DIRECT LIFTS OF COUPLED CELL NETWORKS

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Abstract. In networks of dynamical systems, there are spaces defined in terms of equalities of cell coordinates which are flow-invariant under any dynamical system that has form consistent with the given underlying network structure – the network synchrony subspaces. Given a network and one of its synchrony subspaces, any system with form consistent with the network, restricted to the synchrony subspace, defines a new system which is consistent with a smaller network, called the quotient network of the original network by the synchrony subspace. Moreover, any system associated with the quotient can be interpreted as the restriction to the synchrony subspace of a system associated with the original network. We call the bigger network a lift of the smaller network and a lift can be interpreted as resulting from a cellular splitting of the smaller network. In this paper we address the question of the uniqueness in this lifting process in terms of the networks topologies. A lift G of a given network Q is said to be direct when there are no intermediate lifts of Q between them. We provide necessary and sufficient conditions for a lift of a general network to be direct. Our results characterize direct lifts using the subnetworks of all spitting cells of Q and of all splitted cells of G. We show that G is a direct lift of Q if and only if either the splitted subnetwork is a direct lift or consists into two copies of the splitting subnetwork. These results are then applied to the class of regular uniform networks and to the special classes of ring networks and acyclic networks. We also illustrate that one of the applications of our results is to the lifting bifurcation problem.

Keywords: Coupled cell network, direct lift.

1. Introduction

Many real life aplications are modelled through networks of dynamical systems. See for example Albert and Barabási [6] and Strogatz [16]. The study of these dynamical systems takes into account the internal dynamics of the nodes and the network structure of the interactions between the nodes. This paper follows the framework of the theory of coupled cell networks, developed by Stewart, Golubitsky and co-workers [15, 11] and Field [8]. A *network* is a graph that encodes information concerning the types of nodes and the types of interactions between the nodes. A

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coupled cell system associated with a network is a set of dynamical systems (in general, we take a set of systems of ordinary differential equations) that has structure consistent with the network. It is known that this network structure strongly influences the dynamics of the whole network. Moreover, there are dynamic phenomena that are intrinsic to the specific topology of the network connections and independent of the equations. One such example is the existence of synchrony subspaces, spaces defined in terms of equalities of cell coordinates, which are flow-invariant under any dynamical system that has form consistent with the network structure. These spaces are commonly called *network synchrony subspaces*. Obviously, these spaces have a strong impact at the type of dynamics that can be observed. See an example of that in Aguiar *et al.* [2] and Field [9], where heteroclinic behaviour can occur in a robust way in asymmetric networks of dynamical systems. Moreover, one of the key elements in the occurrence of that phenomena is the existence of the network synchrony subspaces.

It is shown in [15, 11] that, given a network G and a network synchrony subspace Δ, any coupled cell system with form consistent with G, restricted to Δ, defines a new coupled cell system which is consistent with a smaller network, called the *quotient* network, say Q, of G by Δ. Moreover, any coupled cell system associated with Q can be interpreted as the restriction to Δ of some coupled cell system associated with G. We also say that G is a *lift* of Q, and in general, there are many networks G that admit a given network Q has a quotient.

In this paper we consider the process of addition of new nodes in networks within this framework. More precisely, a network G is interpreted as a network that has been obtained from Q by adding nodes (and arrows) to Q with the rule that G has to be a lift of Q, or equivalently, that Q has to be a quotient of G by some of its synchrony subspaces. Using the results in [15, 11, 8], it is described in [14, 4, 5, 1, 7] that this dynamics rule is equivalent to a network rule that we call the *lifting process*, which is similar to the cell division in cellular systems, since each additional node in the bigger network is associated with a unique node in the smaller network. In this sense, each new node is interpreted as resulting from a node's splitting in the smaller network. A lift G of a network Q is resulting from a cellular splitting from some of the cells of Q (the *splitting cells*) and all cells of G that result from these are called the *splitted cells*, provided the following Fundamental property of the splittings is satisfied. Assume that i is a cell of Q that receives k arrows from a cell j of Q. After the splitting, each splitted cell in G associated with cell i receives k cells from the set of splitted cells in G associated with the cell j.

The main question we address in this paper concerns the uniqueness in the lifting process in terms of the networks topologies. Given two networks G and Q where G is a lift of Q, we ask if the process is a one step process, or if, otherwise, the lifting process could be done sequentially in several processes of addition of cells. A lift G of a given network Q is said to be *direct* when there are no intermediate lifts of Q between them (see Definition 2.2 in Section 2). In Section 3 of this work we provide necessary and sufficient conditions for a lift of a general network to be direct. Our results characterize direct lifts using the subnetworks S and S', of all spitting cells of Q and of all splitted cells of G, respectively. We show that G is a direct lift of Q if and only if either S' consists into two copies of S or S' is a direct lift of S. See Theorems 3.1 and 3.4. These results are specialized to the class of regular uniform networks in Corollary 3.8 and to the special classes of ring networks and acyclic networks in Section 4. The lifting bifurcation problem [4, 5] concerns the comparative study of a bifurcation from a fully synchronous equilibrium in two different systems: one associated with a given (quotient) network and the other with a lift of that network. In Section 5 we illustrate one of the usages of the results obtained in Section 3 to the lifting bifurcation problem.

