) and on \( W_1^2 \) are \( \Sigma_q^{II} \)-isomorphic and are \( \Sigma_q^{II} \)-absolutely irreducible. Thus, it is possible to choose a basis of \( W_1 \) such that \((dg)_{(z_0, \lambda_0, \tau_0)}|W_1\) in the new coordinates has the form

\[
(17) \quad \begin{pmatrix} a & \text{Id}_{(q-1) \times (q-1)} \\ c \text{Id}_{(q-1) \times (q-1)} & b \text{Id}_{(q-1) \times (q-1)} \\ d \text{Id}_{(q-1) \times (q-1)} & c \text{Id}_{(q-1) \times (q-1)} \end{pmatrix}
\]

where \( \text{Id}_{(q-1) \times (q-1)} \) is the \((q-1) \times (q-1)\) identity matrix. Furthermore, the eigenvalues of (17) are the eigenvalues of \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) each with multiplicity \( q - 1 \).

With respect to the basis \( B' \) of \( W_1 \) given by

\[
b_1 - b_q, \bar{b}_1 - \bar{b}_q, b_2 - b_q, \bar{b}_2 - \bar{b}_q, \ldots, b_{q-1} - b_q, \bar{b}_{q-1} - \bar{b}_q
\]

we can write \((dg)_{(z_0, \lambda_0, \tau_0)}|W_1\) in the following block diagonal form

\[
(dg)_{(z_0, \lambda_0, \tau_0)}|W_1 = \text{diag}(C_1 - C_6, C_1 - C_6, \ldots, C_1 - C_6).
\]

The eigenvalues of \((dg)_{(z_0, \lambda_0, \tau_0)}|W_1\) are the eigenvalues of \( C_1 - C_6 \), each with multiplicity \( q - 1 \). The eigenvalues of \( C_1 - C_6 \) have negative real part if and only if

\[
\text{tr}(C_1 - C_6) < 0 \text{ and } \det(C_1 - C_6) > 0.
\]

We get

\[
\text{tr}((dg)_{(z_0, \lambda_0, \tau_0)}|W_1) = 2 \text{Re} \left[ \left( 1 + \frac{q}{N} - \frac{q^3}{Np^2} \right) A_1 - q \left( 1 + \frac{q}{p} \right) A_2 \right] |z|^2 + \cdots,
\]

\[
\det((dg)_{(z_0, \lambda_0, \tau_0)}|W_1) = \left| \left( 1 + \frac{q}{N} - \frac{q^3}{Np^2} \right) A_1 - q \left( 1 + \frac{q}{p} \right) A_2 \right|^2 |z|^4 -
\]

\[
\left| A_1 + q \left( 1 + \frac{q}{p} \right) A_2 \right|^2 |z|^4 + \cdots,
\]

(see [15, Chapter 4, Section 4, p.81] for the explicit expressions for \( c_1, c_6, c_1', c_6' \)).
We compute now \((dg)_{(z_0,\lambda_0,\tau_0)}|W_2\) where
\[
W_2 = \{ (0, \ldots, 0, z_{q+1}, \ldots, z_{N-1}, -z_{q+1} - \cdots - z_{N-1}) : z_{q+1}, \ldots, z_{N-1} \in \mathbb{C} \}.
\]
The action of \(\Sigma^H_q\) on \(W_2\) decomposes in the following way
\[
W_2 = W_2^1 \oplus W_2^2
\]
where
\[
W_2^1 = \left\{ \begin{pmatrix} 0, \ldots, 0, x_{q+1}, \ldots, x_{N-1}, -x_{q+1} - \cdots - x_{N-1} \end{pmatrix} : x_{q+1}, \ldots, x_{N-1} \in \mathbb{R} \right\},
\]
\[
W_2^2 = \left\{ \begin{pmatrix} 0, \ldots, 0, ix_{q+1}, \ldots, ix_{N-1}, -ix_{q+1} - \cdots - ix_{N-1} \end{pmatrix} : x_{q+1}, \ldots, x_{N-1} \in \mathbb{R} \right\}.
\]
Moreover, the actions of \(\Sigma^H_q\) on \(W_2^1\) and on \(W_2^2\) are \(\Sigma^H_q\)-isomorphic and are \(\Sigma^H_q\)-absolutely irreducible. Thus, it is possible to choose a basis of \(W_2\) such that \((dg)_{(z_0,\lambda_0,\tau_0)}|W_2\) in the new coordinates has the form
\[
(18) \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \text{Id}_{(N-q-1)\times(N-q-1)} & 0 \\ 0 & \text{Id}_{(N-q-1)\times(N-q-1)} \end{pmatrix}
\]
where \(\text{Id}_{(N-q-1)\times(N-q-1)}\) is the \((N-q-1)\times(N-q-1)\) identity matrix. Furthermore, the eigenvalues of (18) are the eigenvalues of \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) each with multiplicity \(N-q-1\).

