Invariant Theory for Wreath Product Groups

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Abstract. Invariant theory is an important issue in equivariant bifurcation theory. Dynamical systems with wreath product symmetry arise in many areas of applied science. In this paper we develop the invariant theory of wreath product $\mathcal{L} \wr \mathcal{G}$ where $\mathcal{L}$ is a compact Lie group (in some cases, a finite group) and $\mathcal{G}$ is a finite permutation group. For compact $\mathcal{L}$ we find the quadratic and cubic equivariants of $\mathcal{L} \wr \mathcal{G}$ in terms of those of $\mathcal{L}$ and $\mathcal{G}$. These results are sufficient for the classification of generic steady-state branches, whenever the appropriate representation of $\mathcal{L} \wr \mathcal{G}$ is 3-determined. When $\mathcal{L}$ is compact we also prove that the Molien series of $\mathcal{L}$ and $\mathcal{G}$ determine the Molien series of $\mathcal{L} \wr \mathcal{G}$. Finally we obtain 'homogeneous systems of parameters' for rings of invariants and modules of equivariants of wreath products when $\mathcal{L}$ is finite.

1 Introduction

Equivariant bifurcation theory studies the existence and stability of bifurcating branches of steady or periodic solutions of nonlinear dynamical systems with symmetry group Γ. Usually Γ is a compact Lie group, acting linearly on a real vector space V, and the emphasis is on the symmetry group of each bifurcating branch, which typically forms a proper subgroup of Γ. Existing methods rely heavily on being able to determine the equivariant polynomial mappings and invariant polynomial functions for the action of Γ on V, at least up to some suitable degree — cubic order being especially common. The methods used to do

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this have mostly been 'bare hands' calculations for particular groups and representations. Computations can be cumbersome — especially for representations of high dimension. In practice, the results are often limited to equivariants of low degree.

Fortunately, low degree equivariants are sometimes sufficient for the bifurcation analysis. Field *et al.* [7, 8, 9, 10] have developed a theory of 'determinacy' of group representations which guarantees, in some cases, that the picture obtained by truncating the Taylor series to low degree remains accurate if higher degree terms are included — at least for generic branches. However, there are many instances where determinacy either fails, or seems difficult to prove.

Some general algorithms, suitable for implementation within computer algebra packages, have recently been developed [21, 22] for calculating invariants and equivariants, mainly for finite matrix groups. Important ingredients in such algorithms are the use of Gröbner bases, the Cohen-Macaulay property, and the Molien series of the ring of invariants and the module of equivariants. The groups involved are generally assumed to be finite because of the difficulty of representing an infinite group as an exact data structure in a computer.

In this paper we investigate the invariant theory of a class of group representations that arises naturally in the dynamics of systems of coupled nonlinear oscillators (throughout, 'invariant theory' includes both invariant functions and equivariant mappings). These *wreath product* representations were introduced by Golubitsky, Stewart and Dionne [11] and Dionne, Golubitsky and Stewart [5, 6]. The wreath product $\mathcal{L} \wr \mathcal{G}$ is the semidirect product of a number of copies of a 'local' group $\mathcal{L}$, which is a compact Lie group, and a finite 'global' group $\mathcal{G}$ which acts by permuting the copies of $\mathcal{L}$. Applications of wreath product symmetry to dynamical systems include Josephson junction arrays, gauge theory, molecular dynamics, and crystallography: a more extensive list is given in [11]. Other recent papers on bifurcations for wreath product systems include Golubitsky *et. al* [11], Dionne *et. al* [5], and Dias *et. al* [2, 3, 4].

In essence, we shall prove that useful information about the invariants and equivariants of $\mathcal{L} \wr \mathcal{G}$ can be read off in a simple manner from similar information about $\mathcal{L}$ and $\mathcal{G}$. In particular this is the case for quadratic and cubic equivariants, see propositions 3.5 and 3.7. It does not seem possible to describe higher degree invariants and equivariants in such an explicit manner: we describe the complexity involved in such calculations in corollaries 7.11 and 7.12. Although our results do not give a complete description of the invariants and equivariants, they are sufficient for one important case: bifurcation problems with wreath product symmetry that are determined by the third order Taylor expansion of equivariant vector fields — '3-determined' representations in the sense of Field *et al.* [7, 8, 9, 10].

The aforementioned algorithms of Sturmfels [21] and Worfolk [22] make use of the Molien series and of the Cohen-Macaulay property of the corresponding graded ring or module. Their results are developed mainly for finite groups. In theorems 6.4 and 6.7 we obtain formulas for the Molien series of $\mathcal{L} \wr \mathcal{G}$ in terms of Molien series for $\mathcal{L}$ and $\mathcal{G}$. Molien series are generating functions for the dimensions of the spaces of invariants or equivariants of a given degree. In other words, the dimensions of the spaces of invariants or equivariants of degree d can be computed explicitly in terms of the dimensions of the spaces of invariants or equivariants of degree $\leq d$ for $\mathcal{L}$ and for $\mathcal{G}$. We make use of the well known formula for the Molien series as an integral over the group. The invariant case of this formula was originally proved by Molien [16], and the equivariant case by Sattinger [17]. These results give explicit expressions for the relevant Hilbert series in terms of the matrices by which the symmetry group Γ acts. For example Molien's theorem states that the Hilbert series of the ring of invariants is the average (with respect to Haar measure) of the reciprocals of characteristic polynomials of all group elements. At root the results for wreath products are possible because the invariant theory of $\mathcal{L} \wr \mathcal{G}$ is that of $\mathcal{L}$ 'averaged' over $\mathcal{G}$, and this paper constitutes a precise formalisation of this simple remark.

The equivariant mappings for a group action form a module over the ring of invariant functions. When $\mathcal{L}$ is finite, we develop a detailed theory along the lines of Worfolk [22], using the Cohen-Macaulay property of graded rings and modules to organise the structure of the invariants and equivariants, see section 7. In particular we obtain information about 'homogeneous systems of parameters' (h.s.o.p.), which are fundamental to the classification of invariants and equivariants. Specifically, we relate h.s.o.p's of $\mathcal{L} \wr \mathcal{G}$ to h.s.o.p's of $\mathcal{L}$ and $\mathcal{G}$. Such information is useful for the algorithmic approach to computing invariants and equivariants.

The organisation of the paper is as follows. In section 2 we review a few concepts and results concerning linear actions of compact Lie groups on finite-dimensional vector spaces, rings of invariants, and modules of equivariants for those actions. In section 3 we present two of our main results. We introduce a particular class of representations of wreath products, to which our results will apply. We also present a result of Field [7] for wreath products $\mathcal{L} \wr \mathcal{G}$, where the representation of the group of the internal symmetries $\mathcal{L}$ is 'radial'. In this case the ring of invariants and the module of equivariants for $\mathcal{L} \wr \mathcal{G}$ can be completely described from the corresponding information for $\mathcal{L}$ and $\mathcal{G}$.

Next we consider more complicated cases, in which the representation of $\mathcal{L}$ is not radial. The main results of this section are propositions 3.5 and 3.7, which describe the third order terms of general $\mathcal{L} \wr \mathcal{G}$-equivariant vector fields. In section 4 we illustrate these results for a few examples. Section 5 contains the proofs of propositions 3.5 and 3.7. The main results of section 6 are theorems 6.4 and 6.7, which describe the Molien series and equivariant Molien series for wreath products $\mathcal{L} \wr \mathcal{G}$ in terms of the Molien series and equivariant Molien series for $\mathcal{L}$ and $\mathcal{G}$.

Finally, section 7 concerns only finite wreath products, that is, the case when $\mathcal{L}$ is a finite group. In section 7.1 we review some elementary results concerning commutative algebra and combinatorial theory. In section 7.2 we obtain in theorem 7.10, a set of primary invariants for $\mathcal{L} \wr \mathcal{G}$ in terms of primary invariants for $\mathcal{L}$ and for $\mathcal{G}$. By *primary invariants* we mean a set S of polynomial invariants such that the ring (module) of invariants (equivariants) under the group in question is generated freely over the subring generated by S. Corollaries 7.11 and 7.12 estimate the number of invariants and equivariants we need further to generate the all ring (module).

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2 Preliminaries

Throughout sections 2, 3, 4 and 5 we consider real vector spaces. Let V be a finite-dimensional real vector space and Γ a compact Lie group acting linearly and (without loss of generality) orthogonally on V. Recall that a polynomial function $p : V \to \mathbf{R}$ is *invariant* under Γ if $p(\gamma \cdot v) = p(v)$ for all $\gamma \in \Gamma$, and a polynomial mapping $g : V \to V$ is *equivariant* under Γ if $g(\gamma \cdot v) = \gamma . g(v)$ for all $\gamma \in \Gamma$. In applications to bifurcation theory, g is a truncation of the Taylor series of a smooth equivariant vector field, but in this paper we focus only on polynomial functions and mappings in the abstract.

Denote by $\mathcal{P}_V(\Gamma)$ the ring of Γ-invariant polynomials. When Γ is compact the Hilbert-Weyl theorem (theorem [12] XII 4.2) states that there exists a finite set of invariant polynomials $u_1, \ldots, u_s$ such that every invariant polynomial may be written as a polynomial function of $u_1, \ldots, u_s$. Any such set is a *Hilbert basis* for $\mathcal{P}_V(\Gamma)$, and it is *minimal* if no proper subset is a Hilbert basis. Denote by $\mathcal{P}_V^d(\Gamma)$ the vector space of homogeneous Γ-invariant polynomials of degree d.

Let $\vec{\mathcal{P}}_V(\Gamma)$ be the space of Γ-equivariant polynomial mappings from V to V, which is a module over $\mathcal{P}_V(\Gamma)$. We say that $g_1, \ldots, g_r \in \vec{\mathcal{P}}_V(\Gamma)$ *generate* $\vec{\mathcal{P}}_V(\Gamma)$ over $\mathcal{P}_V(\Gamma)$ if every Γ-equivariant g may be written as

$$g = f_1 g_1 + \cdots + f_r g_r \tag{1}$$

for Γ-invariant polynomials $f_1, \ldots, f_r$. For Γ compact there exists a finite set of generators for $\vec{\mathcal{P}}_V(\Gamma)$ (theorem [12] XII 5.2). If $f_1 g_1 + \cdots + f_r g_r \equiv 0$ implies that $f_1 \equiv \cdots \equiv f_r \equiv 0$, then we say that $g_1, \ldots, g_r$ *freely generate* $\vec{\mathcal{P}}_V(\Gamma)$ and $\vec{\mathcal{P}}_V(\Gamma)$ is a *free* module over $\mathcal{P}_V(\Gamma)$. In this case, the expression (1) is unique. Denote by $\vec{\mathcal{P}}_V^d(\Gamma)$ the vector space of homogeneous Γ-equivariant polynomials maps of degree d.

