# About Cartan Geometrization of Non Holonomic Mechanics

**J.N. Tavares**

Centro de Matemática da Universidade do Porto

Dept. Matemática Pura, Faculdade de Ciências do Porto
Rua do Campo Alegre, No. 687, 4169-007 Porto, Portugal.

E-mail adress: jntavar@fc.up.pt

### Abstract

Following the ideas of Élie Cartan (1928), we use Cartan's equivalence method and the notion of Cartan's affine generalized space and development to geometrize non holonomic mechanics.[1]

## 1 Introduction

The purpose of this paper is to give, using modern differential geometrical tools, a detailed version of the ideas of Élie Cartan, exposed in his adress at the 1928 International Congress of Mathematicians (see [5]), about geometrization of non holonomic systems.

This important paper seems forgotten in the mathematical literature devoted to non holonomic systems. To our knowledge, the only exception is due to Jair Koiller and his colaborators, in a recent preprint that has appeared during the preparation of this work (see [12]), in which they also make a tentative to bring at daylight Cartan's paper. However their methods are very different from those we develop here. In fact, they use extensively the traditional Koszul approach to connection theory, based in covariant derivatives, as is explained for example in [19], and they put emphasis in other issues that are not considered here. In this paper we have tryed instead to follow closely the two key ideas of Cartan's approach to geometric structures, namely his equivalence method, or in modern terms the geometry of $G$-structures (see [8], [10], [20]), which hopefully seems the strongest way to treat the geometric structure behind non holonomic systems, and his notion of *"generalized space"*, here space with affine connection (see [6], [7], and for a modern approach, the recent book [18]). These two key ideas were developed by Cartan along several years, in a lot of papers, where he has applied them extensively, for example, to relativity theory (see [6]) and to his program of geometrization of differential equations (see the third volume of his *"Oeuvres complètes"*).

Given a non-holonomic mechanical system $\mathfrak{M}$ with configuration space $Q$, a $n$-dimensional smooth Riemannian manifold, with Riemannian metric $g$ (the kinetic energy), and non-holonomic constraints given by a completely non integrable distribution $\mathcal{D}$ of dimension $d$, the main idea is to associate to $\mathfrak{M}$, an intrinsically defined Euclidean (or metric) connection, in general with torsion, and to use it to develope the space $Q$, along any of its curves, into a fixed affine space $\mathcal{D}_o$, for some fixed point $o \in Q$.

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The tentactive of associating to a non-holonomic mechanical system a connection, goes back to Synge, Vancreanu, and more recently, citing just a few, to Vershik and Gershkovich ([21], [22]), Bates and Sniatycki ([2]) and Bloch and Crouch ([4]). However, in these papers, the connections found are in general neither metric nor unique. In fact, often the choice of connection is based on somewhat ad hoc assumptions which obscures the true geometric realm of the structure of non-holonomic systems. On the contrary, and this one the main differences of the approach we develop, the connection founded here is intrinsically associated to the non-holonomic system, at least for 2-step distributions, and moreover it is a metric connection, though in general with torsion. This difference is very explicit in the example treated in section 4, the constrained particle, which must be compared with ([2], example 2) and ([4], example 6.2). In both these works the connection is not metric.

Another subject that we explore is the following - to the non-holonomic system $\mathfrak{M}$ (in the 2-step situation), we associate a Cartan (affine) connection to the affine frame bundle of $Q$ (in Cartan's terminology [6], [7], a "generalized space" - this is part of Cartan's generalization of Klein's Erlangen programm, as is explained in the recent book [18]), which is then used to develope $Q$, along any of its curves, into a fixed affine space $\mathcal{D}_o$, for some fixed point $o \in Q$. This strongly resembles the analogous situation for holonomic systems, when we roll (eventually with skidding or spinning) a $d$-dimensional submanifold on another $d$-dimensional submanifold (a $d$-plane, for example) in $\mathbb{R}^n$ (see the beautiful paper of Nomizu [14]). However, in general, we have now torsion, whose geometrical meaning is made clear, in our context, in section 3 and more concretely in the example of section 4 - take a "small" loop, based in $o \in Q$, and develop it in $\mathcal{D}_o$ to obtain a curve that starts in $o$. In general, this curve doesn't close, and, to second order, the failure to close is measured by a vector which is exactly the torsion of the connection at $o$ (see section 3 for a rigorous approach).

The paper is organized as follows. In section 1, we use Cartan's equivalence method to geometrize non holonomic mechanics, by associating to such a system an Euclidean connection. For a 2-step generating distribution $\mathcal{D}$, we are able to associate intrinsically two Euclidean connections, in general with torsion, recovering the results of Cartan in [5]. We also include, for pedagogical reasons and also to make the paper as much self-contained as possible, a short exposition about Cartan's equivalence method, following closely references [20] and the very influential recent book [13], since this method seems poorly used in the non-holonomic context. Section 2, gives a detailed version of the notion of Cartan's affine generalized space and also of the notion of development. This is then used to develope $Q$, along any of its curves, into a fixed affine space $\mathcal{D}_o$, for some fixed point $o \in Q$. Finally, in section 3, we ilustrate the previous theory by working out the detailed computations in the example of a constrained particle in $\mathbb{R}^3$.

## 2 Cartan Geometrization of Non Holonomic Mechanics

Consider a non-holonomic mechanical system $\mathfrak{M}$ with configuration space $Q$, a $n$-dimensional smooth Riemannian manifold, with Riemannian metric $g$ (the kinetic energy), a smooth 1-form $F \in \Omega^1(Q)$ (the force field), and non-holonomic constraints[2], given by a smooth rank $d$ completely non integrable vector subundle of $TQ$, i.e, a completely non integrable distribution $\mathcal{D}$ of dimension $d$ in $Q$.

We also assume that $\mathcal{D}$ is brackett generating which, by Chow theorem (see [13]), guarantees that the set of all possible positions of our mechanical system $\mathfrak{M}$ is all of $Q$.

The d'Alembert-Lagrange principle (see [1]) says that the dynamics of $\mathfrak{M}$ obeys the following condition:

$$[L] - F \in \mathcal{D}^{\perp} \tag{2.1}$$

where $[L]$ is the Lagrange derivative of the Lagrangian $L = \frac{1}{2}g$ (see [1], pag. 12) and $\mathcal{D}^{\perp}$ is the anihilator of $\mathcal{D}$ in $T^*Q$.

Hereafter we use the following indices conventions: $i, j, k, \ell = 1, \cdots, d = \dim \mathcal{D}$; $\alpha, \beta, \gamma, \lambda = d+1, \cdots, n = \dim Q$ and $a, b, c = 1, \cdots, n$.