2. Background on networks and lifts

Following [15, 11] or [8], *networks* are graphical representations of *coupled cell systems* which are finite collections of individual dynamical systems, or *cells*, that are coupled together. These cells are often modelled using systems of ordinary differential equations. The general theory associates then to each network a class of admissible vector fields consistent with the network structure. A *regular network* is a network with only one type of cells and edges and where the number of directed edges to each cell is constant (called the *network valency*). If the network has more than one edge type then it is said to be *homogeneous* when all cells have the same type and receive the same number of inputs from each arrow type. It follows then, that an n-cell homogeneous network with $s \geq 2$ types of couplings can be interpreted as the merging of the s n-cell regular networks that are obtained when each arrow type is considered separately. We refer to these networks as the *associated regular networks*.

A *polydiagonal* is a subspace of the total phase space of the network admissible vector fields defined by the equalities of certain cell coordinates. A *network synchrony subspace* is a polydiagonal that is flow-invariant for every coupled cell system with structure consistent with the given network. Golubitsky *et al.* [15, 11] proved that every coupled cell system associated with a network when restricted to a synchrony subspace corresponds to a coupled cell system associated with a smaller network, called the *quotient network*. If Q is a quotient network of a network G then we also say that G is a *lift* of Q.

Example 2.1. Consider the 5-cell network G at the left of figure 1. Any coupled cell system consistent with the structure of G has the form

$$\left\{ \begin{array}{l} \dot{x}_1 = f(x_1; \overline{x_2, x_3}) \\ \dot{x}_2 = f(x_2; \overline{x_1, x_3}) \\ \dot{x}_3 = f(x_3; \overline{x_1, x_2}) \\ \dot{x}_4 = f(x_4; \overline{x_1, x_5}) \\ \dot{x}_5 = f(x_5; \overline{x_1, x_4}) \end{array} \right. , \tag{1}$$

where $x_i \in \mathbb{R}^k$, for some $k \geq 1$. Also, $f : \left(\mathbb{R}^k\right)^3 \to \mathbb{R}^k$ is a smooth function and the overbar indicates the invariance of f under permutation of the coordinates. The space $\Delta = \{x \in \left(\mathbb{R}^k\right)^5 : x_2 = x_4, x_3 = x_5\}$ is a synchrony subspace of the network

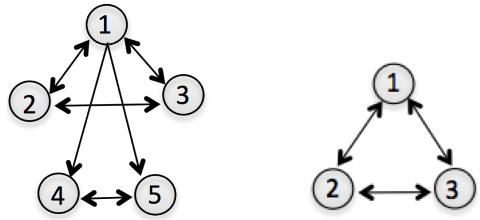

FIGURE 1. The 3-cell network Q on the right is the quotient network of the 5-cell network G on the left by the synchrony subspace $\{x : x_2 = x_4,\ x_3 = x_4\}$ (G is a lift of Q).

G. Restricting the system (1) to Δ we obtain

$$\begin{cases} \dot{x}_1 = f(x_1; \overline{x_2, x_3}) \\ \dot{x}_2 = f(x_2; \overline{x_1, x_3}) \\ \dot{x}_3 = f(x_3; \overline{x_1, x_2}) \end{cases},$$

which is a coupled cell system associated with the 3-cell bidirectional ring Q at the right of figure 1. The network Q is the quotient network of G by Δ. Equivalently, G is a lift of Q. ◊

2.1. **Lifting by splitting of cells.** In [7], a lift G of a network Q is interpreted as resulting from a *cellular splitting* from some of the cells of Q (the *splitting cells*). All cells of G that result from these are called the *splitted cells*. Moreover, G is a lift of Q if and only if the following holds:

Fundamental property of the splittings : Assume that i is a cell that receives k arrows from a cell j. After the splitting, each splitted cell associated with cell i receives k cells from the set of splitted cells associated with the cell j.

