Integrais Gaussianos, Teorema de Wick e Diagramas de Feynman

Peter Gothen

① Motivação

Recordar: T. de campo estatística

- $\Omega = \{ \varphi : M \longrightarrow \mathcal{T} \}$ – campos
- $\mathcal{H} : \Omega \longrightarrow \mathbb{R}$ – Hamiltoniano
- $Z = Z(\beta) = \int_\Omega \mathcal{D}\varphi \, e^{-\beta \mathcal{H}(\varphi)}$ – função de partição

Variáveis aleatórias (suponha $\mathcal{T} = \mathbb{R}^d$)

$$\overset{X_{i,x}}{\varphi_i(x)} : \Omega \longrightarrow \mathcal{T} \qquad , i = 1, \dots, d$$

$$\varphi \longmapsto \varphi_i(x) \qquad \varphi = (\varphi_1, \dots, \varphi_d)$$

Funções de correlação (momentos, covariância..)

$$\langle \varphi_{i_1}(x_1) \dots \varphi_{i_m}(x_m) \rangle = \frac{1}{Z} \int_\Omega \mathcal{D}\varphi \, \varphi_{i_1}(x_1) \dots \varphi_{i_m}(x_m) e^{-\beta \mathcal{H}(\varphi)}$$

Em teoria quântica de campo:

- calcular

M - espaço-tempo

$\varphi(x,t)$

$$\langle \varphi_{i_1}(x_1,t_1) \cdots \varphi_{i_n}(x_n,t_n) \rangle$$

$$= \mathcal{N} \int_\Omega \varphi_{i_1}(x_1,t_1) \cdots \varphi_{i_n}(x_n,t_n)\, e^{iS(\varphi)/\hbar}\, \mathcal{D}\varphi$$

$$S(\varphi) = \int_M \mathcal{L}(\varphi)\, dx\, dt \quad - \text{ acção}$$

densidade Lagrangeana ; e.g.

$$\mathcal{L}(\varphi) = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 - \frac{1}{4!} \lambda \varphi^4$$

- calcular p/ expoente real ("teoria Euclideana") e fazer prolongamento analítico... (rotação de Wick)

Caso + ~~simples~~ simples possível: $M = \{p_0\}$
um ponto

$$\Omega = \{ \{p_0\} \to \tau \} = \tau = \mathbb{R}^d$$

Calcular: $\displaystyle\int_{\mathbb{R}^d} x_{i_1} \cdots x_{i_m} \, e^{-S(x)/\hbar} \, dx \qquad \overset{(-\beta \mathcal{H}(x))}{} \qquad (\lambda \leftrightarrow \hbar \leftrightarrow \tfrac{1}{\beta}) \qquad x \in \mathbb{R}^d$

"steepest descent"
(expansão assintótica em $\hbar$)
$\leftrightarrow$ baixa temperatura.

$$\int_{\mathbb{R}^d} x_{i_1} \cdots x_{i_m} \, e^{-\left(\frac{B(x,x)}{2} + \tilde{S}(x)\right)} dx$$

- $B(x,x) = \langle x, Bx \rangle$ – forma quadrática
- $\tilde{S}(x) =$ termos de ordem ≥ 3 em x,

e.g. $\displaystyle \tilde{S}(x) = \frac{\lambda}{4!} \sum_{i,j,k,l} b_{ijkl} \, x_i x_k x_j x_l$

("teoria φ^4")

② Integrais Gaussianos ($\tilde{S}(x) = 0$)

$\underline{d=1}$:
$$\int_{-\infty}^{\infty} e^{-\frac{bx^2}{2} + jx}\, dx = \sqrt{\frac{2\pi}{b}}\, e^{j^2/(2b)}$$

$b > 0,\ j \in \mathbb{R}$ (ou $\mathbb{C}$)

$\underline{d \geq 1}$ diagonalizar B

$$Bu_i = \lambda_i u_i \quad ; \quad U := [u_1, \dots, u_d]$$

$$U^{-1} B U = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_d \end{bmatrix} \quad ; \quad UU^t = I$$

$J \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} e^{-\frac{1}{2}\langle x, Bx\rangle + \langle J, x\rangle}\, dx$$

$$= \int_{\mathbb{R}^d} e^{-\frac{1}{2}\langle Ux, BUx\rangle + \langle J, Ux\rangle}\, dx$$