Since the homogeneous parts of an invariant polynomial are themselves invariant, we can always choose a *homogeneous* Hilbert basis — one that consists of homogeneous polynomials. Similarly, the g_i for $i = 1, \ldots, p$ can be chosen to have homogeneous components of the same degree. This fact is important for this paper.

In general, for a given group Γ, it may not be easy to find explicit generators $u_1, \ldots, u_s$ and $g_1, \ldots, g_p$. However, it is sometimes sufficient to know the form of a Γ-equivariant mapping up to some degree. The validity of such a truncation relies on the concept of *determinacy.* The idea of determinacy is that if the Taylor expansion of a bifurcation problem is known to degree d, and if certain nondegeneracy conditions are satisfied, then the terms of degree greater than d do not affect the branching pattern. See Field [7, 8] and Field and Richardson [9, 10]. Indeed, several examples of wreath product representations are 3-determined, so degree three truncation gives a complete picture of generic steady-state bifurcations. For example Field [7] shows that this is the case for the group $\mathbf{Z}_2 \wr \mathbf{S}_N$ acting on $\mathbf{R}^N$, where $\mathbf{Z}_2$ acts on $V = \mathbf{R}$ by multiplication by ∓ 1. This is the Weyl group of type B_N, denoted by $W(B_N)$, and it is a reflection group. More generally, Field [7] proves that if V is a *radial* representation for $\mathcal{L}$ (see below for the definition) and $\mathcal{G}$ is a transitive group of $\mathbf{S}_N$, then V^N is 3-determined for $\mathcal{L} \wr \mathcal{G}$.

2.1 Group Action for Wreath Products

Finally we give a precise definition of wreath product, and the class of representations that we study. Suppose that $\mathcal{L} \subseteq \mathbf{O}(k)$, so that there is an action of $\mathcal{L}$ on $V = \mathbf{R}^k$. Assume that $\mathcal{G}$ is a subgroup of $\mathbf{S}_N$. Then we can define $\Gamma = \mathcal{L} \wr \mathcal{G}$ on $W = V^N$ in terms of the actions

$$(l, \sigma).(v_1, \ldots, v_N) = (l_1 \cdot v_{\sigma^{-1}(1)}, \ldots, l_N \cdot v_{\sigma^{-1}(N)})$$

for $l = (l_1, \ldots, l_N) \in \mathcal{L}^N$, $\sigma \in \mathcal{G}$ and $(v_1, \ldots, v_N) \in V^N$. Here the permutations act on $l \in \mathcal{L}^N$ by

$$\sigma(l) = (l_{\sigma^{-1}(1)}, \ldots, l_{\sigma^{-1}(N)}).$$

It follows that the group multiplication in $\mathcal{L} \wr \mathcal{G}$ is given by

$$(h, \tau)(l, \sigma) = (h\tau(l), \tau\sigma).$$

Thus, the homomorphism that sends the permutation group $\mathcal{G}$ into the automorphism group $A(\mathcal{L}^N)$ takes a permutation $\sigma \in \mathcal{G}$ into an automorphism $a_\sigma : \mathcal{L}^N \to \mathcal{L}^N$ defined by $a_\sigma(l_1, \ldots, l_N) = (l_{\sigma^{-1}(1)}, \ldots, l_{\sigma^{-1}(N)})$.

3 Invariant Theory for Wreath Products

We now begin to address the relationship between $\mathcal{P}_{V^N}(\mathcal{L} \wr \mathcal{G})$, $\vec{\mathcal{P}}_{V^N}(\mathcal{L} \wr \mathcal{G})$ and $\mathcal{P}_V(\mathcal{L})$, $\mathcal{P}_{\mathbf{R}^N}(\mathcal{G})$, $\vec{\mathcal{P}}_V(\mathcal{L})$, $\vec{\mathcal{P}}_{\mathbf{R}^N}(\mathcal{G})$. First, we review some results of Field [7] about invariant theory for $\mathcal{L} \wr \mathcal{G}$ when the representation for $\mathcal{L}$ is 'radial'. In this case, the theory for $\mathcal{L} \wr \mathcal{G}$ can be completely described in terms of those for $\mathcal{L}$ and $\mathcal{G}$.

3.1 Radial Representations

First we recall the definition of radial representations:

Definition 3.1

Let $\mathcal{L}$ be a compact subgroup of $\mathbf{O}(V)$. The representation $(V, \mathcal{L})$ is *radial* if the module $\vec{\mathcal{P}}_V(\mathcal{L})$ over the ring $\mathcal{P}_V(\mathcal{L})$ is free with basis the identity map of V.

Remark 3.2

If $(V, \mathcal{L})$ is radial, then $\mathcal{P}_V(\mathcal{L})$ is generated by $\|x\|^2$ and $(V, \mathcal{L})$ is absolutely irreducible. The representation $(V, \mathcal{L})$ is absolutely irreducible because the $\mathcal{L}$-equivariants are generated over $\mathcal{P}_V(\mathcal{L})$ by the identity map Id_V on V. Thus the only matrices that are equivariant under $\mathcal{L}$ are the scalar multiples of Id_V. Since $\mathcal{L} \subseteq \mathbf{O}(V)$, the norm $\|x\|^2$ is always an $\mathcal{L}$-invariant. If there was another non-constant $\mathcal{L}$-invariant, algebraically independent from the norm, then the basis for $\vec{\mathcal{P}}_V(\mathcal{L})$ would not contain only the identity map of V, because the gradient of an $\mathcal{L}$-invariant polynomial gives an $\mathcal{L}$-equivariant mapping.

Examples of radial representations include $\mathbf{O}(p)$ (for $p \geq 1$) and $\mathbf{SO}(p)$ (for $p \geq 3$) in their standard actions on $\mathbf{R}^p$.

The next result is from Field [7]. The notation $A \approx B$ denotes that A and B are isomorphic.

Proposition 3.3 *Suppose that $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$, with $N \geq 1$, and $(V, \mathcal{L})$ is radial. Set $\Gamma = \mathcal{L} \wr \mathcal{G}$ and $\Lambda = \mathbf{Z}_2 \wr \mathcal{G}$. Denote by $\mathcal{G}_1$ the subgroup of $\mathcal{G}$ containing the permutations fixing index 1. Then*
(a) $\mathcal{P}_{V^N}(\Gamma) \approx \mathcal{P}_{\mathbf{R}^N}(\Lambda) \approx \mathcal{P}_{\mathbf{R}^N}(\mathcal{G})$.
(b) $\vec{\mathcal{P}}_{V^N}(\Gamma) \approx \vec{\mathcal{P}}_{\mathbf{R}^N}(\Lambda) \approx \vec{\mathcal{P}}_{\mathbf{R}^N}(\mathcal{G}_1)$.

Proof. See [7] pp 22. For example, for the rings of invariants, if we denote coordinates on $\mathbf{R}^N$ by $(x_1, \ldots, x_N)$ and on V^N by $(X_1, \ldots, X_N)$, then each element of $\mathcal{P}_{\mathbf{R}^N}(\Lambda)$ may be written as a polynomial $p(x_1^2, \ldots, x_N^2)$, where p is $\mathcal{G}$-invariant. Each p determines the Γ-invariant polynomial $p(\|X_1\|^2, \ldots, \|X_N\|^2)$ and every Γ-invariant polynomial can be written uniquely in this form. $\square$

3.2 Simplifying Assumptions

In order to simplify the presentation of our results, we shall impose two restrictions on the wreath product. We assume

- The action of $\mathcal{G}$ is transitive.
- The representation of $\mathcal{L}$ on V is absolutely irreducible (and nontrivial).

Recall that a group $\mathcal{G}$ of permutations of $\{1,\ldots,N\}$ is *transitive* if given any $i,\ j \in \{1,\ldots,N\}$, there exists $\sigma \in \mathcal{G}$ such that $\sigma(i) = j$. (It is sufficient to establish this property when $i = 1$.) A real representation V of $\mathcal{L}$ is *absolutely irreducible* if the only linear equivariants are scalar multiples of the identity. Absolute irreducibility implies irreducibility, but not conversely — see Golubitsky *et al.* [12] chapter XII section 3.

Results similar to those presented here can be obtained without these assumptions, by applying the same methods, but they are cumbersome to state and the extra complexities obscure the ideas. Moreover, in applications to equivariant bifurcation theory, both of the above assumptions usually hold. In particular, generic steady-state bifurcations can be Liapunov-Schmidt reduced on to an absolutely irreducible representations. See Golubitsky *et al.* [12] chapter XIII section 3 proposition 3.2.

When $\mathcal{G}$ is intransitive, its action on $\{1,\ldots,N\}$ can be decomposed into disjoint orbits. The theory developed here can then be applied to each orbit separately, and the results can be reassembled to complete the analysis. When V is not absolutely irreducible, the list of linear equivariants is more extensive, but it can be determined from the isotypic decomposition of V, Golubitsky *et al.* [12] chapter XII section 3.

Remarks 3.4

If $\mathcal{G}$ is a transitive subgroup of $\mathbf{S}_N$, then the space of linear invariants is spanned by $x_1 + \cdots + x_N$.

The assumption that V is absolutely irreducible has the following consequences:

(a) Since $\mathrm{Fix}_{\mathrm{V}}(\mathcal{L})$ is a $\mathcal{L}$-invariant space, then it follows that

$$\mathrm{Fix}_{\mathrm{V}}(\mathcal{L}) = \{0\} \tag{2}$$

and so

$$\mathrm{Fix}_{\mathrm{V^N}}(\mathcal{L} \wr \mathcal{G}) = \{0\}. \tag{3}$$

By proposition [12] XIII 2.2, the only linear $\mathcal{L} \wr \mathcal{G}$-invariant function is the zero function. Recall that, if Σ is a subgroup of $\mathcal{L}$, then $\mathrm{Fix}_{\mathrm{V}}(\Sigma)$, the *fixed-point subspace* of Σ, is formed by the elements of V that are fixed by $\mathcal{L}$. The spaces $\mathrm{Fix}_{\mathrm{V}}(\Sigma)$ are always $\mathcal{L}$-invariant.