If the network is not regular, then the splitted cells must have the same type of the corresponding splitting cell and each arrow type must be considered separately. ◊

The 3-cell bidirectional ring Q on the right of figure 1 is the quotient network of the 5-cell regular network G on the left of figure 1 by the synchrony subspace $\Delta = \{x :\ x_2 = x_4, x_3 = x_5\}$. We have that G is a lift of Q by splitting both cells 2 and 3 of Q, each into exactly two cells: in Q, cells 2 and 3 are the splitting cells; in G, cells 2, 3, 4 and 5 are the splitted cells. In figure 2 we present a 3-cell network which is not regular and a 5-cell lift that results from the splitting of cell 3 followed by the splitting of cell 2.

Considering a network and a subset C of the network set of cells, the *subnetwork* consisting of all cells in C is the digraph whose set of nodes is C and whose arrows are all existing arrows in the original network between the cells in C.

2.2. **Direct lifts.** There are splittings providing lifts that can be decomposed into two (or more) sequential splittings. If it is not the case, then we define:

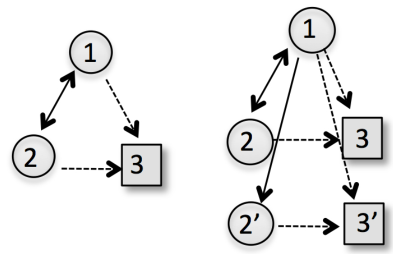

FIGURE 2. A 3-cell network which is not regular (left) and a 5-cell lift (right). The network on the right results from the splitting of cell 3 followed by the splitting of cell 2.

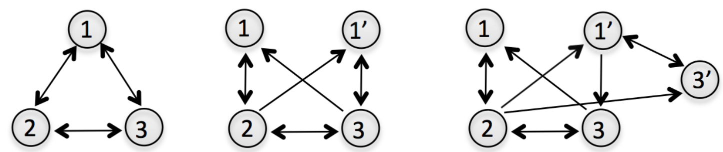

FIGURE 3. The 5-cell network on the right is a lift of the 3-cell bidirectional ring on the left which is not direct: the 4-cell network on the center is a direct lift of the 3-cell network and the 5-cell network is a direct lift of the 4-cell network.

Definition 2.2. Consider a network G with total phase space P, a synchrony subspace Δ of G and Q the corresponding quotient network of G by Δ. We say that G is a *direct lift* of Q if there is no synchrony subspace Δ' of G such that $\Delta \subsetneq \Delta' \subsetneq P$. $\Diamond$

The 5-cell network G at the left of figure 1 is a direct lift of the 3-cell bidirectional ring Q on the right – it is shown in [5] that there is no 4-cell lift of Q that lifts to G. However, the 5-cell lift of Q in figure 3, on the right, is not direct due to the existence of the intermediate 4-cell network presented in that figure, which is a lift of Q and a quotient of G. Also, the network at the right of Figure 2 is a lift of the network on the left which is not direct.

3. Direct lifts: characterization

In this section necessary and sufficient conditions are given for a lift of a general network to be direct. We start by establishing necessary conditions in terms of the subnetworks of all splitting and splitted cells.

Theorem 3.1. *Given a lift G of a network Q, consider the subnetworks S and S' consisting of all splitting cells and of all splitted cells, respectively. If G is a direct lift of Q then:*

(i) *S is strongly connected.*
(ii) *If $\#S > 1$ then in S' each cell receives and sends at least one arrow.*
(iii) *If S' is disconnected then S' consists of two copies of S.*

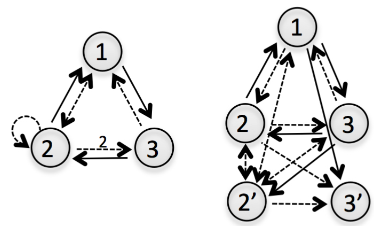

Figure 4. A 3-cell homogeneous network (left) and a 5-cell lift (right). The subnetwork of all splitting cells is strongly connected.

Proof. (i) The subnetwork S is connected, otherwise the splitting could be divided into more than one sequential splittings of each connected component separately. Suppose that this subnetwork is not strongly connected and consider a source C of the condensation of S, that is, a strongly connected component whose cells do not depend on cells from other strongly connected components. All splitted cells associated with a fixed splitting cell depend on the same non-splitting cells. Thus, the splitting can be decomposed into (at least) two splittings: one that splits firstly all cells outside C and, after that, another that splits all cells in C. So, the assumption that S is not strongly connected implies that the splitting can be done sequentially, contradicting the fact that G is a direct lift of Q. Therefore, S is strongly connected.
(ii) Due to (i), each splitted cell receives an arrow from another splitted cell. If there is more than one splitting cell and if there is a splitted cell not sending an arrow then the splitting can be done sequentially, splitting firstly the splitting cell associated with that splitted cell.
(iii) If S' is disconnected then it can be decomposed into (at least) two connected components. Because S is strongly connected, each connected component has at least one splitted cell associated with each splitting cell. So, if S' does not consist of two copies of S then there is at least one splitting cell that is splitted into more than two cells. Hence, the splitting can be done sequentially: starting to obtain two copies of S and then obtaining S' (which is different from the former because there is at least one cell that is splitted into more than two cells). This sequence contradicts the fact that G is a direct lift of Q and so, S' consists of two copies of S. □