$(\xi_1, \dots, \xi_d) := U^{-1} J$

$$= e^{-\frac{1}{2}(\lambda_1 x_1^2 + \dots + \lambda_d x_d^2) + \xi_1 x_1 + \dots + \xi_d x_d}$$

$$= \sqrt{\frac{(2\pi)^d}{\lambda_1 \cdots \lambda_d}}\, e^{\frac{\xi_1^2}{2\lambda_1} + \dots + \frac{\xi_d^2}{2\lambda_d}} \boxed{= \sqrt{\frac{(2\pi)^d}{\det(B)}} \cdot e^{\frac{1}{2}\langle J, B^{-1} J\rangle}}$$

③ Teorema de Wick

Calcular:

$$\int_{\mathbb{R}^d} x_{i_1} \cdots x_{i_m} e^{-\frac{1}{2}\langle x, Bx\rangle} dx$$

Truque:

$$Z(J) := \int_{\mathbb{R}^d} e^{-\frac{1}{2}\langle x, Bx\rangle + \underbrace{\langle J, x\rangle}_{= \sum J_i x_i}} dx$$

$$\frac{\partial Z(J)}{\partial J_i} = \int_{\mathbb{R}^d} x_i e^{-\frac{1}{2}\langle x, Bx\rangle + \langle J, x\rangle} dx$$

$$\left.\frac{\partial^m Z}{\partial J_{i_1} \cdots \partial J_{i_m}}\right|_{J=0} = \int_{\mathbb{R}^d} x_{i_1} \cdots x_{i_m} e^{-\frac{1}{2}\langle x, Bx\rangle + \cancel{\langle J, x\rangle}} dx$$

$$\|$$

$$\frac{\partial^m Z}{\partial J_{i_1} \cdots \partial J_{i_m}} \left(\sqrt{\frac{(2\pi)^d}{\det(B)}}\, e^{\frac{1}{2}\langle J, B^{-1} J\rangle} \right) \Bigg|_{J=0}$$

Cada $\frac{\partial}{\partial J_i}$ leva p/ baixo factores

$$b^{-1}_{ij} J_j \quad (\cdot \exp)$$

ao fazer $J=0$ isto dá zero, a menos que uma derivada $\frac{\partial}{\partial J_j}$ actua, dando

$$b^{-1}_{ij} \; (\cdot \exp)$$

conclusão: emparelhar $i_1, \dots, i_m$; $m = 2k$

em $(p_1, p_2), \dots, (p_{m-1}, p_m)$

Teorema de Wick

$$\int_{\mathbb{R}^d} x_{i_1} \cdots x_{i_m} e^{-\langle x, Bx\rangle/2} \, dx$$

$= 0$ se m é ímpar

$$= \sqrt{\frac{(2\pi)^d}{\det(B)}} \cdot \sum_{\Pi_k} b^{-1}_{p_1, p_2} \cdots \cdots b^{-1}_{p_{m-1}, p_m}$$

emparelhamentos $(p_1, p_2), \dots, (p_{m-1}, p_m)$ de $i_1 \dots i_m$; $m = 2k$

Nota: $|\Pi_k| = \frac{(2k!)}{2^k k!} = (2k-1)(2k-3) \cdots 3 \cdot 1 =: (2k)!!$

Demonstração alternativa $\quad x_{i_j} \leftrightarrow \ell_j : \mathbb{R}^d \to \mathbb{R}$

$\to$ reduzir ao caso $\ell_{i_1} = \dots = \ell_{i_m}$

$\to$ fazer mudança de variável linear

$B = I \;;\; \ell_{i_j}^{(1)} = x_1$

$\to$ caso $d = 1$:

$$\int_{-\infty}^{\infty} x^{2k} e^{-\frac{x^2}{2}} dx$$

$$= 2 \int_0^{\infty} 2^{k-\frac{1}{2}} \cdot u^{\overset{k+\frac{1}{2}-1}{k-\frac{1}{2}}} \cdot e^{-u} du$$

$$= 2^{k+\frac{1}{2}} \cdot \Gamma\left(k+\frac{1}{2}\right)$$

$$= 2^{k+\frac{1}{2}} \cdot \frac{1 \cdot 3 \cdot \ldots \cdot (2k-1)}{2^k} \sqrt{\pi}$$

$$= \sqrt{2\pi} \cdot (2k)!!$$

$u = \frac{x^2}{2}$

$du = x\,dx$

$x^{2k} dx$

$= x^{2k-1} \cdot x\,dx$

$= (2u)^{1/2(2k-1)} \cdot du$

$= 2^{k-\frac{1}{2}} \cdot u^{k-\frac{1}{2}} du$

$\Gamma(s) =$

$= \int_0^{\infty} t^{s-1} e^{-t} dt$

④ Caso $S(x) = \frac{B(x,x)}{2} + \tilde{S}(x)$ – expansão perturbativa.