We denote by $V$ the vector space $\mathbb{R}^n$ of column vectors, with a fixed basis $\{\boldsymbol{\epsilon}_a\}$, and by $V^*$ its dual of row vectors, with the dual basis $\{\boldsymbol{\epsilon}^a\}$, and we also consider the subspace $S$ of $V$, generated by the first $d$

[2] we consider only the time independent case, for simplicity.

vectors $\{\boldsymbol{\epsilon}_i\}_{i=1,\cdots,d}$ of the basis $\{\boldsymbol{\epsilon}_a\}$. By a **0-adapted coframe** $\theta_q$ for $\mathcal{D}_q$, $q \in Q$, we mean an isomorphism $\theta_q : T_qQ \to V$, which satisfies $\theta_q(\mathcal{D}_q) = S$ and $\theta_q^* \langle\, ,\rangle|_S = g_q|_{\mathcal{D}_q}$, where $\langle\, ,\rangle|_S$ is the usual Euclidean inner product on $S \cong \mathbb{R}^d$. Moreover, we denote by $G_0$ the subgroup of $GL(V)$ consisting on the linear isomorphisms of $V$ that fix $S$, and which, when restricted to $S$, are orthogonal transformations of $S$. In terms of the basis $\{\boldsymbol{\epsilon}_a\} = \{\boldsymbol{\epsilon}_i; \boldsymbol{\epsilon}_\alpha\}$ for $V$, $G_0$ is the subgroup of $GL(n)$ given by the following block triangular matrices:

$$\begin{bmatrix} C & B \\ 0 & A \end{bmatrix} \tag{2.2}$$

where $A$ and $B$ are arbitrary real matrices (of functions), respectively $(n-d)\times(n-d)$, $d\times(n-d)$, $C$ is an orthogonal $d \times d$ matrix, and $\det C \det A \neq 0$.

Consider a (local) 0-adapted coframe $\theta$ for $\mathcal{D}$. Put $\theta = \theta^i\boldsymbol{\epsilon}_i + \theta^\alpha\boldsymbol{\epsilon}_\alpha$ and look at $\theta$ as a column vector of 1-forms on $Q$: $\theta = [\theta^a] = \begin{bmatrix} \theta^i \\ \theta^\alpha \end{bmatrix}$. Thus (locally) $\theta^\alpha$ anihilates $\mathcal{D}$ and $g|_{\mathcal{D}} = (\theta^1)^2 + \cdots + (\theta^d)^2\big|_{\mathcal{D}}$. Of course such a coframe is not unique - the indeterminacy is measured by the gauge group $G_0$. Formally, we have a $G_0$-structure $\pi : \mathcal{B}_0 = \mathcal{B}_{G_0} \to Q$, where $G_0$ is the subgroup of $GL(n)$ given by the above mentioned block triangular matrices. The group $G_0$ acts on the right of $\mathcal{B}_0$ by the rule $R_g(\theta) = \theta \cdot g = g^{-1}\theta$ where $\theta = [\theta^a]$, and $g \in G_0 \subset GL(n)$.

If we fix a 0-adapted coframe $\theta = [\theta^a]$, defined on an open set $U \subseteq Q$ (i.e., a local section of $\mathcal{B}_0$ over $U$), then we have a trivialization of the $G_0$-bundle over $U$, given by:

$$\begin{array}{cccc} \tau_\theta : & U \times G_0 & \longrightarrow & \mathcal{B}_0|_U \\ & (q,g) & \longmapsto & g^{-1}\theta_q \end{array} \tag{2.3}$$

that is equivariant in the sense that $\tau_\theta(q, gh) = (gh)^{-1}\theta_q = h^{-1}g^{-1}\theta_q = h^{-1}\tau_\theta(q,g) = \tau_\theta(q,g)\cdot h$.

We now consider the soldering form $\boldsymbol{\Theta}$ (or tautological $V$-valued 1-form), defined on $\mathcal{B}_0$, through the formula:

$$\boldsymbol{\Theta}_\eta(\mathbf{v}) = \eta \circ \pi_*(\mathbf{v}), \qquad \mathbf{v} \in T_\eta\mathcal{B}_0, \quad \eta \in \mathcal{B}_0 \tag{2.4}$$

Note that on the LHS of (2.4), $\eta$ is considered as a point of $\mathcal{B}_0$, while on the RHS is considered as an isomorphism $\eta : T_{\pi(\eta)}Q \to V$, defined by $\eta(v) = [\eta^a(v)]$, $v \in T_{\pi(\eta)}Q$. The soldering form has the following properties (see [10],[13], [20]):

- **Equivariance**: $R_g^*\boldsymbol{\Theta} = g^{-1}\boldsymbol{\Theta}$.
- **Semi-basic**: $\iota_X\boldsymbol{\Theta} = 0$, for every vertical vector field $X$ (tangent to the fibers).
- **Reproducing property**: if $\sigma : U \to \mathcal{B}_0$ is a local section, then $\sigma^*\boldsymbol{\Theta} = \sigma$, where on the RHS $\sigma$ is viewed as a $V$-valued form on $U$.

Using the local trivialization (2.3), it's easy to see that $(\tau_\theta^*\boldsymbol{\Theta})_{(q,g)} = g^{-1}\theta_q$. Let us denote $\tau_\theta^*\boldsymbol{\Theta}$ simply by $\boldsymbol{\Theta}$. Then we have:

$$\boldsymbol{\Theta}_{(q,g)} = \begin{bmatrix} \boldsymbol{\Theta}^i_{(q,g)} \\ \boldsymbol{\Theta}^\alpha_{(q,g)} \end{bmatrix} = \begin{bmatrix} C & B \\ 0 & A \end{bmatrix}^{-1} \begin{bmatrix} \theta^j_q \\ \theta^\beta_q \end{bmatrix}, \qquad \text{where } g = \begin{bmatrix} C & B \\ 0 & A \end{bmatrix} \in G_0 \tag{2.5}$$

Following the equivalence method of E. Cartan (see [10],[13], [20]), we now choose a connection form, that is, an equivariant $\mathfrak{g}_0$-valued 1-form $\boldsymbol{\omega}$ on $\mathcal{B}_0$, where $\mathfrak{g}_0 = Lie(G_0)$, that verifies the following two properties:

- $\boldsymbol{\omega}(X_{\boldsymbol{\xi}}) = \boldsymbol{\xi}$, $\forall\boldsymbol{\xi} \in \mathfrak{g}_0$, where $X_{\boldsymbol{\xi}}$ is the infinitesimal generator of the $G_0$-right action on $\mathcal{B}_0$.
- $R_g^*\boldsymbol{\omega} = g^{-1}\boldsymbol{\omega}g$, $\forall g \in G_0$.