With respect to the basis \(B'\) of \(W_2\) given by
\[
b_{q+1} - b_N, \bar{b}_{q+1} - \bar{b}_N, b_{q+2} - b_N, \bar{b}_{q+2} - \bar{b}_N, \ldots, b_{N-1} - b_N, \bar{b}_{N-1} - \bar{b}_N,
\]
we can write \((dg)_{(z_0,\lambda_0,\tau_0)}|W_2\) in the following block diagonal form
\[
(df)_{(z_0,\lambda_0,\tau_0)}|W_2 = \text{diag}(C_4 - C_5, C_4 - C_5, \ldots, C_4 - C_5).
\]
The eigenvalues of \((dg)_{(z_0,\lambda_0,\tau_0)}|W_2\) are the eigenvalues of \(C_4 - C_5\), each with multiplicity \(N-q-1\). The eigenvalues of \(C_4 - C_5\) have negative real part if and only if
\[
\text{tr}(C_4 - C_5) < 0 \quad \text{and} \quad \det(C_4 - C_5) > 0.
\]
We have
\[
\text{tr}((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = 2\text{Re} \left[ -1 + \frac{q}{N} - \frac{q^3}{Np^2} + \frac{2q^2}{p^2} \right] A_1 - q \left( 1 + \frac{1}{p} \right) A_2 |z|^2 + \cdots,
\]
\[
\det((dg)_{(z_0,\lambda_0,\tau_0)}|W_2) = \left| -1 + \frac{q}{N} - \frac{q^3}{Np^2} + \frac{2q^2}{p^2} \right| A_1 - q \left( 1 + \frac{1}{p} \right) A_2 |z|^4 - \left| \frac{q^2}{p^2} A_1 + q \left( 1 + \frac{q}{p} \right) A_2 \right|^2 |z|^4 + \cdots,
\]
(see [15, Chapter 4, Section 4, p.82] for the explicit expressions for \(c_4, c_5, c_4', c_5'\)).
5. An Example with $N = 5$

In this section we consider Hopf bifurcation with $S_N$-symmetry for the special case $N = 5$.

For general $N$, from Theorem 4.1 we know that the stability of some of the periodic solutions guaranteed by the Equivariant Hopf Theorem in some directions is determined by the fifth degree truncation of the vector field. Moreover, for directions in $W$, when $k > 3$ (recall Table 5), the fifth degree truncation of the vector field is too degenerate to determine their stability. When $N = 5$ the directions in which we need the degree five truncation of the vector field are present in the isotypic decomposition for some of the $C$-axial isotropy subgroups. Moreover, the quintic terms determine completely the stability of the periodic solutions guaranteed by the Equivariant Hopf Theorem.

By Theorem 3.1 and Section 3, we have two types of $C$-axial isotropy subgroups of $S_N \times S^1$: $\Sigma_{q,p}^I = S_q \wr \mathbb{Z}_k \times S_p$ and $\Sigma_{q,p}^{II} = S_q \times S_p$. From Theorem 4.1, we have that if $k > 3$ and $q \geq 2$ in $\Sigma_{q,p}$, then the fifth degree truncation of the vector field is too degenerate to determine the stability of solutions with those symmetries in some directions. In the case $N = 5$, the isotropy subgroups are all of that form with $q < 2$ except one of them (see $\Sigma_1$ in Table 7), but for this one we have $k = 2$. Thus, this is the first case where the fifth degree truncation of the vector field is necessary to determine the stability of such solutions. Moreover, the degree five truncation of a general $S_5$-equivariant vector field determines the stability and the criticality of the branches of periodic solutions guaranteed by the Equivariant Hopf Theorem. We consider so this special case and we give the explicit conditions on the coefficients of the general degree 5 vector field equivariant under $S_5 \times S^1$ determining the stability and the criticality of those solutions.