(b) Dionne *et al.* [5] prove that $\Gamma = \mathcal{L} \wr \mathcal{G}$ acts absolutely irreducible on V^N if and only if $\mathcal{L}$ acts absolutely irreducibly on V.

3.3 Hilbert Bases and Generators

We now state the main results of this section. Proofs will be given in section 5. In section 4 we present some examples illustrating the results. Throughout we assume nontrivial $\mathcal{L}$-representations.

Proposition 3.5 *Let $(V,\mathcal{L})$ be absolutely irreducible. Let $\mathcal{G}$ be a transitive subgroup of $\mathbf{S}_N$. Consider (V^N,Γ) where $\Gamma = \mathcal{L}\wr\mathcal{G}$ and the action of Γ on V^N is defined in section 2.1.*

Consider a minimal homogeneous Hilbert basis for $\mathcal{P}_V(\mathcal{L})$. Denote by $u_1,\ldots,u_s$ the degree two elements of this basis and by $w_1,\ldots,w_p$ those of degree three.

Consider a minimal homogeneous Hilbert basis for $\mathcal{P}_{\mathbf{R}^N}(\mathcal{G})$ and denote by f_1 the element of this basis of degree one.

Then

$$\{f_1(u_j(v_1),\ldots,u_j(v_N));\ j=1,\ldots s\}$$

generates $\mathcal{P}^2_{V^N}(\Gamma)$, and

$$\{f_1(w_j(v_1),\ldots,w_j(v_N));\ j=1,\ldots p\}$$

generates $\mathcal{P}^3_{V^N}(\Gamma)$.

For the proof see section 5.

Corollary 3.6 *With the conditions of proposition 3.5, if $\mathcal{G}=\mathbf{S}_N$, then*

$$\{u_j(v_1)+\cdots+u_j(v_N);\ j=1,\ldots,s\}$$

generates $\mathcal{P}^2_{V^N}(\mathcal{L}\wr\mathbf{S}_N)$ and

$$\{w_j(v_1)+\cdots+w_j(v_N);\ j=1,\ldots,p\}$$

generates $\mathcal{P}^3_{V^N}(\mathcal{L}\wr\mathbf{S}_N)$.

Proof. Since $\mathcal{P}^1_{\mathbf{R}^N}(\mathbf{S}_N)$ is generated for example by $x_1+\ldots+x_N$, the result follows from proposition 3.5. $\Box$

In the statement of the next proposition we use the notation $x\bullet y$ with $x\in\mathbf{R}^N$ and $y\in V^N$, for the vector in V^N with components x_iy_i. Recall that $\mathcal{G}_1$ denotes the subgroup of $\mathcal{G}$ containing the permutations fixing the index 1.

Proposition 3.7 *Let $(V,\mathcal{L})$ be absolutely irreducible. Let $\mathcal{G}$ be a transitive subgroup of $\mathbf{S}_N$. (a) Suppose that $f_1,\ldots,f_p$ generate $\vec{\mathcal{P}}^2_V(\mathcal{L})$. Then*

$$\{(f_i(v_1),\ldots,f_i(v_N));\ i=1,\ldots,p\}$$

generates $\vec{\mathcal{P}}^2_{V^N}(\mathcal{L}\wr\mathcal{G})$.

(b) Let $u_1, \ldots, u_s$ generate $\mathcal{P}^2_V(\mathcal{L})$ and $p_1, \ldots, p_r$ generate $\mathcal{P}^1_{\mathbf{R}^N}(\mathcal{G}_1)$. Suppose also that $h_1, \ldots, h_q$ generate $\vec{\mathcal{P}}^3_V(\mathcal{L})$. For $i = 1, \ldots, s$, denote by $U_i(v) = (u_i(v_1), \ldots, u_i(v_N))$ where $v = (v_1, \ldots, v_N)$. Then $\vec{\mathcal{P}}^3_{V^N}(\mathcal{L} \wr \mathcal{G})$ is generated by:

$$\{(h_i(v_1), \ldots, h_i(v_N));\ j = 1, \ldots, q\}$$

and

$$\{(p_j(U_i(v)), p_j(\sigma_2 U_i(v)), \ldots, p_j(\sigma_N U_i(v))) \bullet v;\ j = 1, \ldots, r,\ i = 1, \ldots, s\},$$

for some permutations $\sigma_i \in \mathcal{G}$ with $i = 2, \ldots, N$ such that $\sigma_i^{-1}(1) = i$.

Since x_1 is $\mathcal{G}_1$-invariant, the previous proposition yields the $\mathcal{L} \wr \mathcal{G}$-equivariants

$$\{(u_i(v_1)v_1, \ldots, u_i(v_N)v_N);\ i = 1, \ldots, s\}.$$

For the proof see section 5.

Corollary 3.8 *Under the conditions of proposition 3.7, if $\mathcal{G} = \mathbf{S}_N$, then*

$$\{(f_i(v_1), \ldots, f_i(v_N));\ i = 1, \ldots, p\}$$

generates $\vec{\mathcal{P}}^2_{V^N}(\mathcal{L} \wr \mathbf{S}_N)$.

4 Examples

Before proving the results stated in section 3 we give some examples. We concentrate on representations (V^N, Γ) to which the theory of Field [7, 8] and Field and Richardson [9, 10] can be applied.

1. $\Gamma = \mathbf{Z}_2 \wr \mathbf{S}_N$

Consider $(\mathbf{R}^N,\ \mathbf{Z}_2 \wr \mathbf{S}_N)$ where $\mathbf{Z}_2$ acts on $\mathbf{R}$ by multiplication by ∓ 1. Then $(\mathbf{R},\ \mathbf{Z}_2)$ is radial, so we can use proposition 3.3. A basis for $\mathcal{P}_{\mathbf{R}}(\mathbf{Z}_2)$ is $\{x^2\}$. Then $\mathcal{P}^2_{\mathbf{R}^N}(\mathbf{Z}_2 \wr \mathbf{S}_N)$ is generated by $\{x_1^2 + \cdots + x_N^2\}$ and there are no $\mathbf{Z}_2 \wr \mathbf{S}_N$-invariants of degree three. Similarly, using the same proposition, there are no $\mathbf{Z}_2 \wr \mathbf{S}_N$-equivariants of degree two and $\vec{\mathcal{P}}^3_{\mathbf{R}^N}(\mathbf{Z}_2 \wr \mathbf{S}_N)$ is generated by $(x_1^3, \ldots, x_N^3)$.
The group $\mathbf{Z}_2 \wr \mathbf{S}_N$ is the Weyl group $W(B_N)$. In [7] it is proved that $W(B_N)$ is 3-determined. Indeed, define $\mathcal{S}_N$ to be the set of subgroups Γ of $\mathbf{Z}_2 \wr \mathbf{S}_N$ such that $(\mathbf{R}^N, \Gamma)$ is absolutely irreducible and $\vec{\mathcal{P}}^2_{\mathbf{R}^N}(\Gamma) = \{0\}$. Field [7] proves that the bifurcation problems defined on representations in the class $\mathcal{S}_N$ are 3-determined.

2. $\Gamma = \mathbf{Z}_2 \wr \mathbf{Z}_3$

Consider $(\mathbf{R}^3,\ \mathbf{Z}_2 \wr \mathbf{Z}_3)$ where $\mathbf{Z}_2$ acts on $\mathbf{R}$ as above. A basis for $\mathcal{P}_{\mathbf{R}}(\mathbf{Z}_2)$ is given by $\{x^2\}$. Note that if $\mathcal{G} = \mathbf{Z}_3$, then $\mathcal{G}_1$ is the group formed by the identity on $\mathbf{R}^3$. Therefore $\mathcal{P}^1_{\mathbf{R}^3}(\mathcal{G}_1)$ is generated by $\{x_1,\ x_2,\ x_3\}$. As $\vec{\mathcal{P}}_{\mathbf{R}}(\mathbf{Z}_2)$ is generated by the identity on $\mathbf{R}$, by proposition 3.7 the space $\vec{\mathcal{P}}^3_{\mathbf{R}^3}(\mathbf{Z}_2 \wr \mathbf{Z}_3)$ is generated by $(x_1^2x_1, x_2^2x_2, x_3^2x_3)$, $(x_2^2x_1, x_3^2x_2, x_1^2x_3)$, $(x_3^2x_1, x_1^2x_2, x_2^2x_3)$.

3. $\Gamma = \mathbf{D}_3 \wr \mathbf{S}_N$

Consider $(\mathbf{C}^N,\ \mathbf{D}_3 \wr \mathbf{S}_N)$ with the standard action of $\mathbf{D}_3$ on $\mathbf{C}$. Now $(\mathbf{C},\ \mathbf{D}_3)$ is not radial, but we can use corollary 3.6. A basis for $\mathcal{P}_{\mathbf{C}}(\mathbf{D}_3)$ is given by $\{|z|^2,\ z^3+\overline{z}^3\}$. Thus $\mathcal{P}^2_{\mathbf{R}^N}(\mathbf{D}_3 \wr \mathbf{S}_N)$ is generated by $\{\mid z_1 \mid^2 + \cdots + \mid z_N \mid^2\}$ and $\mathcal{P}^3_{\mathbf{R}^N}(\mathbf{D}_3 \wr \mathbf{S}_N)$ is generated by $\{z_1^3+\overline{z}_1^3+\cdots+z_N^3+\overline{z}_N^3\}$. In [7] it is proved that $(\mathbf{C}^N,\ \mathbf{D}_3 \wr \mathbf{S}_N)$ is 2-determined.