If we put $g = \exp(t\boldsymbol{\xi})$, $\boldsymbol{\xi} \in \mathfrak{g}_0$ in the equivariance property $R_g^*\boldsymbol{\Theta} = g^{-1}\boldsymbol{\Theta}$, and differentiate for $t = 0$, we obtain $\mathcal{L}_{X_{\boldsymbol{\xi}}}\boldsymbol{\Theta} = -\boldsymbol{\xi}\cdot\boldsymbol{\Theta}$, and since $\boldsymbol{\Theta}(X_{\boldsymbol{\xi}}) = 0$ we get:

$$\begin{aligned}
d\boldsymbol{\Theta}(X_{\boldsymbol{\xi}}, \mathbf{v}) &= \Big(\iota_{X_{\boldsymbol{\xi}}} d\boldsymbol{\Theta}\Big)(\mathbf{v}) \\
&= \Big(\mathcal{L}_{X_{\boldsymbol{\xi}}}\boldsymbol{\Theta} - d\iota_{X_{\boldsymbol{\xi}}}\boldsymbol{\Theta}\Big)(\mathbf{v}) \\
&= -\boldsymbol{\xi}\cdot\boldsymbol{\Theta}(\mathbf{v}) \\
&= -\Big(\boldsymbol{\omega}(X_{\boldsymbol{\xi}})\cdot\boldsymbol{\Theta}(\mathbf{v}) - \boldsymbol{\omega}(\mathbf{v})\cdot\boldsymbol{\Theta}(X_{\boldsymbol{\xi}})\Big) \\
&\overset{\text{def}}{=} -(\boldsymbol{\omega}\wedge\boldsymbol{\Theta})(X_{\boldsymbol{\xi}}, \mathbf{v})
\end{aligned} \tag{2.6}$$

which shows that $d\boldsymbol{\Theta} + \boldsymbol{\omega}\wedge\boldsymbol{\Theta}$ is a $V$-valued semi-basic 2-form on $\mathcal{B}_0$, and thus can be written as:

$$d\boldsymbol{\Theta} + \boldsymbol{\omega}\wedge\boldsymbol{\Theta} = \mathbf{T} \tag{2.7}$$

where $\mathbf{T}$ is a $V$-valued semi-basic 2-form on $\mathcal{B}_0$. This is the so called **Cartan first structure equation**. $\mathbf{T} = \mathbf{T}[\boldsymbol{\omega}]$ is the **torsion** of the connection $\boldsymbol{\omega}$, and can be expanded $\mathbf{T} = \mathbf{T}^a_{ef}\boldsymbol{\Theta}^e\wedge\boldsymbol{\Theta}^f\otimes\boldsymbol{\epsilon}_a$. However, if we put, for each $\eta\in\mathcal{B}_0$, $\ker\boldsymbol{\omega}_\eta = \mathcal{H}_\eta$ we know that $\eta\mapsto\mathcal{H}_\eta$ is a $n$-dimensional distribution transversal to the fibers and that $\boldsymbol{\Theta}_\eta|_{\mathcal{H}_\eta} : \mathcal{H}_\eta \to V$ is an isomorphism. Using this isomorphism we can consider $\mathbf{T}$ as a function:

$$\mathbf{T} : \mathcal{B}_0 \to V\otimes\wedge^2V^* \cong \mathrm{Hom}(\wedge^2V, V), \qquad \mathbf{T} = \mathbf{T}^a_{ef}\boldsymbol{\epsilon}_a\otimes\boldsymbol{\epsilon}^e\wedge\boldsymbol{\epsilon}^f \tag{2.8}$$

that satisfies the equivariance:

$$\mathbf{T}(\eta\cdot g)(v\wedge w) = g^{-1}\,\mathbf{T}(\eta)(gv\wedge gw), \qquad v, w\in V,\ \eta\in\mathcal{B}_0,\ g\in G_0 \tag{2.9}$$

Now we study how the torsion varies with the choice of the connection. So, let us assume that we choose another connection form $\widehat{\boldsymbol{\omega}}$. Then $\widehat{\boldsymbol{\omega}} = \boldsymbol{\omega} + \boldsymbol{\varphi}$, for some $\mathfrak{g}_0$-valued semi-basic 1-form of adjoint type, i.e., $R_g^*\boldsymbol{\varphi} = g^{-1}\boldsymbol{\varphi}g$ and $\iota_X\boldsymbol{\varphi} = 0$, $\forall X$ vertical. Therefore, we can write $\boldsymbol{\varphi} = \boldsymbol{\varphi}_a\boldsymbol{\Theta}^a$, for $\mathfrak{g}_0$-valued functions $\boldsymbol{\varphi}_a$. Using again the above mentioned isomorphism $\boldsymbol{\Theta}_\eta|_{\mathcal{H}_\eta} : \mathcal{H}_\eta\to V$, and in terms of a basis $\{\boldsymbol{\xi}_r\}$ for $\mathfrak{g}_0$ and $\{\boldsymbol{\epsilon}^a\}$ for $V^*$, we can write $\boldsymbol{\varphi}$ as a function:

$$\boldsymbol{\varphi} : \mathcal{B}_0 \to \mathfrak{g}_0\otimes V^* \cong \mathrm{Hom}(V, \mathfrak{g}_0), \qquad \boldsymbol{\varphi} = \boldsymbol{\varphi}^r_a\,\boldsymbol{\xi}_r\otimes\boldsymbol{\epsilon}^a \tag{2.10}$$

for certain functions $\boldsymbol{\varphi}^r_a$ on $\mathcal{B}_0$. Therefore we see that the space $\mathfrak{g}_0\otimes V^*$ parametrizes the ambiguity in the choice of the connection 1-form. We also have that $\boldsymbol{\varphi}$ is $G$-equivariant:

$$\boldsymbol{\varphi}(\eta\cdot g)(v) = g^{-1}\cdot\boldsymbol{\varphi}(\eta)(gv)\cdot g, \qquad v\in V,\ \eta\in\mathcal{B}_0,\ g\in G_0 \tag{2.11}$$

By Cartan first structure equation (2.7), we now have:

$$d\boldsymbol{\Theta} = -\boldsymbol{\omega}\wedge\boldsymbol{\Theta} + \mathbf{T} = -\widehat{\boldsymbol{\omega}}\wedge\boldsymbol{\Theta} + \widehat{\mathbf{T}}$$

where $\widehat{\mathbf{T}}$ is the torsion of $\widehat{\boldsymbol{\omega}}$, and so:

$$\widehat{\mathbf{T}} - \mathbf{T} = (\widehat{\boldsymbol{\omega}} - \boldsymbol{\omega})\wedge\boldsymbol{\Theta} = \boldsymbol{\varphi}\wedge\boldsymbol{\Theta} \tag{2.12}$$

By (2.8), we have, for each $\eta\in\mathcal{B}_0$, that $\widehat{\mathbf{T}}(\eta) - \mathbf{T}(\eta) \in V\otimes\wedge^2V^*$, and by (2.10) $\boldsymbol{\varphi}(\eta)\in\mathfrak{g}_0\otimes V^* \hookrightarrow (V\otimes V^*)\otimes V^*$. Of course $\boldsymbol{\varphi}(\eta)\wedge\boldsymbol{\Theta}(\eta)$ must be in $V\otimes\wedge^2V^*$, and we can prove that, for each $\eta\in\mathcal{B}_0$, we have:

$$\widehat{\mathbf{T}}(\eta) - \mathbf{T}(\eta) = \boldsymbol{\varphi}(\eta)\wedge\boldsymbol{\Theta}(\eta) = \delta(\boldsymbol{\varphi}(\eta)) \tag{2.13}$$

where $\delta$ is the **torsion map** $\delta : \mathfrak{g}_0\otimes V^* \to V\otimes\wedge^2V^*$, obtained by the composition:

$$\delta : \mathfrak{g}_0\otimes V^* \hookrightarrow (V\otimes V^*)\otimes V^* \longrightarrow V\otimes\wedge^2V^* \tag{2.14}$$

where the last map skew-symmetrizes the final two $V^*$-factors. In fact, in terms of the isomorphisms $\mathfrak{g}_0 \otimes V^* \cong \mathrm{Hom}(V, \mathfrak{g}_0)$ and $V \otimes \wedge^2 V^* \cong \mathrm{Hom}(\wedge^2 V, V)$, the torsion map $\delta$ can be written in the useful form:

$$\delta(\psi)(v \wedge w) = \psi(v)\, w - \psi(w)\, v, \qquad v, w \in V,\ \psi \in \mathrm{Hom}(V, \mathfrak{g}_0) \tag{2.15}$$

from where (2.13) is clear.