Consider the action of $S_5 \times S^1$ on $C^{5,0}$ given by

$$\sigma, \theta)(z_1, z_2, z_3, z_4, z_5) = e^{i\theta}(z_{\sigma^{-1}(1)}, z_{\sigma^{-1}(2)}, z_{\sigma^{-1}(3)}, z_{\sigma^{-1}(4)}, z_{\sigma^{-1}(5)})$$

for $\sigma \in S_5, \theta \in S^1$ and $(z_1, z_2, z_3, z_4, z_5) \in C^{5,0}$, with

$$C^{5,0} = \{(z_1, z_2, z_3, z_4, z_5) \in C^5 : z_1 + z_2 + z_3 + z_4 + z_5 = 0\}.$$

From Theorem 3.1 we obtain a description of the $C$-axial subgroups of $S_5 \times S^1$ acting on $C^{5,0}$. They are listed, together with their generators and fixed-point subspaces in Table 7. Recall Table 1. Each of the seven $C$-axial isotropy subgroups of $S_5 \times S^1$ acting on $C^{5,0}$ listed in Table 7 are of the form $\Sigma_{q,p}^I$ or $\Sigma_{q,p}^{II}$. Specifically, we have that $\Sigma_i, i = 1, \ldots, 5$ are of the form $\Sigma_{q,p}^I$ and $\Sigma_6, \Sigma_7$ are of the form $\Sigma_{q,p}^{II}$.

Take $N = 5$ and let $f$ be as in (11). From (12) we write $f = (f_1, f_2, f_3, f_4, f_5)$, where

$$f_1(z, \lambda) = \mu(\lambda)z_1 + f_1^{(3)}(z) + f_1^{(5)}(z) + \cdots$$

$$f_2(z, \lambda) = f_1(z_2, z_1, z_3, z_4, z_5, \lambda)$$

$$\cdots$$

$$f_5(z, \lambda) = f_1(z_5, z_2, z_3, z_4, z_1, \lambda)$$

where
Isotropy Subgroup | Generators | Fixed-Point Subspace
--- | --- | ---
\(\Sigma_1 = \tilde{S}_2 \times \tilde{Z}_2\) | (12), (34), ((13)(24), π) | \(\{ (z_1, z_1, -z_1, -z_1, 0) : z_1 \in \mathbb{C} \}\)
\(\Sigma_2 = \tilde{Z}_2 \times S_3\) | (34), (35), ((12), π) | \(\{ (z_1, -z_1, 0, 0, 0) : z_1 \in \mathbb{C} \}\)
\(\Sigma_3 = \tilde{Z}_3 \times S_2\) | (45), ((123), \(\frac{2\pi}{3}\)) | \(\{ (z_1, \xi z_1, \xi^2 z_1, 0, 0) : z_1 \in \mathbb{C} \}, \xi = e^{2\pi i/3}\)
\(\Sigma_4 = \tilde{Z}_4\) | ((1234), \(\frac{\pi}{2}\)) | \(\{ (z_1, iz_1, -z_1, -iz_1, 0) : z_1 \in \mathbb{C} \}\)
\(\Sigma_5 = \tilde{Z}_5\) | ((12345), \(\frac{2\pi}{5}\)) | \(\{ (z_1, \xi z_1, \xi^2 z_1, \xi^3 z_1, \xi^4 z_1) : z_1 \in \mathbb{C} \}, \xi = e^{2\pi i/5}\)
\(\Sigma_6 = S_2 \times S_3\) | (12), (34), (35) | \(\{ (z_1, z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1, -\frac{2}{3}z_1) : z_1 \in \mathbb{C} \}\)
\(\Sigma_7 = S_4\) | (23), (24), (25) | \(\{ (z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1, -\frac{1}{4}z_1) : z_1 \in \mathbb{C} \}\)

**Table 7.** C-axial isotropy subgroups of \(S_5 \times S^1\) acting on \(\mathbb{C}^{5,0}\), generators and fixed-point subspaces.

\[
\Sigma_1 \lambda = -(A_{1r} + 4A_{2r} + 4A_{3r})|z|^2 + \cdots \\
\Sigma_2 \lambda = -(A_{1r} + 2A_{2r} + 2A_{3r})|z|^2 + \cdots \\
\Sigma_3 \lambda = -(A_{1r} + 3A_{3r})|z|^2 + \cdots \\
\Sigma_4 \lambda = -(A_{1r} + 4A_{3r})|z|^2 + \cdots \\
\Sigma_5 \lambda = -(A_{1r} + 5A_{3r})|z|^2 + \cdots \\
\Sigma_6 \lambda = -\frac{1}{3}(\frac{7}{3}A_{1r} + 10A_{2r} + 10A_{3r})|z|^2 + \cdots \\
\Sigma_7 \lambda = -\frac{1}{4}(\frac{15}{4}A_{1r} + 5A_{2r} + 5A_{3r})|z|^2 + \cdots 
\]