$\underline{1^{\underline{o}}}$: $$S(x) = \frac{B(x,x)}{2} + \frac{\lambda}{4!} \underbrace{\sum_{i,j,k,l} b_{ijkl} x_i x_j x_k x_l}_{=: B_4(x)}$$

Calcular:

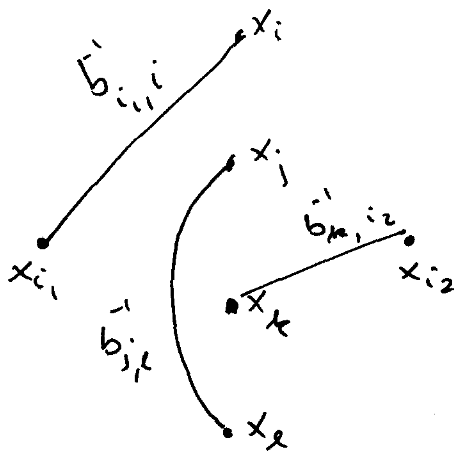
$$\langle x_{i_1} x_{i_2} \rangle = \int_{\mathbb{R}^d} x_{i_1} x_{i_2} e^{-\frac{\langle x, Bx \rangle}{2} - \frac{\lambda}{4!} B_4(x)} dx$$

$$= \int_{\mathbb{R}^d} x_{i_1} x_{i_2} e^{-\frac{\langle x, Bx \rangle}{2}} \cdot \sum_{n=0}^{\infty} \frac{1}{n!} \left(-\frac{\lambda}{4!} B_4(x) \right)^n dx$$

$$=: \sum_{n=0}^{\infty} I_n \cdot \lambda^n$$

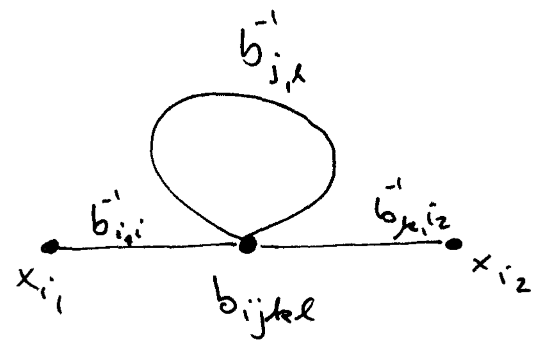
$$I_0 = b^{-1}_{i_1 i_2} \cdot \sqrt{\frac{(2\pi)^d}{\det(B)}} \quad (\text{T. de Wick})$$

$$I_1 = \frac{-1}{4!} \int_{\mathbb{R}^d} x_{i_1} x_{i_2} e^{-\frac{\langle x, Bx \rangle}{2}} \sum_{i,j,k,l} b_{ijkl} x_i x_j x_k x_l \, dx$$

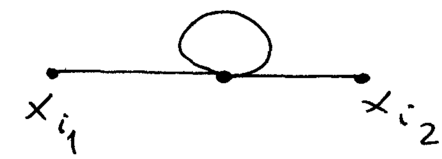


b_{ijkl}

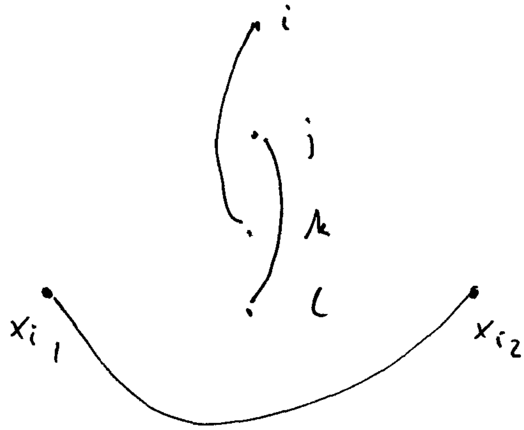
. . . .