So we see that, under a change $\boldsymbol{\omega} \longmapsto \widehat{\boldsymbol{\omega}} = \boldsymbol{\omega} + \boldsymbol{\varphi}$, the torsion changes acording to $\mathbf{T} \longmapsto \widehat{\mathbf{T}} = \mathbf{T} - \delta(\boldsymbol{\varphi})$, which suggests studying the kernel and cokernel of the torsion map $\delta$:

$$\ker \delta \stackrel{\mathrm{def}}{=} \mathfrak{g}_0^{(1)} \qquad\qquad \operatorname{coker} \delta \stackrel{\mathrm{def}}{=} H^{0,2}(\mathfrak{g}_0) \tag{2.16}$$

$\mathfrak{g}_0^{(1)}$ is called the **first prolongation** of $\mathfrak{g}_0$, and $H^{0,2}(\mathfrak{g}_0)$ the **intrinsic torsion space** of $\mathfrak{g}_0$. Because the map $\delta$ is $G_0$-equivariant, it follows that these two vector spaces have natural induced $G_0$-actions $\rho^{(1)} : G_0 \to GL(\mathfrak{g}_0^{(1)})$ and $\rho^{0,2} : G_0 \to GL(H^{0,2}(\mathfrak{g}_0))$.

For an element $t \in V \otimes \wedge^2 V^*$, denote by $[t] \in H^{0,2}(\mathfrak{g}_0)$, its projection into the intrinsic torsion space. Then the computation above shows that $[\widehat{\mathbf{T}}] = [\mathbf{T}]$, as maps from $\mathcal{B}_0$ to $H^{0,2}(\mathfrak{g}_0)$. In other words, the map $[\mathbf{T}] : \mathcal{B}_0 \to H^{0,2}(\mathfrak{g}_0)$ is independent of the choice of the connection $\boldsymbol{\omega}$, and so defines an **intrinsic torsion function** of the $G_0$-structure $\mathcal{B}_0$, which is equivariant:

$$[\mathbf{T}](\eta \cdot g) = \rho^{0,2}(g^{-1})([\mathbf{T}](\eta)), \qquad \forall \eta \in \mathcal{B}_o,\ \forall g \in G_0$$

One of the main steps in Cartan's equivalence method is to choose the connection $\boldsymbol{\omega}$, using the freedom $\boldsymbol{\omega} \mapsto \boldsymbol{\omega} + \boldsymbol{\varphi}$, so that the torsion simplifies as much as possible (this is usually called "torsion absorsion"). In our case we have:

$$\boldsymbol{\omega} = \begin{bmatrix} \boldsymbol{\omega}^i_j & \boldsymbol{\omega}^i_\beta \\ 0 & \boldsymbol{\omega}^\beta_\gamma \end{bmatrix} \in \Omega^1(\mathcal{B}_0; \mathfrak{g}_0)$$

that coincides with the left-invariant Maurer-Cartan form on each fiber $\pi^{-1}(q) \cong G_0$, $q \in Q$, and Cartan structure equation (2.7) takes the form:

$$\begin{bmatrix} d\boldsymbol{\Theta}^i \\ d\boldsymbol{\Theta}^\alpha \end{bmatrix} = -\begin{bmatrix} \boldsymbol{\omega}^i_j & \boldsymbol{\omega}^i_\beta \\ 0 & \boldsymbol{\omega}^\alpha_\beta \end{bmatrix} \wedge \begin{bmatrix} \boldsymbol{\Theta}^j \\ \boldsymbol{\Theta}^\beta \end{bmatrix} + \begin{bmatrix} \mathbf{T}^i_{jk}\boldsymbol{\Theta}^j \wedge \boldsymbol{\Theta}^k + \mathbf{T}^i_{k\beta}\boldsymbol{\Theta}^k \wedge \boldsymbol{\Theta}^\beta + \mathbf{T}^i_{\beta\gamma}\boldsymbol{\Theta}^\beta \wedge \boldsymbol{\Theta}^\gamma \\ \mathbf{T}^\alpha_{jk}\boldsymbol{\Theta}^j \wedge \boldsymbol{\Theta}^k + \mathbf{T}^\alpha_{k\beta}\boldsymbol{\Theta}^k \wedge \boldsymbol{\Theta}^\beta + \mathbf{T}^\alpha_{\beta\gamma}\boldsymbol{\Theta}^\beta \wedge \boldsymbol{\Theta}^\gamma \end{bmatrix} \tag{2.17}$$

where $\boldsymbol{\omega}^i_j = -\boldsymbol{\omega}^j_i$. But we are free to add arbitrary semi-basic parts $\boldsymbol{\varphi}^i_\beta$ and $\boldsymbol{\varphi}^\alpha_\beta$, respectivelly to $\boldsymbol{\omega}^i_\beta$ and $\boldsymbol{\omega}^\alpha_\beta$. If we expand these semi-basic parts $\boldsymbol{\varphi}^i_\beta = \boldsymbol{\varphi}^i_{\beta k}\boldsymbol{\Theta}^k + \boldsymbol{\varphi}^i_{\beta\gamma}\boldsymbol{\Theta}^\gamma$ and $\boldsymbol{\omega}^\alpha_\beta = \boldsymbol{\omega}^\alpha_{\beta j}\boldsymbol{\Theta}^j + \boldsymbol{\omega}^\alpha_{\beta\gamma}\boldsymbol{\Theta}^\gamma$, and substitute in the structure equation (2.17), we see that we can choose these $\boldsymbol{\varphi}'s$ so that the $\mathbf{T}^i_{k\beta}, \mathbf{T}^i_{\beta\gamma}, \mathbf{T}^\alpha_{k\beta}$ and $\mathbf{T}^\alpha_{\beta\gamma}$ all vanish. Now add a semi-basic part $\boldsymbol{\varphi}^i_j$, with $\boldsymbol{\varphi}^i_j + \boldsymbol{\varphi}^j_i = 0$, to $\boldsymbol{\omega}^i_j$. We expand $\boldsymbol{\varphi}^i_j = \boldsymbol{\varphi}^i_{jk}\boldsymbol{\Theta}^k + \boldsymbol{\varphi}^i_{j\gamma}\boldsymbol{\Theta}^\gamma$, and we can assume already that $\boldsymbol{\varphi}^i_{j\gamma} = 0$. Now note that we can also assume that $\boldsymbol{\varphi}^i_{jk} = -\boldsymbol{\varphi}^i_{kj}$, since any three tensor $\boldsymbol{\varphi}^i_{jk}$ skew in two indices and symmetric in the other two (i.e., $\boldsymbol{\varphi}^i_{jk} = \boldsymbol{\varphi}^i_{kj} = -\boldsymbol{\varphi}^j_{ik}$), must be zero (this is called the $S_3$ lemma). So, if we choose $\boldsymbol{\varphi}^i_{jk} = \frac{1}{2}(\mathbf{T}^i_{jk} - \mathbf{T}^j_{ik})$ we vanish (absorve) the $\mathbf{T}^i_{jk}$ torsion terms.