**Table 8.** Branching equations for \(S_5\) Hopf bifurcation. Subscript \(r\) on the coefficients refer to the real part and + \(\cdots\) stands for higher order terms.
<table>
<thead>
<tr>
<th>Subgroup</th>
<th>$\Delta_0$</th>
<th>$\Delta_1, \ldots, \Delta_r$</th>
</tr>
</thead>
</table>
| $\Sigma_1$ | $A_1r + 4A_2r + 4A_3r$ | $-\frac{3}{5}A_1r - 4A_2r$  
$- | - \frac{3}{5}A_1 - 4A_2|^2 - |\frac{1}{5}A_1 + 4A_2|^2 |$  
$A_1r - 4A_2r$  
$- (|A_1 - 4A_2|^2 - |A_1 + 4A_2|^2)$ |
| $\Sigma_2$ | $A_1r + 2A_2r + 2A_3r$ | $\frac{1}{5}A_1r - 2A_2r$  
$- (\frac{1}{5}A_1 - 2A_2|^2 - |\frac{2}{5}A_1 + 2A_2|^2 |$  
$A_1r - 2A_2r$  
$- (|A_1 + 2A_2|^2 - |2A_2|^2)$ |
| $\Sigma_3$ | $A_1r + 3A_3r$ | $-A_1r$  
$- (|A_1 + 6A_2|^2 - |\frac{2}{5}A_1|^2 |$  
$A_1r + 6A_2r$  
$A_1r$  
$- |A_1|^2$ |
| $\Sigma_4$ | $A_1r + 4A_3r$ | $- |A_1|^2$  
$- A_1r$  
$A_1r$  
$- (|A_1 + 8A_2|^2 - |A_1|^2 |$  
$A_1r + 8A_2r$ |
| $\Sigma_5$ | $A_1r + 5A_3r$ | $- |A_1|^2$  
$A_1r$  
$- \text{Re}[A_1(\bar{\xi}_1 - \bar{\xi}_2)] |$  
$A_1r + 10A_2r$  
$- (|A_1 + 10A_2|^2 - |A_1|^2)$ |
| $\Sigma_6$ | $\frac{7}{3}A_1r + 10A_2r + 10A_3r$ | $\frac{11}{3}A_1r - 10A_2r$  
$- (\frac{1}{3} (\frac{11}{3}A_1 - 10A_2)^2 - |A_1 + \frac{10}{3}A_2|^2 |$  
$\frac{1}{3}A_1r - 8A_2r$  
$- (\frac{1}{3} (\frac{1}{3}A_1 - 8A_2)^2 - \frac{1}{3} (\frac{4}{3}A_2 + 10A_2)^2 |)$ |
| $\Sigma_7$ | $\frac{13}{7}A_1r + 5A_2r + 5A_3r$ | $\text{Re}(-\frac{55}{36}A_1 - \frac{5}{4}A_2)$  
$- (\frac{1}{3} (-\frac{55}{36}A_1 - \frac{5}{4}A_2)^2 - |\frac{1}{16}A_1 + \frac{5}{4}A_2|^2)$ |

**Table 9.** Stability for $S_5$ Hopf bifurcation. Here $\xi_1 = 2A_4 + 10A_{14}$ and $\xi_2 = 2A_4 + 5A_{11} + 5A_{14}$. For each $\Sigma_i$, the corresponding branch of periodic solutions is supercritical if $\Delta_0 < 0$ and subcritical if $\Delta_0 > 0$. If $\Delta_j > 0$ for some $j = 0, \ldots, r$, then the corresponding branch of periodic solutions is unstable. If $\Delta_j < 0$ for all $j$, then the solutions are stable near $\lambda = 0$ and $z = 0$. Note that solutions with $\Sigma_3$ and $\Sigma_4$ symmetry are always unstable.
### Isotropy subgroup and Orbit Representative