Remark 3.2. (a) Condition (i) of Theorem 3.1 is quite restrictive. See for example Lemma 4.1.
(b) All types of couplings must be considered together in the analysis of the connectedness of the subnetwork of all splitting cells. For instance, in the splitting illustrated in figure 4, the subnetwork of all splitting cells is strongly connected. ◊

Example 3.3. In figure 5, we present a 6-cell network Q, an 11-cell lift G of Q and the corresponding subnetworks S of all splitting cells and S' of all splitted cells. A simple look at these subnetworks allows to verify that S is not strongly connected and that the splitted cell $6''$ does not send any arrow. Thus, using Theorem 3.1, G is not a direct lift of Q. We show in figure 6 two lifts G_1 and G_2 of Q such that G

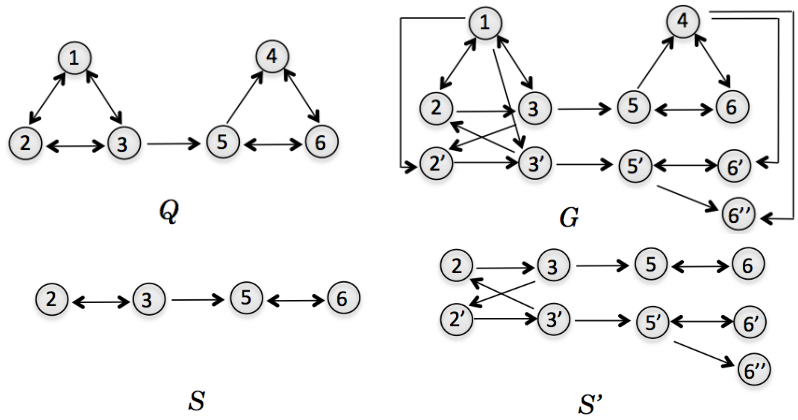

FIGURE 5. A regular network Q, a lift G of it, and the corresponding subnetworks S and S' of all splitting cells and of all splitted cells, respectively.

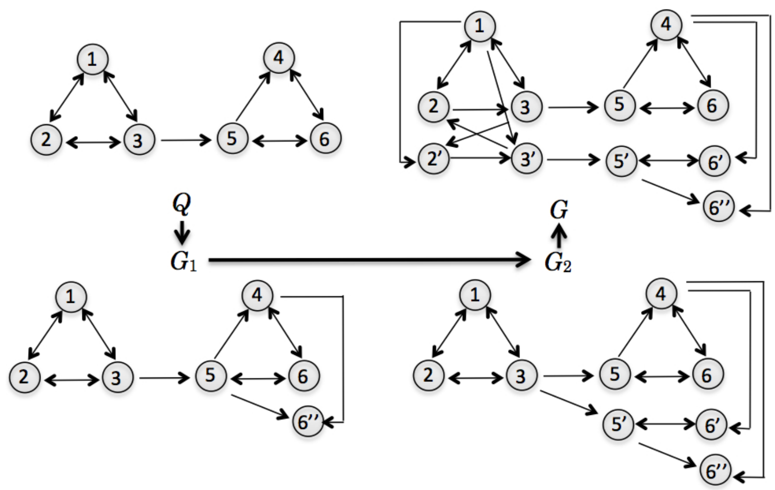

FIGURE 6. A sequence of direct lifts: G_1 is a direct lift of Q; G_2 is a direct lift of G_1; and G is a direct lift of G_2.

is a direct lift of G_2, the network G_2 is a direct lift of G_1 and G_1 is a direct lift of Q. This sequence illustrates the proof of Theorem 3.1. ◊

The following result characterizes direct lifts using the subnetworks of all spitting cells and of all splitted cells. It shows that in order to identify direct lifts, there is no need to compare the entire quotient with the entire lift.

Theorem 3.4. *Given a lift G of a network Q, consider the subnetworks S and S' consisting of all splitting cells and of all splitted cells, respectively. Then the network G is a direct lift of Q if and only if S' is either two copies of S or S' is a connected direct lift of S.*

Proof. When S' is connected, there is an intermediate lift of S between S and S' if and only if there is an intermediate lift of Q between Q and G. Thus, in this case,

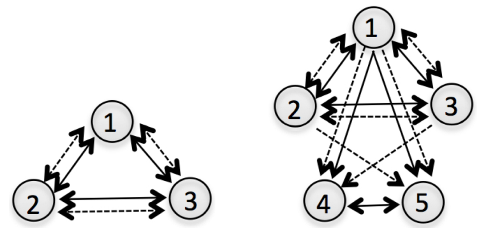

FIGURE 7. The 5-cell homogeneous network (right) is a direct lift of the 3-cell homogeneous network (left). Considering each arrow type separately, the regular lift corresponding to the solid arrow type is a direct lift of the corresponding regular quotient.