Therefore by an appropriate choice of connection we can reduce the torsion terms in (2.17) to the form:

$$\begin{bmatrix} 0 \\ \mathbf{T}^\alpha_{jk}\boldsymbol{\Theta}^j \wedge \boldsymbol{\Theta}^k \end{bmatrix}$$

and, with this choice of connection the corresponding structure equation is (omiting the $\widehat{\cdot}$):

$$\begin{bmatrix} d\boldsymbol{\Theta}^i \\ d\boldsymbol{\Theta}^\alpha \end{bmatrix} = -\begin{bmatrix} \boldsymbol{\omega}^i_j & \boldsymbol{\omega}^i_\beta \\ 0 & \boldsymbol{\omega}^\alpha_\beta \end{bmatrix} \wedge \begin{bmatrix} \boldsymbol{\Theta}^j \\ \boldsymbol{\Theta}^\beta \end{bmatrix} + \begin{bmatrix} 0 \\ \mathbf{T}^\alpha_{jk}\boldsymbol{\Theta}^j \wedge \boldsymbol{\Theta}^k \end{bmatrix} \tag{2.18}$$

or (compare with [5], eq. (5)):

$$\left\{ \begin{array}{lllllll} d\boldsymbol{\Theta}^i & = & -\boldsymbol{\omega}^i_j \wedge \boldsymbol{\Theta}^j & - & \boldsymbol{\omega}^i_\beta \wedge \boldsymbol{\Theta}^\beta & & \\ d\boldsymbol{\Theta}^\alpha & = & & - & \boldsymbol{\omega}^\alpha_\beta \wedge \boldsymbol{\Theta}^\beta & + & \mathbf{T}^\alpha_{jk}\, \boldsymbol{\Theta}^j \wedge \boldsymbol{\Theta}^k \end{array} \right. \tag{2.19}$$

The second equation in (2.19) can be written in the form:

$$d\mathbf{\Theta}^\alpha = \mathbf{T}^\alpha_{jk}\,\mathbf{\Theta}^j\wedge\mathbf{\Theta}^k \quad \mod\{\mathbf{\Theta}^\alpha\} \tag{2.20}$$

which reveals that $\mathbf{T}^\alpha_{jk}$ are the components of the **structure tensor** of the distribution $\mathcal{D}$. More precisely, if we choose a 0-adapted (local) coframe $\theta = [\theta^a] = \begin{bmatrix}\theta^i\\ \theta^\alpha\end{bmatrix}$ to the distribution $\mathcal{D}$, with dual frame $\{X_i; X_\alpha\}$, then, pulling back (2.41) via $\theta$, we obtain:

$$d\theta^\alpha = T^\alpha_{jk}\,\theta^j\wedge\theta^k \quad \mod\{\theta^\alpha\} \tag{2.21}$$

where $T^\alpha_{jk}(q) = \mathbf{T}^\alpha_{jk}(\theta(q))$, and so:

$$\begin{aligned} 2T^\alpha_{jk} &= d\theta^\alpha(X_j, X_k)\\ &= T^\alpha_{jk}\,\theta^j\wedge\theta^k(X_j, X_k)\\ &= -\theta^\alpha([X_j, X_k]) \end{aligned} \tag{2.22}$$

Incidentally, the previous computations shows that the intrinsic torsion space $H^{0,2}(\mathfrak{g}_0)$ is $V/S\otimes\wedge^2S^*\cong \mathrm{Hom}(S\wedge S, V/S)$, where we recall that $S$ is the subspace of $V$ generated by the first $d$ vectors $\{\boldsymbol{\epsilon}_i\}_{i=1,\cdots,d}$ of the basis $\{\boldsymbol{\epsilon}_a\}$ for $V$:

$$H^{0,2}(\mathfrak{g}_0) = (V\otimes\wedge^2V^*)/\mathrm{Im}\,\delta = V/S\otimes\wedge^2S^*\cong \mathrm{Hom}(S\wedge S, V/S) \tag{2.23}$$

Now choose an adapted 0-coframe $\theta = [\theta^a] = \begin{bmatrix}\theta^i\\ \theta^\alpha\end{bmatrix}$, and consider the Riemannian space with Riemannian metric:

$$ds^2 = \sum_i(\theta^i)^2 + \sum_\alpha(\theta^\alpha)^2$$

Consider also the connection 1-form given, in the gauge $\theta$, by the $\mathfrak{g}_0$-valued 1-form $\omega = \theta^*\boldsymbol{\omega}$, with structure equations the pull-back to the base of (2.19):

$$d\theta = -\omega\wedge\theta + T \tag{2.24}$$

(recall the reproducing property $\theta^*\mathbf{\Theta} = \theta$. We have also puted $T = \theta^*\mathbf{T}$). Let $X_a$ denote the frame dual to $\theta^a$. If we consider a trajectory $\gamma: I\to Q$, the corresponding velocity is the vector field $V$, along $\gamma$:

$$V(t) = v^a(t)X_a(\gamma(t)), \qquad \text{where} \quad v^a(t) = \theta^a(\dot\gamma)$$

and its ($\omega$-covariant) acceleration is given by:

$$\begin{aligned} \frac{DV}{dt} &= \frac{dv^a}{dt}X_a + v^a(t)\frac{DX_a}{dt}(\gamma(t))\\ &= \frac{dv^a}{dt}X_a + v^a(t)\nabla_{\dot\gamma}X_a\\ &= \frac{dv^b}{dt}X_b + v^a(t)\,\omega^b_a(\dot\gamma)X_b\\ &= \left(\frac{dv^b}{dt} + v^a(t)\,\omega^b_a(\dot\gamma)\right)X_b \end{aligned} \tag{2.25}$$

In particular, if we assume that $\gamma$ is horizontal. i.e., $\dot\gamma\in\mathcal{D}_{\gamma(t)}$, $\forall t$, then splitting again the indices $a = (i;\alpha)$, we have that $v^\alpha = \theta^\alpha(\dot\gamma) = 0$, and so, since $\omega^\alpha_j(\dot\gamma) = 0$:

$$\begin{aligned} \frac{DV}{dt} &= \left(\frac{dv^i}{dt} + v^j(t)\omega^i_j(\dot\gamma)\right)X_i\\ &= \left(\frac{d\theta^i(\dot\gamma)}{dt} + \omega^i_j(\dot\gamma)\,\theta^j(\dot\gamma)\right)X_i(\gamma(t)) \end{aligned} \tag{2.26}$$