4. $\Gamma = \Gamma = C_{p,N}(\mathbf{S}_N) = (\mathbf{Z}_2 \wr \mathbf{S}_p) \wr \mathbf{S}_N$

We consider here $(\mathbf{R}^p, \mathbf{Z}_2 \wr \mathbf{S}_p)$ as in the first example. This representation is absolutely irreducible but not radial. The ring $\mathcal{P}_{\mathbf{R}^p}(\mathbf{Z}_2 \wr \mathbf{S}_p)$ is generated by $\{x_1^2+\cdots+x_p^2, \ldots, x_1^{2p}+\cdots+x_p^{2p}\}$. By corollary 3.6 the space $\mathcal{P}^2_{\mathbf{R}^{pN}}(\Gamma)$ is generated by $\{x_{1,1}^2+\cdots+x_{p,1}^2+\cdots+x_{1,N}^2+\cdots+x_{p,N}^2\}$. There are no nontrivial Γ-invariants of degree three.

By [7] bifurcation problems for $(\mathbf{R}^{pN}, C_{p,N}(\mathbf{S}_N))$ are 3-determined.

5 Proof of Propositions 3.5 and 3.7

Throughout this section, we consider minimal homogeneous Hilbert bases for the rings $\mathcal{P}_V(\Gamma)$.

Proof of proposition 3.5.

Let $\Gamma = \mathcal{L} \wr \mathcal{G}$ and $p \in \mathcal{P}^2_{V^N}(\mathcal{L} \wr \mathcal{G})$. Then p is $\mathcal{L}^N$-invariant, so it is $\mathcal{L}$-invariant in v_i for $i = 1, \ldots, N$. Since there are no $\mathcal{L}$-invariants of degree one (see remark 3.4), p is a linear combination of degree two $\mathcal{L}$-invariants. Therefore

$$p(v_1, \ldots, v_N) = p_1(v_1) + \cdots + p_N(v_N)$$

where $p_i(v_i) = a_{i1}u_1(v_i) + \cdots + a_{is}u_s(v_i)$.

Also p is $\mathcal{G}$-invariant, so $p(\sigma v) = p(v)$ for all $\sigma \in \mathcal{G}$. Therefore $p_i(v_{\sigma^{-1}(i)}) = p_{\sigma^{-1}(i)}(v_{\sigma^{-1}(i)})$ for $i = 1, \ldots, N$ (and $v_j \in V$). That is, $p_i(v) = p_{\sigma^{-1}(i)}(v)$ for all $v \in V$ and $\sigma \in \mathcal{G}$. Therefore, $a_{ij} = a_{\sigma^{-1}(i)j}$ for $i = 1, \ldots, N$ and $j = 1, \ldots, s$, and for all $\sigma \in \mathcal{G}$. That is, if we define $f_i : \mathbf{R}^N \to \mathbf{R}$ as $f_i(x_1, \ldots, x_N) = a_{1i}x_1 + \ldots + a_{Ni}x_N$ for $i = 1, \ldots, s$, then each f_i is $\mathcal{G}$-invariant (of degree one). The same method applies to the Γ-invariants of degree three. $\square$

Proof of proposition 3.7

Let $g \in \vec{\mathcal{P}}_{V^N}(\Gamma)$, so that $g : V^N \to V^N$ is $\mathcal{L}^N$-equivariant and $\mathcal{G}$-equivariant. Write g as $(g_1, \dots, g_N)$. Since $\mathcal{G}$ is transitive there are permutations $\sigma_j \in \mathcal{G}$ $(j = 2, \dots, N)$ such that

$$\sigma_j^{-1}(1) = j. \tag{4}$$

Since g commutes with $\mathcal{G}$, the identity

$$g(\sigma_j(v_1, \dots, v_N)) = \sigma_j g(v_1, \dots, v_N)$$

implies that

$$g_j(v_1, \dots, v_N) = g_1(v_j, v_{\sigma_j^{-1}(2)}, \dots, v_{\sigma_j^{-1}(N)})$$

for $j = 2, \dots, N$. Thus

$$g = (g_1(v), g_1(\sigma_2(v)), \dots, g_1(\sigma_N(v))) \tag{5}$$

where $v = (v_1, \dots, v_N)$. So g is Γ-equivariant if and only if $g_1 : \ V^N \to V$ is equivariant under $\mathcal{L}$ in v_1, invariant under $\mathcal{L}$ in $v_2, \dots, v_N$, and g is $\mathcal{G}$-equivariant. Thus g_1 is a sum of terms $p_k f_k(v_1)$ where each p_k is a $\mathcal{L}$-invariant polynomial function (of $v_1, \dots, v_N$) and f_k is $\mathcal{L}$-equivariant (in v_1).

Case (a)

Let $g \in \vec{\mathcal{P}}^2_{V^N}(\Gamma)$. Since any polynomial in the $\mathcal{L}$-invariants has degree ≥ 2 (remark 3.4 (a)) and g has degree two components, then the p_k must be constant, so the f_k must be $\mathcal{L}$-equivariants with components of degree two. From (5), g is a sum of functions of the type

$$(f_i(v_1), f_i(v_2), \dots, f_i(v_N)),$$

(which are $\mathcal{G}$-equivariants), so we need consider only the generators f_i of $\vec{\mathcal{P}}^2_V(\mathcal{L})$.

Case (b)

Let $g \in \vec{\mathcal{P}}^3_{V^N}(\Gamma)$. There are two subcases: p_k constant or not. If p_k is constant then we get Γ-equivariants

$$(h_i(v_1), h_i(v_2), \dots, h_i(v_N))$$

with $h_i \in \vec{\mathcal{P}}^3_V(\mathcal{L})$. If p_k is not constant then as g has components of degree three, the polynomial p_k must be a linear combination of $\mathcal{L}$-invariants of degree two. Then f_k is an $\mathcal{L}$-equivariant function of v_1 with components of degree one, hence a scalar multiple of the identity on V. In this case $p_k f_k$ is Γ-equivariant if and only if

$$r(v) = (p_k(\sigma_1 v), p_k(\sigma_2 v), \dots, p_k(\sigma_N v))$$

is $\mathcal{G}$-equivariant. Here σ_1 is the identity of $\mathcal{G}$ and $\sigma_2, \dots, \sigma_N$ are as in (4) . We refer to r as a function of $v = (v_1, v_2, \dots, v_N)$ to make the notation easier, but in fact the p_k and r depend on $u_i(v_j)$ for $i = 1, \dots, s$ and $j = 1, \dots, N$. Recall that $u_1, \dots, u_s$ generate $\mathcal{P}^2_V(\mathcal{L})$.

Let $\tau \in \mathcal{G}$ and denote $\tau^{-1}(j) = i_j$ for $j = 1, \ldots, N$. From $r(\tau v) = \tau r(v)$ we obtain

$$(p_k(\sigma_1 \tau v), p_k(\sigma_2 \tau v), \ldots, p_k(\sigma_N \tau v)) = (p_k(\sigma_{i_1} v), p_k(\sigma_{i_2} v), \ldots, p_k(\sigma_{i_N} v)),$$

so

$$p_k(v) = p_k(\sigma_{i_1} \tau^{-1} \sigma_1^{-1} v) = p_k(\sigma_{i_2} \tau^{-1} \sigma_2^{-1} v) = \cdots = p_k(\sigma_{i_N} \tau^{-1} \sigma_N^{-1} v)$$

for all $\tau \in \mathcal{G}$. Moreover,

$$(\sigma_{i_j} \tau^{-1} \sigma_j^{-1})^{-1}(1) = \sigma_j \tau \sigma_{i_j}^{-1}(1) = \sigma_j \tau(i_j) = \sigma_j(j) = 1.$$

If we define the subgroup $\mathcal{G}_1 = \{\tau \in \mathcal{G} \ : \ \tau(1) = 1\}$ then

$$\{\sigma_{i_j} \tau^{-1} \sigma_j^{-1}; \ j = 1, \ldots, N\} \subseteq \mathcal{G}_1.$$

In particular if $\tau \in \mathcal{G}_1$ then $i_1 = \tau(1) = 1$, so p_k is invariant under τ. Thus p_k is $\mathcal{G}_1$-invariant. And if p_k is $\mathcal{G}_1$-invariant then r is $\mathcal{G}$-equivariant.

We describe now the $\mathcal{G}_1$-invariant polynomials p_k. Recall that p_k is a linear combination of invariants $u_i(v_j)$. If we write p_k as

$$q_1(u_1(v_1), \ldots, u_1(v_N)) + \cdots + q_s(u_s(v_1), \ldots, u_s(v_N)),$$

then p_k is $\mathcal{G}_1$-invariant if and only if the $q_i(x_1, \ldots, x_N)$ are $\mathcal{G}_1$-invariants of degree one, for $i = 1, \ldots, s$. $\square$

6 Molien Series

The aim of this section is to construct the Molien series for the ring $\mathcal{P}(\mathcal{L} \wr \mathcal{G})$ and module $\vec{\mathcal{P}}(\mathcal{L} \wr \mathcal{G})$ from the corresponding series for $\mathcal{L}$ and $\mathcal{G}$.

6.1 Review of Molien Series

We review Molien series for the rings of invariants and modules of equivariants for general compact Lie groups.

It will be convenient to change from a real representation to a complex representation, and we briefly explain why this useful step produces no extra complications. Let Γ be a compact Lie group acting on $V = \mathbf{R}^k$, so that $\gamma \in \Gamma$ acts as a matrix M_γ. The matrix M_γ has real entries, and we can view it as a matrix acting on $\mathbf{C}^k$. If $(x_1, \ldots, x_k)$ denote real coordinates on $\mathbf{R}^k$, $x_j \in \mathbf{R}$, then we obtain complex coordinates on $\mathbf{C}^k$ by permitting the x_j to be complex. Moreover, there is a natural inclusion $\mathbf{R}[x_1, \ldots, x_k] \subseteq \mathbf{C}[x_1, \ldots, x_k]$ where these are the rings of polynomials in the x_j with coefficients in $\mathbf{R}$, $\mathbf{C}$ respectively.

Every real-valued Γ-invariant in $\mathbf{R}[x_1, \ldots, x_k]$ is also a complex-valued Γ-invariant in $\mathbf{C}[x_1, \ldots, x_k]$. Conversely the real and imaginary parts of a complex-valued invariant are real invariants (because the matrices M_γ have real entries). Therefore a basis over $\mathbf{R}$ for the real vector space of degree d real-valued invariants is also a basis over $\mathbf{C}$ for the complex vector space of degree d $\mathbf{C}$-valued invariants. That is, the 'real' and 'complex' Molien series are the same. Similar remarks apply to the equivariants. Bearing these facts in mind, we 'complexify' the entire problem, reducing it to the following situation.