S' is a direct lift of S if and only if G is a direct lift of Q. Assume, now, that S' is disconnected. If G is a direct lift of Q then Theorem 3.1 guarantees that S' consists of two copies of S. Conversely, if S' consists of two copies of S then G is a direct lift of Q, obtained with the splitting of each cell in S into exactly two. □

Example 3.5. Taking the networks Q and G in Example 3.3 and using Theorem 3.4, we show in figure 6 a sequence of direct lifts leading from Q to G. The network G_1 is a direct lift of Q and G_2 is a direct lift of G_1 because, in both cases, the subnetwork of all splitted cells consists of two copies of the subnetwork of all splitting cells (which is a trivial strongly connected component, in both cases). Moreover, the network G is a direct lift of G_2 because the 4-ring is a direct lift of the 2-ring. ♢

Observe that if G is an homogeneous network with $s \geq 2$ different types of arrows and $G_1, \ldots, G_s$ are the associated regular networks, then a polydiagonal subspace is a synchrony subspace of G if and only if it is a synchrony subspace of $G_1, \cdots, G_s$. See Corollary 4.3 of [3].

Theorem 3.6. *Let G and Q be two homogeneous networks with $s \geq 2$ different types of arrows where Q is the quotient network of G by a synchrony subspace S. Let $G_1, \ldots, G_s$ be the associated regular networks of G, and $Q_1, \ldots, Q_s$ be the corresponding quotient networks of G_i by S. If, for some $1 \leq i \leq s$, the network G_i is a direct lift of Q_i, then G is a direct lift of Q.*

Proof. Trivially, if we consider a synchrony subspace S' of G containing properly S, then, as S' is also a synchrony subspace of G_i and G_i is a direct lift of Q_i, it follows that S' is the total phase space of G_i (and G). As a consequence, G is a direct lift of Q. □

In figure 7, it is presented a 3-cell homogeneous network and a 5-cell direct lift of it. In fact, the 5-cell regular network G obtained from this lift by considering only the solid arrow type is precisely the network depicted in figure 1 which as referred previously is a direct lift of the 3-cell bidirectional ring. Therefore, by Theorem 3.6, the 5-cell network in figure 7 is a direct lift of the 3-cell homogeneous network in the same figure.

Remark 3.7. The reciprocal of Theorem 3.6 is false. For example, consider the two homogeneous networks in figure 8. The 5-cell network is a lift of the 3-cell network

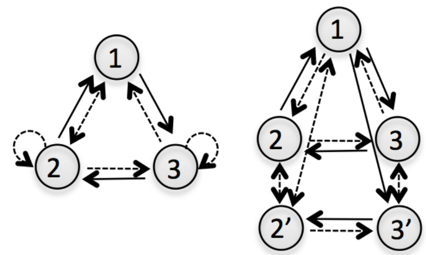

FIGURE 8. A 3-cell homogeneous network with two arrow types (left) and a 5-cell direct lift (right). Considering each arrow type separately, none of the regular lifts is a direct lift of the corresponding regular quotient.

and results from the splitting of both cells 2 and 3. This lift is direct because neither cells 2 and $2'$ have the same input set, nor do cells 3 and $3'$. Thus, the splitting can not be done sequentially and so the lift is direct. However, considering each arrow type separately, none of the regular lifts is a direct lift of the corresponding quotient. ♢

A network is said to be *uniform* when it has no loops and no multiple arrows. The following result is a corollary of Theorems 3.1 and 3.4 applied to regular uniform networks.

Corollary 3.8. *Given a lift G of a regular uniform network Q, consider the subnetworks S and S' of all splitting cells and of all splitted cells, respectively.*

(1) *If $\#S = 1$ then G is a direct lift of Q if and only if G has exactly one cell more than Q.*
(2) *If $\#S = 2$ then G is a direct lift of Q if and only if S' either consists of two copies of S or is a $(2p)$-ring, with p prime.*

Proof. The proof follows directly from Theorems 3.1 and 3.4. Recall that the network Q is uniform and so it has no loops. Also, S is the 2-ring when $\#S = 2$. □

4. Ring networks and acyclic networks

We apply the results of the previous section to characterize the direct lifts of two classes of networks: rings and acyclic networks.

Rings. For positive integers q and s, the q-*ring* and the $(q+s)$-*chain with feedback* are the networks depicted in figure 9, left and right, respectively.