$\mathbf{F}_1$

$=$ 

$4 \cdot 3 = 12$ emparelhamentos



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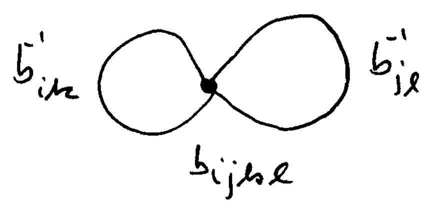
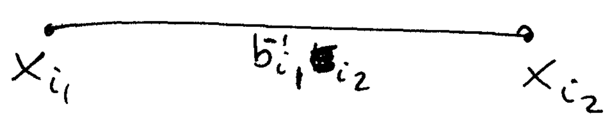
$\mathbf{F}_2$





3 emparelhamentos

$$\boxed{\text{Grafos de Feynman etiquetados}}$$

Total: $12 + 3 = 15 = 6!!$

$$I_1 = \sqrt{\frac{(2\pi)}{\det(B)}} \left(\overset{A(\Gamma_1)}{\underbrace{\frac{1}{2} \cancel{F_1}\, \Gamma_1}_{\substack{12 \cdot d^4 \\ \text{parcelas}}}} + \overset{A(\Gamma_2)}{\underbrace{\frac{1}{8} \cancel{F_2}\, \Gamma_2}_{\substack{3 \cdot d^4 \\ \text{parcelas}}}} \right)$$

$$A(\Gamma_1) = \frac{1}{4!} \sum_{ijkl} \sum_{\substack{\text{emparelhamentos} \\ (p_i,p_j)\,(p_k,p_l)}} b^{-1}_{i_1 p_i}\, b^{-1}_{p_j p_k}\, b^{-1}_{p_l i_2}\, \underbrace{b_{ijkl}}_{=\, b_{p_i p_j p_k p_l}}$$

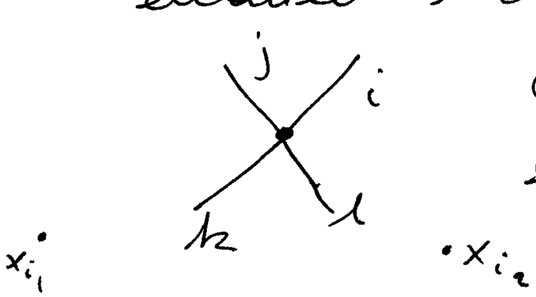
só existem d^4 parcelas distintas — cada uma repetida <u>12 vezes</u>.

$$= \underbrace{\frac{12}{4!}}_{=\,\frac{1}{2}} \underbrace{\sum_{ijkl} b^{-1}_{i_1 i}\, b^{-1}_{jk}\, b^{-1}_{l i_2}\, b_{ijkl}}_{=\, F_{\Gamma_1}}$$

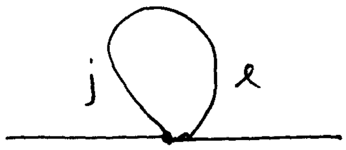
$$A(\Gamma_2) = \underbrace{\frac{3}{4!}}_{=\,\frac{1}{8}} \underbrace{\sum_{i,j,k,l} b^{-1}_{ik}\, b^{-1}_{jl}\, b^{-1}_{i_1 i_2} \cdot b_{ijkl}}_{=\, F_{\Gamma_2}}$$

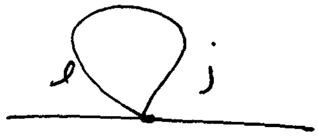
<u>O factor de simetria</u>

— em princípio cada permutação das arestas emanentes do vértice interno

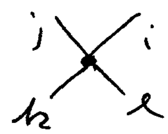


corresponde a um emparelhamento diferente.