Let V be a k-dimensional vector space over $\mathbf{C}$, and let $x_1, \ldots, x_k$ denote coordinates relative to a basis for V. Let $\Gamma \subseteq \mathbf{GL}(V)$ be a compact Lie group. Let $\mathbf{C}[x_1, \ldots, x_k]$ denote the ring of polynomials over $\mathbf{C}$ in $x_1, \ldots, x_k$. Consider an action of Γ on V and let $\mathcal{P}_V(\Gamma)$ denote the subalgebra of $\mathbf{C}[x_1, \ldots, x_k]$ formed by the invariant polynomials under Γ (over $\mathbf{C}$). Note that $\mathbf{C}[x_1, \ldots, x_k]$ is graded:

$$\mathbf{C}[x_1, \ldots, x_k] = R_0 \oplus R_1 \oplus R_2 \oplus \cdots$$

where R_i consists of all homogeneous polynomials of degree i. If $f(x) \in R_i$ for some i then $f(\gamma x) \in R_i$ for all $\gamma \in \Gamma$. Therefore for any subgroup Γ of $\mathbf{GL}(V)$ the space $\mathcal{P}_V(\Gamma)$ has the structure

$$\mathcal{P}_V(\Gamma) = \mathcal{P}_V^0(\Gamma) \oplus \mathcal{P}_V^1(\Gamma) \oplus \mathcal{P}_V^2(\Gamma) \oplus \cdots$$

of an N-graded $\mathbf{C}$-algebra given by $\mathcal{P}_V^i(\Gamma) = \mathcal{P}_V(\Gamma) \cap R_i$.

The *Hilbert series* or *Poincaré series* of the graded algebra $\mathcal{P}_V(\Gamma)$ is a generating function for the dimension of the vector space of invariants at each degree and is defined to be

$$\Phi_\Gamma(z) = \sum_{d=0}^{\infty} \dim(\mathcal{P}_V^d(\Gamma)) z^d. \qquad (6)$$

Since Γ is compact, consider the normalized Haar measure μ defined on Γ [12] and denote the integral with respect to μ of a continuous function f defined on Γ by

$$\int_\Gamma f.$$

Recall that if Γ is finite, then the normalized Haar integral on Γ is

$$\int_\Gamma f \equiv \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} f(\gamma),$$

where $|\Gamma|$ denotes the order of Γ [12].

There is a famous explicit formula:

Theorem 6.1 (Molien's theorem) *Let Γ be a compact Lie group. Then the Hilbert series of $\mathcal{P}_V(\Gamma)$ is*

$$\Phi_\Gamma(z) = \int_\Gamma \frac{1}{\det(1_\Gamma - \gamma z)}.$$

Proof. See [16] for the original proof of the finite case, and [17] for the extension to a compact group. □

Now we turn to equivariants, the module analogue of the ring of invariants. Equivariants can be interpreted as invariants with respect to a different group action on a different space. The *Hilbert series* of the graded module $\vec{\mathcal{P}}_V(\Gamma)$ over the ring $\mathcal{P}_V(\Gamma)$ is the generating function

$$\Psi_\Gamma(z) = \sum_{d=0}^{\infty} \dim(\vec{\mathcal{P}}_V^d(\Gamma))z^d.$$

Again there is an explicit formula:

Theorem 6.2 (Equivariant Molien theorem) *Let Γ be a compact Lie group. Then the module $\vec{\mathcal{P}}_V(\Gamma)$ over the ring $\mathcal{P}_V(\Gamma)$ has a Hilbert series given by*

$$\Psi_\Gamma(z) = \int_\Gamma \frac{\operatorname{tr}(\gamma^{-1})}{\det(1_\Gamma - \gamma z)}.$$

Proof. See [17]. □

Note that for orthogonal group representations, in which $\Gamma \subseteq \mathbf{O}(k)$, we have $\operatorname{tr}(\gamma) = \operatorname{tr}(\gamma^{-1})$.

6.2 Molien Series for Wreath Products

We now obtain the Molien series $\Phi_{\mathcal{L}\wr\mathcal{G}}$ and $\Psi_{\mathcal{L}\wr\mathcal{G}}$ from the Molien series $\Phi_{\mathcal{L}}$, $\Phi_{\mathcal{G}}$, $\Psi_{\mathcal{L}}$, $\Psi_{\mathcal{G}}$.

Recall that the *cycle type* of a permutation σ of $\mathbf{S}_N$ is the integer vector $k(\sigma) = (k_1, \ldots, k_N)$, where k_i counts the number of cycles of length i in the cycle decomposition of σ.

Remark 6.3

Let σ be a permutation of $\mathbf{S}_N$ with cycle type $k(\sigma) = (k_1, \ldots, k_N)$. Then

$$\det(\mathrm{I}_N - \sigma z) = \prod_{i=1}^{N} (1 - z^i)^{k_i}.$$

Theorem 6.4 *Let $\mathcal{L} \subseteq \mathbf{O}(k)$ be a compact Lie group acting on a complex k-dimensional vector space V. Suppose that $\mathcal{G}$ is a subgroup of $\mathbf{S}_N$ acting on $\mathbf{C}^N$. Consider the group $\Gamma = \mathcal{L}\wr\mathcal{G}$ acting on V^N. Denote by $\Phi_{\mathcal{L}}(z)$ the Molien series for $\mathcal{P}_V(\mathcal{L})$, and by $\Phi_{\mathcal{G}}(z)$ the Molien series for $\mathcal{P}_{\mathbf{C}^N}(\mathcal{G})$. Then*

$$\Phi_{\mathcal{L}\wr\mathcal{G}}(z) = \frac{1}{|\mathcal{G}|} \sum_{\sigma\in\mathcal{G}} \left[\prod_{i=1}^{N} (\Phi_{\mathcal{L}}(z^i))^{k_i} \right].$$

The proof is given below after the next lemma.

Remark 6.5

Explicitly,

$$\Phi_{\mathcal{G}}(z) = \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} \frac{1}{\prod_{i=1}^{N} (1 - z^i)^{k_i}},$$

where k_i denotes the number of cycles of length i in the cycle decomposition of σ, for $i = 1, \ldots, N$.

Lemma 6.6 *Let $l_1, \ldots, l_r \in \mathbf{M}_k$, the set of $k \times k$ matrices, and let $z \in \mathbf{C}$. Denote by $A(z, l_1, \ldots, l_r)$ the matrix*

$$A(z, l_1, \ldots, l_r) = \begin{pmatrix} \mathrm{I}_k & 0 & 0 & \cdots & 0 & -zl_1 \\ -zl_2 & \mathrm{I}_k & 0 & \cdots & 0 & 0 \\ 0 & -zl_3 & \mathrm{I}_k & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & -zl_r & \mathrm{I}_k \end{pmatrix}.$$

Then

$$\det(A(z, l_1, \ldots, l_r)) = \det(\mathrm{I}_k - z^r l_1 l_r l_{r-1} \ldots l_2).$$

Proof. Note that

$$\det(A(z, l_1, \ldots, l_r)) = \det(A(z, l_1, \ldots, l_r)) \det \begin{pmatrix} \mathrm{I}_k & 0 & 0 & \cdots & 0 & 0 \\ X_1 & \mathrm{I}_k & 0 & \cdots & 0 & 0 \\ X_2 & 0 & \mathrm{I}_k & \cdots & 0 & 0 \\ & & & \cdots & & \\ X_{r-1} & 0 & 0 & \cdots & 0 & \mathrm{I}_k \end{pmatrix},$$

for any matrices $X_1, \ldots, X_{r-1}$ in $\mathbf{M}_k$. Thus

$$\det(A(z, l_1, \ldots, l_r)) = \det \begin{pmatrix} \mathrm{I}_k - zl_1 X_{r-1} & 0 & 0 & \cdots & 0 & -zl_1 \\ -zl_2 + X_1 & \mathrm{I}_k & 0 & \cdots & 0 & 0 \\ -zl_3 X_1 + X_2 & -zl_3 & \mathrm{I}_k & \cdots & 0 & 0 \\ & & & \cdots & & \\ -zl_r X_{r-2} + X_{r-1} & 0 & 0 & \cdots & -zl_r & \mathrm{I}_k \end{pmatrix}.$$

Set $X_1 = zl_2$ and $X_i = z^i l_{i+1} l_i \ldots l_2$ for $2 \leq i \leq r-1$ to get

$$\det(A(z,l_1,\ldots,l_r)) = \det\begin{pmatrix} \mathrm{I}_k - z^r l_1 l_r l_{r-1}\ldots l_2 & 0 & 0 & \cdots & 0 & -zl_1 \\ 0 & \mathrm{I}_k & 0 & \cdots & 0 & 0 \\ 0 & -zl_3 & \mathrm{I}_k & \cdots & 0 & 0 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & -zl_r & \mathrm{I}_k \end{pmatrix}.$$

Thus

$$\det(A(z,l_1,\ldots,l_r)) = \det(\mathrm{I}_k - z^r l_1 l_r l_{r-1}\ldots l_2). \quad \square$$

Proof of theorem 6.4. Write $\mathcal{L}^N = \mathcal{L}_1 \times \cdots \times \mathcal{L}_N$ where each $\mathcal{L}_j \cong \mathcal{L}$. Haar measure on $\mathcal{L}^N$ is the product of Haar measures on the $\mathcal{L}_j$. Moreover, since $\mathcal{L} \wr \mathcal{G}$ is the semidirect product, we have

$$\begin{aligned} \int_{\mathcal{L}\wr\mathcal{G}} f &= \int_{\mathcal{G}} \int_{\mathcal{L}^N} f \\ &= \frac{1}{|\mathcal{G}|} \sum_{\mathcal{G}} \int_{\mathcal{L}_1} \cdots \int_{\mathcal{L}_N} f. \end{aligned}$$