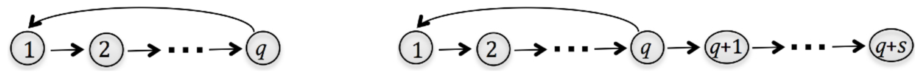

FIGURE 9. The q-ring (left) and the $(q+s)$-chain with feedback (right).

Lemma 4.1. *For a positive integer q, a (connected) direct lift of the q-ring is a $(q+1)$-chain with feedback or a (pq)-ring with p prime.*

Proof. Note that for every positive integer t, the (tq)-ring is a lift of the q-ring [13]. Among these, the direct lifts are the (pq)-rings, with p prime. In fact, if $t = pl$, for some positive integers p and l, with p prime, then we can split all cells of the (pq)-ring into exactly l cells and obtain the (plq)-ring, that is, the (tq)-ring. Also, for every positive integer s, the $(q + s)$-chain with feedback is a lift of the q-ring. Among these, only the case $s = 1$ is direct. Indeed, if $s > 1$ then the $(q + s)$-chain with feedback can be lifted from the $(q + s - 1)$-chain with feedback as follows: in this last network, split cell q, which is the unique cell having two outputs, into exactly two cells in such a way that one of the splitted cells sends an arrow to cell 1 and the other splitted cell sends an arrow to cell $(q + 1)$. Thus, the $(q + 1)$-chain with feedback and the (pq)-rings are direct lifts of the q-ring, with p prime.

Using now condition (i) of Theorem 3.1, we show that the unique forms of (connected) direct lifts of the q-ring are the $(q + 1)$-chains with feedback and the (pq)-rings, with p prime. Since the subnetwork of all splitting cells S has to be strongly connected, we have only two possible situations: S either consists of a unique cell or it is the whole network. In the first case, as we proved in Corollary 3.8, we obtain only the $(q+1)$-chain with feedback as direct lift. In the second situation, we obtain (tq)-rings as possible lifts, for every positive integer t. □

Remark 4.2. The result of Lemma 4.1 can be extended to valency-1 regular networks. Indeed, these networks have a unique nontrivial strongly connected component, which is a unique q-cycle, with $q \geq 1$ [13]. Therefore, they have only two types of connected direct lifts: those that result from splitting the q-cycle into a (pq)-cycle, with p prime, and those that result from splitting exactly one cell into two cells (which is equivalent to add a unique vertex with 0 outdegree). ♢

Directed acyclic networks. A *(directed) acyclic network* is a network having no cycles (in particular, it has no self-loops). An example of a connected acyclic network (a *directed tree*) is given in figure 10.

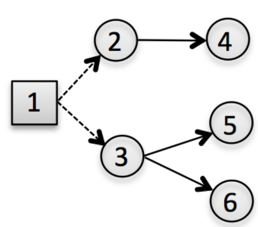

FIGURE 10. An acyclic network.

Lemma 4.3. *A lift of an acyclic network is acyclic.*

Proof. If G is a lift of an acyclic network Q and it contains a k-cycle, say, $i_1 \to i_2 \to \cdots \to i_k \to i_1$, then we obtain the path $[i_1] \to [i_2] \to \cdots \to [i_k] \to [i_1]$ in Q, where $[i]$ denotes the splitting cell associated with cell i if this is a splitted cell, and denotes cell i if this is a non-splitted cell. Moreover, this path contains a p-cycle with $1 \leq p \leq k$. If $p = 1$ then Q has a self-loop. Then, if Q is acyclic then G is also acyclic. □

Proposition 4.4. *Given a lift G of an acyclic network Q, if the lift is direct then the subnetwork S consists of a unique splitting cell.*

Proof. If a directed graph is acyclic then it has no strongly connected subnetworks with more than one vertex. Now, by Theorem 3.1, S is strongly connected. Thus, S consists into a unique cell. □

Corollary 4.5. *Let G be a lift of an acyclic network Q. Suppose G has n cells and Q has k cells, where $n > k$. Then there are networks $G_k = Q, G_{k+1}, \cdots, G_n = G$ such that G_{k+j} has $k+j$ cells and it is a direct lift of G_{k+j-1}, for $j = 1, \cdots, n-k$.*

Proof. By Proposition 4.4, for any direct lift of Q, the subnetwork S consists of a unique cell. By Theorem 3.4, S' has to consist into two copies of S, that is, the disconnected graph consisting of only two cells. Thus, any direct lift of Q has $k+1$ cells and, by Lemma 4.3 is also acyclic. Now, applying the argument, recursively, it follows the result. □

Remark 4.6. A similar result can be stated for a lift of a nontrivial valency-1 regular network, that preserves the number of cells in the cycle. ♢

5. Direct lifts and the lifting bifurcation problem

Given an n-cell regular network G, consider a 1-parameter system of ordinary differential equations

$$\dot{x} = F(x, \lambda), \tag{2}$$

representing a coupled cell system with structure consistent with G and depending at a (real) bifurcation parameter λ. If k is the dimension of the internal dynamics then $x = (x_1, \ldots, x_n) \in (\mathbb{R}^k)^n$ and we consider that $F : (\mathbb{R}^k)^n \times \mathbb{R} \to (\mathbb{R}^k)^n$ is smooth.