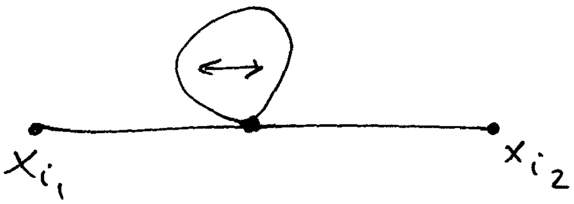
Mas: 



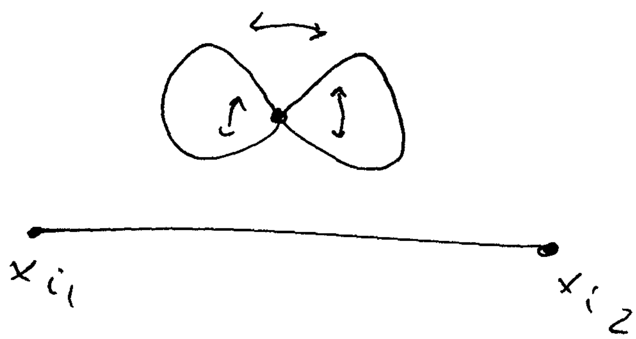
se houver simetria no diagrama haverá menos emparelhamentos:

$$\mathrm{Aut}(\Gamma) = \{ \text{grupos de permutações} $$
de 

que preserve $\Gamma \}$



$|\mathrm{Aut}(\Gamma)| = 2$



$|\mathrm{Aut}(\Gamma)| = 8$

– o número de emparelhamentos distintos é:

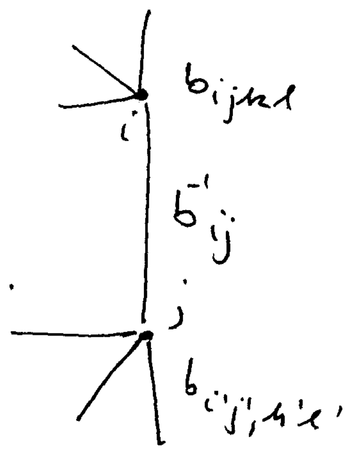
$$\frac{4!}{|\mathrm{Aut}(\Gamma)|}$$

Ordens superiores:

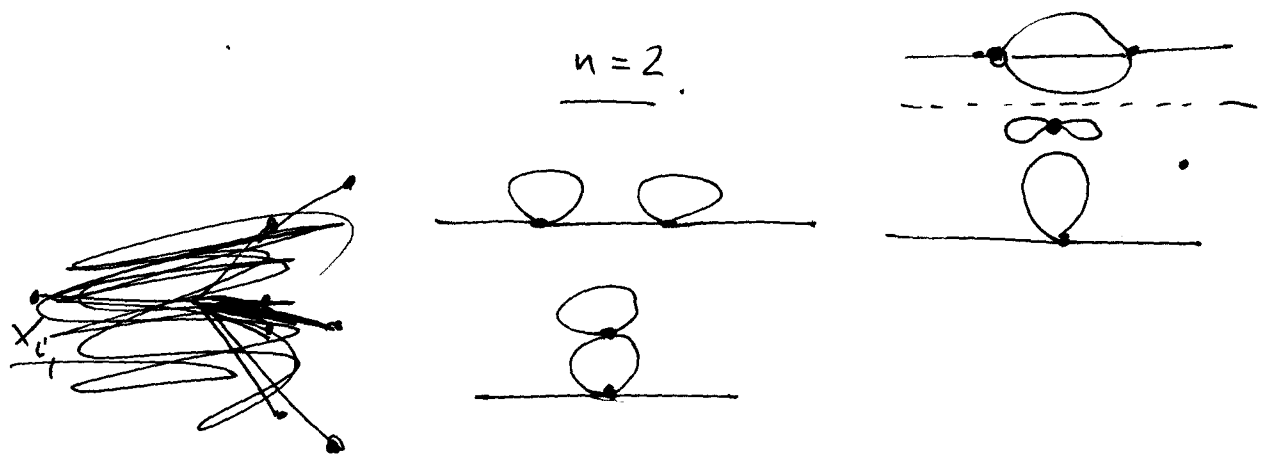
$$I_n = \sqrt{\frac{(2\pi)^d}{\det(B)}} \sum_{\substack{\Gamma \text{ c/ } n \\ \text{vértices} \\ \text{internos}}} \frac{1}{|\mathrm{Aut}(\Gamma)|} F_\Gamma$$

onde

$$F_\Gamma = \sum_{\substack{(i_1,j_1,k_1,l_1) \\ \vdots \\ (i_n,j_n,k_n,l_n)}} \overbrace{b^{-1}_{..} \cdots b^{-1}_{..}}^{n+1} \cdot b_{i_1j_1k_1l_1} \cdots b_{....}$$