By theorem 6.1 the Molien series for $\mathcal{P}_{V^N}(\mathcal{L} \wr \mathcal{G})$ is

$$\begin{aligned} \Phi_{\mathcal{L}\wr\mathcal{G}}(z) &= \int_{(l_1,\ldots,l_N)\,\in\,\mathcal{L}^N,\ \sigma\,\in\,\mathcal{G}} \frac{1}{\det(\mathrm{I}_{Nk} - ((l_1,\ldots,l_N),\sigma)z)} \\ &= \frac{1}{|\mathcal{G}|} \sum_{\sigma\in\mathcal{G}} \int_{(l_1,\ldots,l_N)\,\in\,\mathcal{L}^N} \frac{1}{\det(\mathrm{I}_{Nk} - ((l_1,\ldots,l_N),\sigma)z)}. \end{aligned}$$

Let σ be a permutation of $\mathcal{G}$ with cycle type $k(\sigma) = (k_1,\ldots,k_N)$. Denote by $c^j_{k_i} = (p_1^{j,k_i},\ldots,p_i^{j,k_i})$ the jth cycle of σ with length i (for $j = 1,\ldots,k_i$), and by $l^j_{k_i}$ the vector in $\mathcal{L}^j_{k_i} = \mathcal{L}_{p_1^{j,k_i}} \times \cdots \times \mathcal{L}_{p_i^{j,k_i}}$ with the components of l

corresponding to the i indices $p_1^{j,k_i},\ldots,p_i^{j,k_i}$ in $c_{k_i}^j$. Then by lemma 6.6

$$\begin{aligned}\det(\mathrm{I}_{Nk}-((l_1,\ldots,l_N),\sigma)z) &= \prod_{i=1}^{N}\prod_{j=1}^{k_i}\det(\mathrm{I}_{ik}-(l_{k_i}^j,c_{k_i}^j)z)\\ &= \prod_{i=1}^{N}\prod_{j=1}^{k_i}\det(\mathrm{I}_k-(l_{k_i}^j)^*z^i),\end{aligned}$$

where $(l_1,\ldots,l_i)^* = l_1 l_i l_{i-1}\ldots l_2$.

Note that

$$\begin{aligned}\int_{l_{k_i}^j\,\in\,\mathcal{L}_{k_i}^j}\frac{1}{\det(\mathrm{I}_k-(l_{k_i}^j)^*z^i)} &= \int_{(l_1,\ldots,l_i)\,\in\,\mathcal{L}^i}\frac{1}{\det(\mathrm{I}_k-l_1\,l_i\,l_{i-1}\ldots l_2\,z^i)}\\ &= \int_{l_1\,\in\,\mathcal{L}_1}\cdots\int_{l_i\,\in\,\mathcal{L}_i}\frac{1}{\det(\mathrm{I}_k-l_1\,l_i\,l_{i-1}\ldots l_2\,z^i)}\\ &= \Phi_{\mathcal{L}}(z^i).\end{aligned}$$

It follows that

$$\begin{aligned}\int_{(l_1,\ldots,l_N)\,\in\,\mathcal{L}^N}\frac{1}{\prod_{i=1}^{N}\prod_{j=1}^{k_i}\det(\mathrm{I}_k-(l_{k_i}^j)^*z^i)} &= \prod_{i=1}^{N}\prod_{j=1}^{k_i}\left(\int_{l_{k_i}^j\,\in\,\mathcal{L}_{k_i}^j}\frac{1}{\det(\mathrm{I}_k-(l_{k_i}^j)^*z^i)}\right)\\ &= \prod_{i=1}^{N}\prod_{j=1}^{k_i}\Phi_{\mathcal{L}}(z^i)\\ &= \prod_{i=1}^{N}(\Phi_{\mathcal{L}}(z^i))^{k_i}. \quad \square\end{aligned}$$

Theorem 6.7 *Let $\mathcal{L}\subseteq\mathbf{O}(k)$ be a compact Lie group acting on a complex k-dimensional vector space V. Suppose that $\mathcal{G}$ is a subgroup of $\mathbf{S}_N$ acting on $\mathbf{C}^N$. Consider the group $\Gamma=\mathcal{L}\wr\mathcal{G}$ acting on V^N. Denote by $\Phi_{\mathcal{L}}(z)$ the Molien series* *for $\mathcal{P}_V(\mathcal{L})$ and by $\Psi_{\mathcal{L}}(z)$ the Molien series for $\vec{\mathcal{P}}_V(\mathcal{L})$. Let $\Psi_{\mathcal{G}}(z)$ be the Molien series for $\vec{\mathcal{P}}_{\mathbf{C}^N}(\mathcal{G})$. Then*

$$\Psi_{\mathcal{L}\wr\mathcal{G}}(z) = \frac{1}{|\mathcal{G}|}\sum_{\sigma\in\mathcal{G}}\left[(k_1\Psi_{\mathcal{L}}(z))\left(\Phi_{\mathcal{L}}(z)\right)^{k_1-1}\right)\left(\prod_{i=2}^{N}(\Phi_{\mathcal{L}}(z^i))^{k_i}\right)\right].$$

Remark 6.8

Explicitly,

$$\Psi_{\mathcal{G}}(z) = \frac{1}{|\mathcal{G}|}\sum_{\sigma\in\mathcal{G}}\frac{k_1}{\prod_{i=1}^{N}(1-z^i)^{k_i}},$$

where k_i denotes the number of cycles of length i in the cycle decomposition of σ, for $i = 1, \ldots, N$.

Proof. By theorem 6.2

$$\Psi_{\mathcal{L}\wr\mathcal{G}}(z) = \int_{\mathcal{L}\wr\mathcal{G}}\frac{\mathrm{tr}((l_1,\ldots,l_N),\sigma))}{\det(\mathrm{I}_{Nk} - ((l_1,\ldots,l_N),\sigma)z)}.$$

If σ is a permutation of $\mathcal{G}$ with cycle type $k(\sigma) = (k_1, \ldots, k_N)$ and $l^j_{k_1}$ the component of l corresponding to the index in the jth cycle of σ with length 1 (for $j = 1, \ldots, k_1$), then

$$\mathrm{tr}((l_1,\ldots,l_N),\sigma)) = \mathrm{tr}(l^1_{k_1}) + \cdots + \mathrm{tr}(l^{k_1}_{k_1}).$$

The rest follows as in the proof of theorem 6.4. $\square$

Examples.

1. Consider $(V,\mathcal{L}) = (\mathbf{R},\mathbf{Z}_2)$, where $\mathbf{Z}_2$ acts by multiplication by ∓ 1. Then

$$\begin{aligned}\Phi_{\mathbf{Z}_2}(z) &= \tfrac{1}{2}\left(\tfrac{1}{1-z}+\tfrac{1}{1+z}\right) = \tfrac{1}{1-z^2}\\ &= 1 + z^2 + z^4 + z^6 + \cdots,\end{aligned}$$

$$\begin{aligned}\Psi_{\mathbf{Z}_2}(z) &= \tfrac{1}{2}\left(\tfrac{1}{1-z}-\tfrac{1}{1+z}\right) = \tfrac{z}{1-z^2}\\ &= z + z^3 + z^5 + z^7 + \cdots.\end{aligned}$$

(a) $\mathbf{Z_2}\wr\mathbf{S}_3$

$$\begin{aligned}\Phi_{\mathbf{S}_3}(z) &= \tfrac{1}{6}\left[\tfrac{1}{(1-z)^3}+\tfrac{2}{(1-z^3)}+\tfrac{3}{(1-z^2)(1-z)}\right]\\ &= 1 + z + 2z^2 + 3z^3 + 4z^4 + 5z^5 + 7z^6 + 8z^7 + \cdots,\end{aligned}$$

$$\begin{aligned}\Psi_{\mathbf{S}_3}(z) &= \tfrac{1}{6}\left[\tfrac{3}{(1-z)^3}+\tfrac{3}{(1-z^2)(1-z)}\right]\\ &= 1 + 2z + 4z^2 + 6z^3 + 9z^4 + 12z^5 + 16z^6 + 20z^7 + \cdots,\end{aligned}$$

$$\begin{aligned}\Phi_{\mathbf{Z}_2\wr\mathbf{S}_3}(z) &= \tfrac{1}{6}\left[(\Phi_{\mathbf{Z}_2}(z))^3 + 2\Phi_{\mathbf{Z}_2}(z^3) + 3\Phi_{\mathbf{Z}_2}(z^2)\Phi_{\mathbf{Z}_2}(z)\right]\\ &= 1 + z^2 + 2z^4 + 3z^6 + \cdots,\end{aligned}$$

$$\begin{aligned}\Psi_{\mathbf{Z}_2\wr\mathbf{S}_3}(z) &= \tfrac{1}{6}\left[3\Psi_{\mathbf{Z}_2}(z)(\Phi_{\mathbf{Z}_2}(z))^2 + 3\Psi_{\mathbf{Z}_2}(z)\Phi_{\mathbf{Z}_2}(z^2)\right]\\ &= z + 2z^3 + 5z^5 + 6z^7 + \cdots.\end{aligned}$$

(b) $\mathbf{Z_2} \wr \mathbf{Z}_3$

$$\begin{aligned}\Phi_{\mathbf{Z}_3}(z) &= \tfrac{1}{3}\left[\tfrac{1}{(1-z)^3} + \tfrac{2}{(1-z^3)}\right]\\ &= 1 + z + 2z^2 + 4z^3 + 5z^4 + 7z^5 + 10z^6 + 12z^7 + \cdots,\end{aligned}$$

$$\begin{aligned}\Psi_{\mathbf{Z}_3}(z) &= \tfrac{1}{3}\left[\tfrac{3}{(1-z)^3}\right]\\ &= 1 + 3z + 6z^2 + 10z^3 + 15z^4 + 21z^5 + 28z^6 + 36z^7 + \cdots,\end{aligned}$$

$$\begin{aligned}\Phi_{\mathbf{Z}_2\wr\mathbf{Z}_3}(z) &= \tfrac{1}{3}\left[(\Phi_{\mathbf{Z}_2}(z))^3 + 2\Phi_{\mathbf{Z}_2}(z^3)\right]\\ &= 1 + z^2 + 2z^4 + 4z^6 + \cdots,\end{aligned}$$

$$\begin{aligned}\Psi_{\mathbf{Z}_2\wr\mathbf{Z}_3}(z) &= \tfrac{1}{3}\left[3\Psi_{\mathbf{Z}_2}(z)(\Phi_{\mathbf{Z}_2}(z))^2\right]\\ &= z + 3z^3 + 6z^5 + 10z^7 + \cdots.\end{aligned}$$