Suppose that there exists a synchronous equilibrium in the *fully synchronous subspace* $\{x : x_1 = \cdots = x_n\}$, which we assume, after a change of coordinates, to be the origin for $\lambda = 0$, that is, $F(0,0) = 0$. Codimension-one local bifurcations of (2) divide into *steady-state* and *Hopf bifurcations*, depending on when the Jacobian $J = (dF)_{0,0}$ has a zero eigenvalue or a pair of purely imaginary eigenvalues, respectively. Moreover, each of these bifurcation types divide into *synchrony-preserving* and *synchrony-breaking*, depending whether the center subspace is contained or not, respectively, in the fully synchronous subspace.

The eigenvalues of the Jacobian J are directly related with the eigenvalues of the network adjacency matrix [12, 10]. More precisely, if $\mu_1, \ldots, \mu_n$ are the eigenvalues of the network adjacency matrix then the kn eigenvalues of the Jacobian J are the union of the eigenvalues of the $k \times k$ matrices $\alpha + \beta\mu_i$, for $1 \leq i \leq n$, where α is the $k \times k$ matrix of the linearized internal dynamics at the origin and β is the $k \times k$ matrix of the linearized coupling at the origin, both matrices found by differentiating F. In [10] it is proved that when $k \geq 2$, then generically the center subspace at a synchrony breaking bifurcation is isomorphic to the real part of a generalized eigenspace of the network adjacency matrix. Note that for $k = 1$, every eigenvalue of the Jacobian J has the form $\alpha + \beta\mu_i$, where now $\alpha, \beta \in \mathbb{R}$, and so, two eigenvalues $\alpha + \beta\mu_i$ and $\alpha + \beta\mu_j$ of the Jacobian J lie on the imaginary axis if and only if the eigenvalues μ_i and μ_j of the adjacency matrix have the same real part.

All eigenvalues of the adjacency matrix of a regular network are also eigenvalues of the adjacency matrix of any of its lifts, including multiplicities [15, 11]. Hence, if n and m are the number of cells in Q and G, respectively, then all (complex) eigenvalues of A_G are precisely the n (complex) eigenvalues of A_Q together with more $m - n$ (complex) eigenvalues, to which we call *extra eigenvalues*.

In [4, 5], it is addressed the issue concerning a comparative study of a bifurcation from a fully synchronous equilibrium in two different systems: one associated with a given (quotient) network and the other with a lift of that network. Assuming a bifurcation occurs for a coupled cell system restricted to a fixed (quotient) network, examples are given where new bifurcating solution branches occur for some lifts. Moreover, a necessary condition for that to happen, in the general setup, is the increasing of the dimension of the center subspace of J comparatively to $J|_\Delta$. It follows then that the issue of preserving or not the number of eigenvalues of J comparatively to $J|_\Delta$ in the imaginary axis is so translated, in terms of the preservation or not of the number of eigenvalues with a specific real part of the network adjacency matrices of G and Q. From the point of view of bifurcations of coupled cell networks, it is then of interest to compare the spectrum of the adjacency matrices of the lifts and of Q. Using the results of Section 3, we describe a method for listing all extra eigenvalues of direct lifts with a prescribed additional number of cells of regular uniform networks.

We recall a useful result that simplifies the calculation of the extra eigenvalues by considering *just* the subnetwork of all splitted cells: Given a lift G of a regular network Q, consider the subnetworks S and S' of all splitting cells and of all splitted cells, respectively. It is proved in [7] that the extra eigenvalues of G with respect to Q are precisely the extra eigenvalues of S' with respect to S.