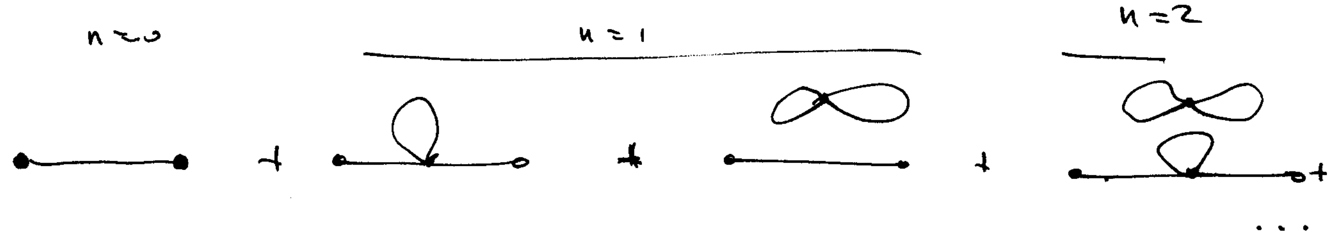
$\underline{n=2}$.

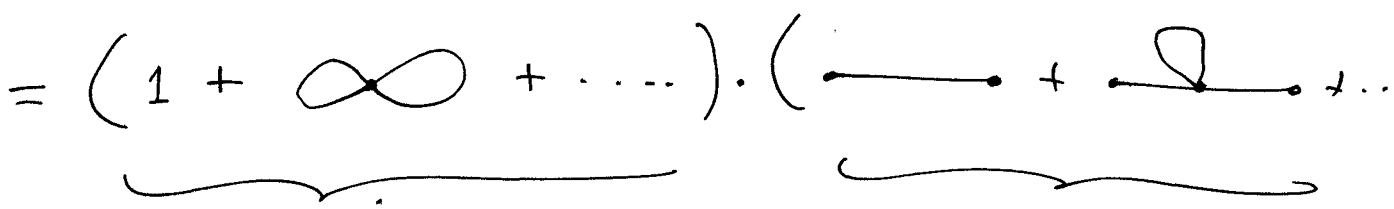


e $\mathrm{Aut}(\Gamma)$ = automorfismos ~~da~~ permitindo permutação de arestas e vértices internos.

(normalmente ~~[illegible]~~
igual ao factor $\frac{1}{n!}$ da exponencial

Nota final: Podemos "factorizar" a sequência de diagramas:





sem vértices externos - grafo do "vácuo"

c/ 2 vértices externos

$$\frac{\langle x_{i_1} \cdots x_{i_n} \rangle}{\langle \emptyset \rangle} = \sum_{\substack{\text{grafos c/} \\ n \text{ vértices} \\ \text{internos e sem} \\ \text{componente de "vácuo"}}} \frac{F_\Gamma}{|\mathrm{Aut}(\Gamma)|} \cdot \lambda^n$$

$$\langle \emptyset \rangle = \int_{\mathbb{R}^d} e^{-\frac{\langle x, Bx \rangle}{2} - \frac{\lambda}{4!} B(x)} \, dx$$

Fórmula geral p/ calcular função

$$\langle x_{i_1} \cdots x_{i_m} \rangle = \int_{\mathbb{R}^d} x_{i_1} \cdots x_{i_m} \, e^{-\langle x, Bx \rangle/2 + \tilde{S}(x)} \, dx$$

com $$\tilde{S}(x) = \sum_{r \geq 3} \frac{\lambda^{r/2-1}}{r!} \underbrace{B_r(x)}_{\text{grau } r}$$

$$\langle x_{i_1} \cdots x_{i_m} \rangle = \sqrt{\frac{(2\pi)^d}{\det(B)}} \cdot \sum_{\Gamma} \frac{1}{|\mathrm{Aut}(\Gamma)|} F_\Gamma \cdot \lambda^{b(\Gamma)}$$

$$b(\Gamma) = \sum_{\substack{\text{vértices}\\ \text{internos}}} \left(\deg(v)/2 - 1 \right)$$

$$F_\Gamma = \sum_{\substack{\text{índices}\\ \text{internos } I}} F_\Gamma(I)$$

calculado a partir de B e B_m