2. Consider $(V, \mathcal{L}) = (\mathbf{R}^2, \mathbf{Z}_4)$, where the action of $\mathbf{Z}_4$ is generated by $\rho = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then

$$\begin{aligned}\Phi_{\mathbf{Z}_4}(z) &= \tfrac{1}{4}\left[\tfrac{1}{(1-z)^2} + \tfrac{2}{1+z^2} + \tfrac{1}{(1+z)^2}\right]\\ &= 1 + z^2 + 3z^4 + 3z^6 + \cdots,\end{aligned}$$

$$\begin{aligned}\Psi_{\mathbf{Z}_4}(z) &= \tfrac{1}{4}\left[\tfrac{2}{(1-z)^2} + \tfrac{-2}{(1+z)^2}\right]\\ &= 2z + 4z^3 + 6z^5 + 8z^7 + \cdots.\end{aligned}$$

(a) $\mathbf{Z_4} \wr \mathbf{S}_3$

$$\begin{aligned}\Phi_{\mathbf{Z}_4\wr\mathbf{S}_3}(z) &= \tfrac{1}{6}\left[(\Phi_{\mathbf{Z}_4}(z))^3 + 2\Phi_{\mathbf{Z}_4}(z^3) + 3\Phi_{\mathbf{Z}_4}(z^2)\Phi_{\mathbf{Z}_4}(z)\right]\\ &= 1 + z^2 + 4z^4 + 7z^6 + \cdots,\end{aligned}$$

$$\begin{aligned}\Psi_{\mathbf{Z}_4\wr\mathbf{S}_3}(z) &= \tfrac{1}{6}\left[3\Psi_{\mathbf{Z}_4}(z)(\Phi_{\mathbf{Z}_4}(z))^2 + 3\Psi_{\mathbf{Z}_4}(z)\Phi_{\mathbf{Z}_4}(z^2)\right]\\ &= 2z + 6z^3 + 18z^5 + 42z^7 + \cdots.\end{aligned}$$

(b) $\mathbf{Z_4} \wr \mathbf{Z}_3$

$$\begin{aligned}\Phi_{\mathbf{Z}_4 \wr \mathbf{Z}_3}(z) &= \tfrac{1}{3}\left[(\Phi_{\mathbf{Z}_4}(z))^3 + 2\Phi_{\mathbf{Z}_4}(z^3)\right] \\ &= 1 + z^2 + 4z^4 + 10z^6 + \cdots,\end{aligned}$$

$$\begin{aligned}\Psi_{\mathbf{Z}_4 \wr \mathbf{Z}_3}(z) &= \tfrac{1}{3}\left[3\Psi_{\mathbf{Z}_4}(z)(\Phi_{\mathbf{Z}_4}(z))^2\right] \\ &= 2z + 8z^3 + 28z^5 + 72z^7 + \cdots.\end{aligned}$$

7 Finite Wreath Products

We consider now finite wreath products, that is, groups $\mathcal{L} \wr \mathcal{G}$ where $\mathcal{L}$ is finite, and derive some results on higher-order invariant theory along the lines of Worfolk [22]. The aim of this section is to construct the theory for $\mathcal{L} \wr \mathcal{G}$ from the corresponding theories for $\mathcal{L}$ and $\mathcal{G}$. In particular we consider primary and secondary invariants, the Cohen-Macaulay property, and homogeneous sets of parameters. We give formulas for the number of secondary invariants (corollary 7.11) and fundamental equivariants (corollary 7.12).

7.1 Preliminaries

In this section we review some elementary results related to combinatorial theory and commutative algebra. For proofs and more extensive discussions see Stanley [20], Sturmfels [21], and Worfolk [22]. The restriction to finite groups seems necessary for their methods to apply, although some — but not all — of their results extend to the compact case using other methods.

For finite groups Γ the ring $\mathcal{P}_V(\Gamma)$ is generated by a set of primary and secondary invariants (see below for definitions). The equivariant results are analogous to those for the invariants. Note that $g : V \to V$ is Γ-invariant if and only if the function $f : V \times V \to \mathbf{R}$ defined by $f(x,y) = < g(x), y >$ is Γ-invariant. Here the action of Γ on $V \times V$ is defined by $\gamma \cdot (x,y) = (\gamma \cdot x, \gamma \cdot y)$, and the functions g and f are related by $g(x) = (d_y f)^{\mathrm{T}}_{x,0}$, where T indicates the transpose.

As before, let V be a k-dimensional vector space over $\mathbf{C}$, and let $x_1, \ldots, x_k$ denote coordinates relative to a basis for V. Let $\Gamma \subseteq \mathbf{GL}(V)$ be a finite matrix group of order $|\Gamma|$ and $\mathbf{C}[x_1, \ldots, x_k]$ the graded algebra of polynomials over $\mathbf{C}$ in $x_1, \ldots, x_k$. Consider an action of Γ on V and $\mathcal{P}_V(\Gamma)$ the N-graded subalgebra of $\mathbf{C}[x_1, \ldots, x_k]$ formed by the invariant polynomials under Γ (over $\mathbf{C}$). Thus $\mathcal{P}_V(\Gamma) = \mathcal{P}^0_V(\Gamma) \oplus \mathcal{P}^1_V(\Gamma) \oplus \mathcal{P}^2_V(\Gamma) \oplus \cdots$.

Let $\dim \mathcal{P}_V(\Gamma)$ be the *Krull dimension* of $\mathcal{P}_V(\Gamma)$, that is, the maximum number of elements of $\mathcal{P}_V(\Gamma)$ that are algebraically independent over $\mathbf{C}$.

Theorem 7.1 *If* $\dim V = k$ *then there exist* k*, but not* $k+1$*, algebraically independent invariants over* $\mathbf{C}$*. Equivalently* $\mathcal{P}_V(\Gamma)$ *has Krull dimension* k*.*

Proof. See [1]. $\square$

Consider the subring $\mathbf{C}[p_1, \ldots, p_k]$ of $\mathcal{P}_V(\Gamma)$ generated by k algebraically independent elements $p_1, p_2, \ldots, p_k$.

Lemma 7.2 *The subring $\mathbf{C}[p_1, \ldots, p_k]$ of $\mathcal{P}_V(\Gamma)$ has Hilbert series*

$$H(\mathbf{C}[p_1, \ldots, p_k], z) = \frac{1}{(1 - z^{d_1}) \cdots (1 - z^{d_n})}$$

where d_i is the degree of p_i.

Proof. See [21]. $\square$

A set $\{p_1, \ldots, p_k\}$ of k homogeneous invariant polynomials with positive degree is a *homogeneous system of parameters* (*h.s.o.p.*) of $\mathcal{P}_V(\Gamma)$ if $\dim \mathcal{P}_V(\Gamma) = k$ and $\mathcal{P}_V(\Gamma)$ is a finitely generated module over $\mathbf{C}[p_1, \ldots, p_k]$.

Theorem 7.3 *A set $\{p_1, \ldots, p_k\}$ of k homogeneous invariant polynomials is an h.s.o.p. for $\mathcal{P}_V(\Gamma)$, where $k = \dim \mathcal{P}_V(\Gamma)$, if and only if the polynomials in $\{p_1, \ldots, p_k\}$ have no common zeros except 0.*

Proof. See for example [15] pp 104-105. $\square$

A ring is *Cohen-Macaulay* if it is finitely generated as a free module over the ring determined by any h.s.o.p. For finite groups Γ, the rings $\mathcal{P}_V(\Gamma)$ form Cohen-Macaulay rings:

Theorem 7.4 *For any finite $\Gamma \subset GL(V)$, the ring $\mathcal{P}_V(\Gamma)$ is Cohen-Macaulay. That is, there are k homogeneous invariant polynomials $p_1, \ldots, p_k$ and homogeneous $q_1, \ldots, q_m$, all in $\mathcal{P}_V(\Gamma)$, such that $\mathcal{P}_V(\Gamma)$ is finitely generated as a free module over the subring $\mathbf{C}[p_1, \ldots, p_k]$:*

$$\mathcal{P}_V(\Gamma) = \bigoplus_{i=1}^{m} q_i \mathbf{C}[p_1, \ldots, p_k]. \tag{7}$$

Thus the set $\{p_1, \ldots, p_k\}$ is an h.s.o.p.

Proof. See [21]. $\square$

Remark 7.5

Theorem 7.4 also holds for compact Lie groups Γ by Hochster and Roberts [13].

The decomposition (7) is called a *Hironaka decomposition* of the Cohen-Macaulay algebra $\mathcal{P}_V(\Gamma)$. Then the Hilbert series of $\mathcal{P}_V(\Gamma)$ equals

$$\Phi_\Gamma(z) = (\sum_{i=1}^{m} z^{\deg q_i}) / \prod_{j=1}^{k} (1 - z^{\deg p_j})$$

by corollary [21] 2.3.4.

Thus, given an h.s.o.p. $\{p_1, \ldots, p_k\}$, there is a finite set of homogeneous invariants $\{q_1, \ldots, q_m\}$ such that any invariant may be written *uniquely* as a linear combination of the q_i with polynomials in the p_i as coefficients. Usually the p_i are called *primary invariants* and the q_i *secondary invariants*. The p_i and q_i form a set of *fundamental invariants* for Γ. The primary invariants are not unique, and neither are their degrees. However, once the degrees of the primary invariants are fixed, the number and degrees of the secondary invariants are also fixed.

Theorem 7.6 *For Γ finite and $p_1, \ldots, p_k$ a set of primary invariants for $\mathcal{P}_V(\Gamma)$ with degrees $d_1, \ldots, d_k$, the number of secondary invariants is*

$$m = \frac{d_1 \cdots d_k}{|\Gamma|}.$$

The degrees with multiplicity of the secondary invariants are the exponents of the generating function

$$\Phi_\Gamma(z) / H(\mathbf{C}[p_1, \ldots, p_k], z) = z^{e_1} + \cdots + z^{e_m},$$

where $H(\mathbf{C}[p_1, \ldots, p_k], z)$ is the Hilbert series of $\mathbf{C}[p_1, \ldots, p_k]$.