Our first observation in this section concerns regular networks with common strongly connected subnetworks:

Proposition 5.1. *Regular networks having in common a strongly connected subnetwork admit lifts with the same extra eigenvalues, including multiplicities. Moreover, if they have the same strongly connected subnetworks then their direct lifts have exactly the same extra eigenvalues.*

Proof. If two networks have in common a strongly connected subnetwork then it is possible to split the cells of this subnetwork, in both networks, precisely in the same way, any finite number of times. The second part follows from the fact that, by Theorem 3.1, the direct lifts result from the splitting of a unique strongly connected subnetwork. In both cases, as the extra eigenvalues of G with respect to Q are precisely the extra eigenvalues of S' with respect to S, we have that the corresponding lifts have the same extra eigenvalues, including multiplicities. □

Example 5.2. Consider the 4-cell and the 10-cell networks of figure 11. There are lifts of both these two networks which have the same extra eigenvalues. This follows from the fact that they have common strongly connected subnetworks. Moreover,

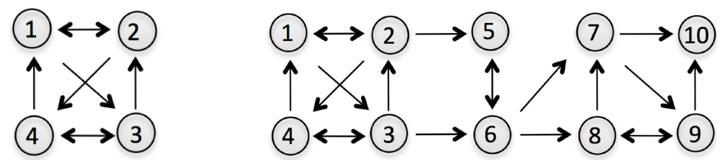

FIGURE 11. Regular networks admitting lifts with the same extra eigenvalues. Moreover, the direct lifts of these networks have precisely the same extra eigenvalues.

as they have precisely the same strongly connected subnetworks, their direct lifts also have the same extra eigenvalues, including multiplicities. ◊

It also follows directly from Corollary 3.8 the next observation:

Corollary 5.3. *Given a direct lift G of a regular uniform network Q, consider the subnetworks S and S' of all splitting cells and of all split cells, respectively. We have that:*
(i) If $\#S = 1$ then the extra eigenvalue is 0.
(ii) If $\#S = 2$ then either S' consists of two copies of S and so the extra eigenvalues are the roots of $x^2 - 1$, or S' is a $(2p)$-ring, with p prime, and the extra eigenvalues are the complex roots of $(x^{2p} - 1)/(x^2 - 1)$.

Method for listing all extra eigenvalues in direct lifts. The main results of Section 3 allow to list all extra eigenvalues of direct lifts of a given regular uniform network Q, considering that it is fixed the maximum number n of additional cells in these lifts. For each $1 \leq i \leq n$, it is possible to obtain all extra eigenvalues of direct lifts with exactly i cells more than Q proceeding as follows:

(1) Consider all possible configurations of strongly connected subnetworks S of all splitting cells, having at most i cells;
(2) For each configuration distinct from Q (if Q is one of the possible configurations), calculate the corresponding eigenvalues.
(3) For each configuration, find all possible connected direct lifts with exactly i additional cells and calculate the corresponding extra eigenvalues.

The eigenvalues obtained in steps (2)-(3) of this method are precisely the extra eigenvalues of direct lifts having at most n cells more than Q. ◊

Remark 5.4. Notice that if the number of cells in S is small then the above calculations are simpler. Indeed, for example, due to Corollary 3.8, the case $\#S = 1$ just leads to one direct lift, and the case $\#S = 2$ just leads to direct lifts with 2 or $2p$ additional cells, with p prime. Clearly, a higher number of configurations of S implies a higher number of forms for S' and of calculations that are involved. ◊

Example 5.5. Consider the 4-cell regular uniform network in figure 12. There are only two possible forms of nontrivial strongly connected subnetworks, namely, the two subnetworks that are depicted in the same figure (up to relabeling cells). Suppose we consider the possible direct lifts of this 4-cell network having at most three more cells. It is easy to list all possible connected direct lifts of these two

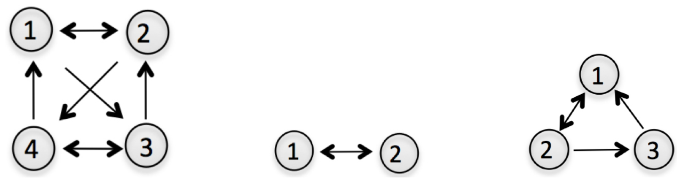

FIGURE 12. A 4-cell network (left) and the unique two possible forms of nontrivial strongly connected subnetworks (center and right), up to relabeling cells.

forms of subnetworks having at most three cells more, and conclude that the extra eigenvalues are the complex roots of the following polynomials: x, $x^2 \pm 1$ and $x^3 \pm 1$ and $x^3 \pm x \pm 1$. $\Diamond$

List of all extra eigenvalues in lifts of a general regular uniform network. The previous method also allows to obtain the list of all extra eigenvalues in direct lifts of a general regular uniform network (admitting that a finite maximum number of cells is fixed), and thus, it allows to obtain the list of all extra eigenvalues of lifts of a general regular uniform network. In fact, if G is a lift of Q, it is possible to construct a chain of lifts of Q, say G_i for $i = 0, \ldots, k+1$, where $G_0 = Q$, $G_{k+1} = G$ and for all $1 \leq j \leq k$, the network G_{j+1} is a direct lift of G_j. Therefore, the extra eigenvalues of G with respect to Q are the union of the extra eigenvalues of G_{j+1} with respect to G_j. $\Diamond$

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