But what happens if we change the gauge? To see this, let us differentiate the equation $R_g^*\mathbf{\Theta} = g^{-1}\mathbf{\Theta}$, for a fixed $g \in G_0$, with $\mathbf{\Theta} = \begin{bmatrix} \mathbf{\Theta}^i \\ \mathbf{\Theta}^\alpha \end{bmatrix}$ and $g^{-1} = \begin{bmatrix} C & B \\ 0 & A \end{bmatrix}^{-1} = \begin{bmatrix} C^{-1} & -C^{-1}BA^{-1} \\ 0 & A^{-1} \end{bmatrix}$. We then have:

$$R_g^* \begin{bmatrix} \mathbf{\Theta}^i \\ \mathbf{\Theta}^\alpha \end{bmatrix} = \begin{bmatrix} C^{-1} & -C^{-1}BA^{-1} \\ 0 & A^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{\Theta}^i \\ \mathbf{\Theta}^\alpha \end{bmatrix} = \begin{bmatrix} C^{-1}\mathbf{\Theta}^i - C^{-1}BA^{-1}\mathbf{\Theta}^\alpha \\ A^{-1}\mathbf{\Theta}^\alpha \end{bmatrix}$$

and so, using the structure equations (2.19):

$$\begin{aligned}
d\widehat{\mathbf{\Theta}}^i &= d(R_g^*\mathbf{\Theta}^i) \\
&= d\left(C^{-1}\mathbf{\Theta}^i - C^{-1}BA^{-1}\mathbf{\Theta}^\alpha\right) \\
&= C^{-1}\, d\mathbf{\Theta}^i - C^{-1}BA^{-1}\, d\mathbf{\Theta}^\alpha \\
&= (C^{-1})^i_j \left(-\boldsymbol{\omega}^j_k \wedge \mathbf{\Theta}^k - \boldsymbol{\omega}^j_\beta \wedge \mathbf{\Theta}^\beta\right) - (C^{-1}BA^{-1})^i_\alpha \left(-\boldsymbol{\omega}^\alpha_\gamma \wedge \mathbf{\Theta}^\gamma + \mathbf{T}^\alpha_{jk}\mathbf{\Theta}^j \wedge \mathbf{\Theta}^k\right) \\
&\simeq -(C^{-1})^i_j\, \boldsymbol{\omega}^j_k \wedge \mathbf{\Theta}^k - (C^{-1}BA^{-1})^i_\alpha\, \mathbf{T}^\alpha_{\ell k}\, \mathbf{\Theta}^\ell \wedge \mathbf{\Theta}^k \\
&\simeq -(C^{-1})^i_j\, \boldsymbol{\omega}^j_k \wedge \mathbf{\Theta}^k - (C^{-1})^i_j B^j_\beta (A^{-1})^\beta_\alpha\, \mathbf{T}^\alpha_{\ell k}\, \mathbf{\Theta}^\ell \wedge \mathbf{\Theta}^k \\
&\simeq -(C^{-1})^i_j \left(\boldsymbol{\omega}^j_k - B^j_\beta (A^{-1})^\beta_\alpha \mathbf{T}^\alpha_{\ell k}\mathbf{\Theta}^\ell\right) \wedge \mathbf{\Theta}^k \\
&\simeq -(C^{-1})^i_j \left(\boldsymbol{\omega}^j_k - B^j_\beta (A^{-1})^\beta_\alpha \mathbf{T}^\alpha_{\ell k}\mathbf{\Theta}^\ell\right) \wedge C^k_m \widehat{\mathbf{\Theta}}^m \\
&\simeq -(C^{-1})^i_j \left(\boldsymbol{\omega}^j_k C^k_m - B^j_\beta (A^{-1})^\beta_\alpha \mathbf{T}^\alpha_{\ell k} C^k_m \mathbf{\Theta}^\ell\right) \wedge \widehat{\mathbf{\Theta}}^m \\
&\simeq -(C^{-1})^i_j \left(\boldsymbol{\omega}^j_k C^k_m - B^j_\beta (A^{-1})^\beta_\alpha \mathbf{T}^\alpha_{\ell k} C^k_m C^\ell_p \widehat{\mathbf{\Theta}}^p\right) \wedge \widehat{\mathbf{\Theta}}^m \\
&\simeq -(C^{-1})^i_j \left(\boldsymbol{\omega}^j_k C^k_m - B^j_\beta (A^{-1})^\beta_\alpha \mathbf{T}^\alpha_{\ell k} C^k_m C^\ell_p \widehat{\mathbf{\Theta}}^p\right) \wedge \widehat{\mathbf{\Theta}}^m \\
&\simeq -\left((C^{-1})^i_j \boldsymbol{\omega}^j_k C^k_m\right) \wedge \widehat{\mathbf{\Theta}}^m - (C^{-1})^i_j B^j_\beta \widehat{\mathbf{T}}^\beta_{pm}\, \widehat{\mathbf{\Theta}}^p \wedge \widehat{\mathbf{\Theta}}^m
\end{aligned} \tag{2.27}$$

where $\simeq$ means $=\ \mathrm{mod}\,\{\mathbf{\Theta}^\alpha\}$, and we have also used (2.34). But, on the other side:

$$d\widehat{\mathbf{\Theta}}^i = -\widehat{\boldsymbol{\omega}}^i_m \wedge \widehat{\mathbf{\Theta}}^m \qquad\qquad \mathrm{mod}\,\{\mathbf{\Theta}^\alpha\} \tag{2.28}$$

and $\widehat{\boldsymbol{\omega}}^i_m = (C^{-1})^i_j \boldsymbol{\omega}^j_k C^k_m, \quad \mathrm{mod}\,\{\mathbf{\Theta}^\alpha\}$, and so, comparing (2.27) with (2.28), we deduce that, in order to preserve covariance of the covariant acceleration (2.26), for horizontal curves, we must have:

$$(C^{-1})^i_j B^j_\beta \widehat{\mathbf{T}}^\beta_{mp} = 0$$

or equivalently (compare with equation (6) in [5]):

$$B^j_\beta \widehat{\mathbf{T}}^\beta_{mp} = 0 \tag{2.29}$$

But this restricts the set of admissible coframes. In fact, if for example $\mathbf{T}(\eta) \in \mathrm{Hom}(S \wedge S, V/S)$ is surjective $\forall \eta \in \mathcal{B}_0$, which means that $\mathcal{D}$ is a 2-step brackett generating distribution, then (2.29) implies that $B^j_\beta = 0$, and so we must reduce the gauge group to[3]:

$$G_1 = \left\{ \begin{bmatrix} C & 0 \\ 0 & A \end{bmatrix} : C \in SO(d), \quad A \in Gl(n-d) \right\} \subset G_0$$

Hereafter we assume that $\mathcal{D}$ is a 2-step generating distribution. We then consider the corresponding $G_1$-structure $\pi : \mathcal{B}_1 = \mathcal{B}_{G_1} \to Q$, and we choose a $\mathfrak{g}_1$-connection form $\boldsymbol{\omega}$, on $\mathcal{B}_1$, with structure equations:

$$\begin{bmatrix} d\mathbf{\Theta}^i \\ d\mathbf{\Theta}^\alpha \end{bmatrix} = \begin{bmatrix} \boldsymbol{\omega}^i_j & 0 \\ 0 & \boldsymbol{\omega}^\alpha_\beta \end{bmatrix} \wedge \begin{bmatrix} \mathbf{\Theta}^j \\ \mathbf{\Theta}^\beta \end{bmatrix} + \begin{bmatrix} \mathbf{T}^i_{jk}\mathbf{\Theta}^j \wedge \mathbf{\Theta}^k + \mathbf{T}^i_{j\beta}\mathbf{\Theta}^j \wedge \mathbf{\Theta}^\beta + \mathbf{T}^i_{\beta\gamma}\mathbf{\Theta}^\beta \wedge \mathbf{\Theta}^\gamma \\ \mathbf{T}^\alpha_{jk}\mathbf{\Theta}^j \wedge \mathbf{\Theta}^k + \mathbf{T}^\alpha_{j\beta}\mathbf{\Theta}^j \wedge \mathbf{\Theta}^\beta + \mathbf{T}^\alpha_{\beta\gamma}\mathbf{\Theta}^\beta \wedge \mathbf{\Theta}^\gamma \end{bmatrix} \tag{2.30}$$

[3]this is the case treated in [5].