Proof. See [21]. $\square$

Consider now the graded module $\vec{\mathcal{P}}_V(\Gamma)$ over the ring $\mathcal{P}_V(\Gamma)$. A set $\{p_1, \ldots, p_k\}$ of k homogeneous invariant polynomials with positive degree is a *homogeneous system of parameters (h.s.o.p.)* of the module $\vec{\mathcal{P}}_V(\Gamma)$ over the ring $\mathcal{P}_V(\Gamma)$ if $\dim \mathcal{P}_V(\Gamma) = k$ and $\vec{\mathcal{P}}_V(\Gamma)$ is a finitely generated module over $\mathbf{C}[p_1, \ldots, p_k]$. A module is *Cohen-Macaulay* if it is finitely generated as a free module over the ring of any h.s.o.p. The modules $\vec{\mathcal{P}}_V(\Gamma)$ for finite groups Γ are Cohen-Macaulay:

Theorem 7.7 *For finite $\Gamma \subset GL(V)$ there are $k = \dim V$ homogeneous invariant polynomials $p_1, \ldots, p_k$ such that $\vec{\mathcal{P}}_V(\Gamma)$ is finitely generated as a free module over the ring $\mathbf{C}[p_1, \ldots, p_k]$. There exist $g_1, \ldots, g_s$ in $\vec{\mathcal{P}}_V(\Gamma)$ with each g_i homogeneous, such that*

$$\vec{\mathcal{P}}_V(\Gamma) = \bigoplus_{i=1}^{s} g_i \mathbf{C}[p_1, \ldots, p_k]. \tag{8}$$

That is, $\{p_1, \ldots, p_k\}$ is an h.s.o.p. and $\vec{\mathcal{P}}_V(\Gamma)$ is Cohen-Macaulay.

Proof. See [22]. □

Remark 7.8

The proof of this proposition not only shows that the module is Cohen-Macaulay, but also shows that the set of primary invariants that forms an h.s.o.p. for the ring $\mathcal{P}_V(\Gamma)$ is also an h.s.o.p. for the module $\vec{\mathcal{P}}_V(\Gamma)$.

Call the g_i a set of *fundamental equivariants* for the module $\vec{\mathcal{P}}_V(\Gamma)$. Again, as for the secondary invariants, once the degrees of the primary invariants are fixed, the number and degrees of the fundamental equivariants are also fixed:

Theorem 7.9 *For Γ finite and $p_1, \ldots, p_k$ a set of primary invariants for $\mathcal{P}_V(\Gamma)$ with degrees $d_1, \ldots, d_k$, the number of fundamental equivariants is*

$$s = k\frac{d_1 \cdots d_k}{|\Gamma|}.$$

The degrees with multiplicity of the fundamental equivariants are the exponents of the generating function

$$\Psi_\Gamma(z)/H(\mathbf{C}[p_1, \ldots, p_k], z) = z^{e_1} + \cdots + z^{e_s},$$

where $H(\mathbf{C}[p_1, \ldots, p_k], z)$ is the Hilbert series of $\mathbf{C}[p_1, \ldots, p_k]$.

Proof. See [21]. □

Given a module decomposition as presented in this theorem, any equivariant g may be written uniquely in the form

$$g(x) = \sum_{i=1}^{s} h_i(p_1(x), \ldots, p_k(x))g_i(x),$$

where the g_i are a fundamental set of equivariants and the h_i are polynomials in the primary invariants. This result does *not* generalise to the compact case: see Kostant [14], Schwarz [18, 19]. Schwarz calls groups satisfying the above uniqueness condition *cofree*, and studies the more general case of 'reductive' groups.

7.2 Primary and Secondary Invariants

We return now to wreath products. We restrict attention to groups $\mathcal{L} \wr \mathcal{G}$, where $\mathcal{L}$ is a finite subgroup of the orthogonal group $\mathbf{O}(k)$. In this section we obtain two types of result. We construct a set of primary invariants for $\mathcal{P}_{V^N}(\mathcal{L} \wr \mathcal{G})$ from a set of primary invariants for $\mathcal{P}_V(\mathcal{L})$ and $\mathcal{P}_{\mathbf{C}^N}(\mathcal{G})$. Moreover, the set chosen for $\mathcal{P}_{\mathbf{C}^N}(\mathcal{G})$ can be any set of primary invariants generating $\mathcal{P}_{\mathbf{C}^N}(\mathbf{S}_N)$. With this information we get the number of secondary invariants and the number of fundamental equivariants needed to generate the modules $\mathcal{P}_{V^N}(\mathcal{L} \wr \mathcal{G})$ and $\vec{\mathcal{P}}_{V^N}(\mathcal{L} \wr \mathcal{G})$ over the ring generated by the set of primary invariants constructed for $\mathcal{P}_{V^N}(\mathcal{L} \wr \mathcal{G})$, respectively.

Theorem 7.10 *Let $\mathcal{L} \subset \mathbf{O}(k)$ be a finite group acting on a complex k-dimensional vector space V. Suppose that $\mathcal{G}$ is a subgroup of $\mathbf{S}_N$ acting on $\mathbf{C}^N$. Consider the group $\Gamma = \mathcal{L} \wr \mathcal{G}$ acting as usual on V^N. Let $\{u_1, \ldots, u_k\}$ be a h.s.o.p. for $\mathcal{L}$, and $\{f_1, \ldots, f_N\}$ for $\mathcal{G}$. Then*

$$\{f_j(u_i(v_1), \ldots, u_i(v_N));\ i = 1, \ldots, k;\ j = 1, \ldots, N\}$$

is an h.s.o.p. for $\mathcal{L} \wr \mathcal{G}$.

Proof. By theorem 7.3, the polynomials $u_1(v_i), \ldots, u_k(v_i)$ have no common zeros besides $v_i = 0$. Also $f_1(x_1, \ldots, x_N), \ldots, f_N(x_1, \ldots, x_N)$ have no common zeros except $(x_1, \ldots, x_N) = (0, \ldots, 0)$. Using theorem 7.3, it suffices to prove that the polynomials in

$$S = \{f_j(u_i(v_1), \ldots, u_i(v_N));\ i = 1, \ldots, k;\ j = 1, \ldots, N\}$$

have no common zero, since by theorem 7.1 the Krull dimension of $\mathcal{P}_{V^N}(\mathcal{L} \wr \mathcal{G})$ is Nk, and S contains Nk polynomials. Suppose that there is a common zero $(v_1^0, \ldots, v_N^0)$, and suppose that $v_1^0 \neq 0$. Let $X^i = (u_i(v_1^0), \ldots, u_i(v_N^0))$ for $i = 1, \ldots, k$. Since the set $\{u_j(v)\}$ has no common zero, then it cannot happen that $u_1(v_1^0) = \cdots = u_k(v_1^0) = 0$. Therefore $X^i \neq 0$ for some i. But then X^i is a nontrivial zero of $f_1(x_1, \ldots, x_N), \ldots, f_N(x_1, \ldots, x_N)$, a contradiction. Thus S is an h.s.o.p. for $\mathcal{L} \wr \mathcal{G}$. $\square$

Applying the previous theorem and theorem 7.6, we can relate the number of invariants and equivariants for $\mathcal{L} \wr \mathcal{G}$ to the numbers for $\mathcal{L}$ and $\mathcal{G}$. If $d_1, \ldots, d_k$ denote the degrees of $u_1, \ldots, u_k$ and $D_1, \ldots, D_N$ the degrees of $f_1, \ldots, f_N$, then let d be the number of secondary invariants for $\mathcal{L}$ and D the number of secondary invariants for $\mathcal{G}$.

Corollary 7.11 *With the conditions of theorem 7.10 and the above notation, the number of secondary invariants for $\mathcal{L} \wr \mathcal{G}$ (using the h.s.o.p. S) is $d^N D (D_1 \cdots D_N)^{k-1}$.*

Proof. Note that $\Gamma = \mathcal{L}^N \dot{+} \mathcal{G}$ and so $|\Gamma| = |\mathcal{L}|^N |\mathcal{G}|$. By theorem 7.6

$$d = \frac{d_1 \cdots d_k}{|\mathcal{L}|}, \quad D = \frac{D_1 \cdots D_N}{|\mathcal{G}|}.$$

The rest follows by theorem 7.6. $\square$

Let r denote the number of fundamental equivariants for $\mathcal{L}$ and R the number of fundamental equivariants for $\mathcal{G}$.

Corollary 7.12 *With the conditions of theorem 7.10 and the above notation, the number of fundamental equivariants for $\mathcal{L} \wr \mathcal{G}$ (using the h.s.o.p. S) is $rd^{N-1}R(D_1 \cdots D_N)^{k-1}$.*

Proof. By theorem 7.9

$$r = k\frac{d_1 \cdots d_k}{|\mathcal{L}|}, \quad R = N\frac{D_1 \cdots D_N}{|\mathcal{G}|}.$$

The rest follows using theorem 7.9 and the fact that an h.s.o.p. for $\mathcal{P}_{V^N}(\mathcal{L} \wr \mathcal{G})$ is also a h.s.o.p. for $\vec{\mathcal{P}}_{V^N}(\mathcal{L} \wr \mathcal{G})$ (by remark 7.8). □

Remark 7.13

A good choice of primary invariants for $\mathcal{G}$ consists of the elementary symmetric functions $\sigma_1, \sigma_2, \ldots, \sigma_N$, or any algebra basis for the ring of symmetric polynomials $\mathcal{P}(\mathbf{S}_N)$. Another possibility is the set of the kth power sums $x_1^k + x_2^k + \cdots + x_N^k$ for $1 \leq k \leq N$. If we choose either of these sets for the primary invariants, then with the conditions of corollaries 7.11 and 7.12 the number of secondary invariants for $\mathcal{L} \wr \mathcal{G}$ is $d^N D(N!)^{k-1}$ and the number of fundamental equivariants is $rd^{N-1}R(N!)^{k-1}$. Here $D = N!/|\mathcal{G}|$ and $R = (NN!)/|\mathcal{G}|$.

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