<table>
<thead>
<tr>
<th>Subgroup</th>
<th>Representative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma_1$</td>
<td>$\overline{S_2 \times Z_2}$</td>
</tr>
<tr>
<td>$z = (z_1, z_1, -z_1, -z_1, 0)$</td>
<td>$\Sigma_2 = \overline{Z_2 \times S_3}$</td>
</tr>
<tr>
<td>$z = (z_1, -z_1, 0, 0, 0)$</td>
<td>$\Sigma_3 = \overline{Z_3 \times S_2}$</td>
</tr>
<tr>
<td>$z = (z_1, \xi z_1, \xi^2 z_1, 0, 0)$</td>
<td>$\Sigma_4 = \overline{Z_4}$</td>
</tr>
<tr>
<td>$\xi = e^{2\pi i/3}$</td>
<td>$\Sigma_5 = \overline{Z_5}$</td>
</tr>
<tr>
<td>$z = (z_1, \xi z_1, \xi^2 z_1, \xi^3 z_1, \xi^4 z_1)$</td>
<td>$\Sigma_6 = S_2 \times S_3$</td>
</tr>
<tr>
<td>$\xi = e^{2\pi i/5}$</td>
<td>$\Sigma_7 = S_4$</td>
</tr>
<tr>
<td>$z = (z_1, z_1, -\frac{2}{3} z_1, -\frac{2}{3} z_1, -\frac{2}{3} z_1)$</td>
<td></td>
</tr>
<tr>
<td>$z = (z_1, -\frac{1}{4} z_1, -\frac{1}{4} z_1, -\frac{1}{4} z_1, -\frac{1}{4} z_1)$</td>
<td></td>
</tr>
</tbody>
</table>

Table 10. Isotypic decomposition of $C^{5,0}$ for the action of each of the isotropy subgroups listed in Table 7.

\[
f_1^{(3)}(z) = \sum_{i=1}^{3} A_j f_{1,j}(z),
\]

\[
f_1^{(5)}(z) = \sum_{i=4, i \neq 12}^{15} A_j f_{1,j}(z),
\]

with $z_5 = -z_1 - z_2 - z_3 - z_4$. The coefficients $A_i, i = 1, \ldots, 15, i \neq 12$ are complex smooth functions of $\lambda$, $\mu(0) = i$ and $\text{Re}(\mu'(0)) \neq 0$. For $N = 5$ we obtain only eleven
linearly independent $S_5 \times S^1$-equivarians with homogeneous polynomial components of degree five (see [2] and [15, Remark 4.12 p.60]). Thus, we don’t need to include the term $f_{1,12}$ and the other $f_{1,i}$ are given by

\begin{align*}
  f_{1,1}(z) &= \left[ \frac{4}{5} |z_1|^2 z_1 - \frac{1}{5} (|z_2|^2 z_2 + |z_3|^2 z_3 + |z_4|^2 z_4 + |z_5|^2 z_5) \right] \\
  f_{1,2}(z) &= \frac{1}{2} \left( z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 \right) \\
  f_{1,3}(z) &= \frac{1}{2} (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2) \\
  f_{1,4}(z) &= \frac{4}{5} |z_1|^2 z_1 - \frac{1}{5} (|z_2|^2 z_2 + |z_3|^2 z_3 + |z_4|^2 z_4 + |z_5|^2 z_5) \\
  f_{1,5}(z) &= z_1 (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2) \\
  f_{1,6}(z) &= z_1 (z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2) (\frac{1}{2} + \frac{1}{2} + \frac{3}{2} + \frac{3}{2} + \frac{3}{2}) \\
  f_{1,7}(z) &= z_1 (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2) \\
  f_{1,8}(z) &= \frac{1}{2} (|z_1|^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2) \\
  f_{1,9}(z) &= \frac{1}{2} (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2) \\
  f_{1,10}(z) &= \frac{1}{2} (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2) \\
  f_{1,11}(z) &= \frac{1}{2} (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2) \\
  f_{1,13}(z) &= \frac{1}{2} (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2) \\
  f_{1,14}(z) &= \frac{1}{2} (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2) \\
  f_{1,15}(z) &= \frac{1}{2} (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2) \\
  &\quad - \frac{1}{2} (z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2) \\
  &\quad - \frac{1}{2} (|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2) \\
  &\quad - \frac{1}{2} (|z_2|^2 + |z_3|^2 + |z_4|^2 + |z_5|^2).
\end{align*}

Recall the isotypic decomposition for each type of the isotropy subgroups $\Sigma_{q,p}^I$ and $\Sigma_q^{II}$ in Table 6. For the seven isotropy subgroups $\Sigma_i$, for $i = 1, \ldots, 7$, in Table 7, it is possible to put the Jacobian matrix $(dg)_{\Sigma_0}$ into block diagonal form. We do this by decomposing $C^{5,0}$ into isotypic components for the action of each isotropy subgroup $\Sigma_i$. Table 10 gives the isotypic decomposition of $C^{5,0}$ for each of the isotropy subgroups $\Sigma_i$ listed in Table 7.

From Theorem 4.1 we get Tables 8 and 9, which give respectively the branching equations and the stability for Hopf bifurcation with $S_5$-symmetry. Note that in particular it follows that solutions with $\Sigma_3$ and with $\Sigma_4$-symmetry are always unstable.

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References


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