Arguing as before, we add an arbitrary semibasic form $\boldsymbol{\varphi}^{\alpha}_{\beta} = \boldsymbol{\varphi}^{\alpha}_{\beta j}\boldsymbol{\Theta}^{j} + \boldsymbol{\varphi}^{\alpha}_{\beta\gamma}\boldsymbol{\Theta}^{\gamma}$ to $\boldsymbol{\omega}^{\alpha}_{\beta}$, and using this freedom, we absorve the $\mathbf{T}^{\alpha}_{j\beta}$ and $\mathbf{T}^{\alpha}_{\beta\gamma}$ torsion terms. Analogously, adding an arbitrary semibasic form $\boldsymbol{\varphi}^{i}_{j} = \boldsymbol{\varphi}^{i}_{jk}\boldsymbol{\Theta}^{k}$ to $\boldsymbol{\omega}^{i}_{j}$, with $\boldsymbol{\varphi}^{i}_{j} + \boldsymbol{\varphi}^{j}_{i} = 0$, we absorve the $\mathbf{T}^{i}_{jk}$ torsion terms. With this choice of connection form, the structure equation (2.30) reduces to:

$$\left[\begin{array}{c} d\boldsymbol{\Theta}^{i} \\ d\boldsymbol{\Theta}^{\alpha} \end{array}\right] = \left[\begin{array}{cc} \boldsymbol{\omega}^{i}_{j} & 0 \\ 0 & \boldsymbol{\omega}^{\alpha}_{\beta} \end{array}\right] \wedge \left[\begin{array}{c} \boldsymbol{\Theta}^{j} \\ \boldsymbol{\Theta}^{\beta} \end{array}\right] + \left[\begin{array}{c} \mathbf{T}^{i}_{j\alpha}\boldsymbol{\Theta}^{j}\wedge\boldsymbol{\Theta}^{\alpha} + \mathbf{T}^{i}_{\alpha\beta}\boldsymbol{\Theta}^{\alpha}\wedge\boldsymbol{\Theta}^{\beta} \\ \mathbf{T}^{\alpha}_{jk}\boldsymbol{\Theta}^{j}\wedge\boldsymbol{\Theta}^{k} \end{array}\right] \tag{2.31}$$

or (compare with [5], eq. (8)):

$$\left\{\begin{array}{lllllll} d\boldsymbol{\Theta}^{i} & = & \boldsymbol{\omega}^{i}_{j}\wedge\boldsymbol{\Theta}^{j} & + & \mathbf{T}^{i}_{k\alpha}\boldsymbol{\Theta}^{k}\wedge\boldsymbol{\Theta}^{\alpha} & + & \mathbf{T}^{i}_{\alpha\beta}\boldsymbol{\Theta}^{\alpha}\wedge\boldsymbol{\Theta}^{\beta} \\ d\boldsymbol{\Theta}^{\alpha} & = & \boldsymbol{\omega}^{\alpha}_{\beta}\wedge\boldsymbol{\Theta}^{\beta} & + & \mathbf{T}^{\alpha}_{jk}\boldsymbol{\Theta}^{j}\wedge\boldsymbol{\Theta}^{k} & & \end{array}\right. \tag{2.32}$$

Finally we can add an arbitrary semibasic form $\boldsymbol{\varphi}^{i}_{j} = \boldsymbol{\varphi}^{i}_{j\alpha}\boldsymbol{\Theta}^{\alpha}$ to $\boldsymbol{\omega}^{i}_{j}$, with $\boldsymbol{\varphi}^{i}_{j} + \boldsymbol{\varphi}^{j}_{i} = 0$, and arrange things so that:

$$\mathbf{T}^{i}_{k\alpha} = \mathbf{T}^{k}_{i\alpha} \tag{2.33}$$

and in this way the $\boldsymbol{\omega}^{i}_{j}$-part of the connection $\boldsymbol{\omega}$ is uniquely defined.

Now look at the $\mathbf{T}^{\alpha}_{jk}$-part of the torsion $\mathbf{T} = \mathbf{T}[\boldsymbol{\omega}]$, defined by the second equation in (2.32). Denote this torsion part simply by $\widetilde{\mathbf{T}}$, and recall that we may see $\widetilde{\mathbf{T}}$ as a map $\eta \in \mathcal{B}_0 \mapsto \widetilde{\mathbf{T}}(\eta) \in \mathrm{Hom}(S\wedge S, V/S) \cong \wedge^2 S^*\otimes V/S$. To see explicitly how $G_0$ acts on this torsion part $\widetilde{\mathbf{T}}$, let us take a fixed $g \in G_0$. Then, with $g^{-1} = \left[\begin{array}{cc} C^{-1} & -C^{-1}BA^{-1} \\ 0 & A^{-1} \end{array}\right]$, the equation $\widehat{\boldsymbol{\Theta}} \stackrel{\text{def}}{=} R^*_g\boldsymbol{\Theta} = g^{-1}\boldsymbol{\Theta}$ implies that: $\widehat{\boldsymbol{\Theta}}^{i} = (C^{-1})^{i}_{k}\boldsymbol{\Theta}^{k} \mod\{\boldsymbol{\Theta}^{\beta}\}$, $\widehat{\boldsymbol{\Theta}}^{\alpha} = (A^{-1})^{\alpha}_{\beta}\boldsymbol{\Theta}^{\beta}$, and so $\boldsymbol{\Theta}^{\alpha}$ changes to $(A^{-1})^{\alpha}_{\beta}\boldsymbol{\Theta}^{\beta}$, when we apply the gauge transformation $g$. On the other side, $\boldsymbol{\Theta}^{j}\wedge\boldsymbol{\Theta}^{k}$ changes to $(C^{-1})^{j}_{\ell}(C^{-1})^{k}_{m}\boldsymbol{\Theta}^{l}\wedge\boldsymbol{\Theta}^{m} \mod\{\boldsymbol{\Theta}^{\beta}\}$. So, in one hand we have:

$$d\widehat{\boldsymbol{\Theta}}^{\alpha} = \widehat{\mathbf{T}}^{\alpha}_{\ell m}\widehat{\boldsymbol{\Theta}}^{\ell}\wedge\widehat{\boldsymbol{\Theta}}^{m} \quad \mod\{\widehat{\boldsymbol{\Theta}}^{\alpha}\} = \mod\{\boldsymbol{\Theta}^{\alpha}\}$$

and on the other:

$$\begin{array}{rcll} d\widehat{\boldsymbol{\Theta}}^{\alpha} & = & (A^{-1})^{\alpha}_{\beta}\, d\boldsymbol{\Theta}^{\beta} & \\ & = & (A^{-1})^{\alpha}_{\beta}\,\mathbf{T}^{\beta}_{ik}\,\boldsymbol{\Theta}^{i}\wedge\boldsymbol{\Theta}^{k} & \mod\{\boldsymbol{\Theta}^{\alpha}\} \\ & = & (A^{-1})^{\alpha}_{\beta}\,\mathbf{T}^{\beta}_{ik}\,C^{i}_{\ell}C^{k}_{m}\,\widehat{\boldsymbol{\Theta}}^{l}\wedge\widehat{\boldsymbol{\Theta}}^{m} & \mod\{\boldsymbol{\Theta}^{\alpha}\} \end{array}$$

which means that the torsion part, that we are considering, changes according to:

$$\mathbf{T}^{\alpha}_{\ell m}(\eta\cdot g) = (A^{-1})^{\alpha}_{\beta}\,\mathbf{T}^{\beta}_{ik}(\eta)\,C^{i}_{\ell}C^{k}_{m}, \qquad \eta\in\mathcal{B}_0, \quad g = \left[\begin{array}{cc} C & B \\ 0 & A \end{array}\right] \tag{2.34}$$

In particular we see that $G_1$ acts exactly in the same way, since the $B's$ have no appearance in (2.34).

When $\mathcal{D}$ is a 2-step generating distribution, we can "normalize" the torsion part $\widetilde{\mathbf{T}}$ in the following way. In this case, we know that $\widetilde{\mathbf{T}}(\eta) : S\wedge S \to V/S$ is surjective $\forall\eta\in\mathcal{B}_1$, and thus, for each $\alpha = d+1,\cdots,n$, we can choose a bicovector $\mathbf{B}^{\alpha} = \mathbf{T}^{\alpha}_{ij}(\eta)\,\boldsymbol{\epsilon}^{i}\wedge\boldsymbol{\epsilon}^{j} \in \wedge^2 S^*$, such that $\widetilde{\mathbf{T}}(\eta)(\mathbf{B}^{\alpha})$ form a basis for $V/S$. But in $\wedge^2 S^*$ we have a metric, since $S$ is Euclidean, and we can choose the linearly independent $\mathbf{B}^{\alpha}$ orthonormal, with respect to that metric, acting if necessary with an appropriate $C$-part of $g$ (recall that $C\in SO(d)$). This imposes the conditions (compare with (11) in [5]):

$$\sum_{ij}\mathbf{T}^{\alpha}_{ij}\mathbf{T}^{\beta}_{ij} = \delta^{\alpha\beta} \tag{2.35}$$

Which $g's$ preserve this $\widetilde{\mathbf{T}}$-torsion normalization ? Of course those for which $A\in SO(n-d)$. So we must reduce the group to:

$$G_2 = \left\{ g = \left[\begin{array}{cc} C & 0 \\ 0 & A \end{array}\right], \quad C\in SO(d), \quad A\in SO(n-d) \right\} \tag{2.36}$$

and this new $\mathcal{B}_2$-structure defines an intrinsic metric for the normal bundle $TM/\mathcal{D}$.

We proceed as before, choosing a connection for $\mathcal{B}_2$ and doing torsion absortion. The structure equation has now the form:

$$\begin{bmatrix} d\boldsymbol{\Theta}^i \\ d\boldsymbol{\Theta}^\alpha \end{bmatrix} = \begin{bmatrix} \boldsymbol{\omega}^i_j & 0 \\ 0 & \boldsymbol{\omega}^\alpha_\beta \end{bmatrix} \wedge \begin{bmatrix} \boldsymbol{\Theta}^j \\ \boldsymbol{\Theta}^\beta \end{bmatrix} + \begin{bmatrix} \mathbf{T}^i_{jk}\boldsymbol{\Theta}^j \wedge \boldsymbol{\Theta}^k + \mathbf{T}^i_{j\beta}\boldsymbol{\Theta}^j \wedge \boldsymbol{\Theta}^\beta + \mathbf{T}^i_{\beta\gamma}\boldsymbol{\Theta}^\beta \wedge \boldsymbol{\Theta}^\gamma \\ \mathbf{T}^\alpha_{jk}\boldsymbol{\Theta}^j \wedge \boldsymbol{\Theta}^k + \mathbf{T}^\alpha_{j\beta}\boldsymbol{\Theta}^j \wedge \boldsymbol{\Theta}^\beta + \mathbf{T}^\alpha_{\beta\gamma}\boldsymbol{\Theta}^\beta \wedge \boldsymbol{\Theta}^\gamma \end{bmatrix} \tag{2.37}$$

with $\boldsymbol{\omega}^i_j + \boldsymbol{\omega}^j_i = 0$ and $\boldsymbol{\omega}^\alpha_\beta + \boldsymbol{\omega}^\beta_\alpha = 0$. Arguing as before (the $S_3$-lemma), we absorve the $\mathbf{T}^i_{jk}$ and the $\mathbf{T}^\alpha_{\beta\gamma}$ torsion terms. Then we can add an arbitrary semibasic form $\boldsymbol{\varphi}^i_j = \boldsymbol{\varphi}^i_{j\alpha}\boldsymbol{\Theta}^\alpha$ to $\boldsymbol{\omega}^i_j$, with $\boldsymbol{\varphi}^i_j + \boldsymbol{\varphi}^j_i = 0$, and arrange things so that:

$$\mathbf{T}^i_{k\alpha} = \mathbf{T}^k_{i\alpha} \tag{2.38}$$

and in this way the $\boldsymbol{\omega}^i_j$-part of the connection $\boldsymbol{\omega}$ is uniquely defined. Analousgly, we can add an arbitrary semibasic form $\boldsymbol{\varphi}^\alpha_\beta = \boldsymbol{\varphi}^\alpha_{\beta i}\boldsymbol{\Theta}^i$ to $\boldsymbol{\omega}^\alpha_\beta$, with $\boldsymbol{\varphi}^\alpha_\beta + \boldsymbol{\varphi}^\beta_\alpha = 0$, and arrange things so that:

$$\mathbf{T}^\alpha_{j\beta} = \mathbf{T}^\beta_{j\alpha} \tag{2.39}$$

and in this way the $\boldsymbol{\omega}^\alpha_\beta$-part of the connection $\boldsymbol{\omega}$ is also uniquely defined. With this choice of connection form, the structure equation (2.37) finally reduces to:

$$\begin{bmatrix} d\boldsymbol{\Theta}^i \\ d\boldsymbol{\Theta}^\alpha \end{bmatrix} = \begin{bmatrix} \boldsymbol{\omega}^i_j & 0 \\ 0 & \boldsymbol{\omega}^\alpha_\beta \end{bmatrix} \wedge \begin{bmatrix} \boldsymbol{\Theta}^j \\ \boldsymbol{\Theta}^\beta \end{bmatrix} + \begin{bmatrix} \mathbf{T}^i_{j\alpha}\boldsymbol{\Theta}^j \wedge \boldsymbol{\Theta}^\alpha + \mathbf{T}^i_{\alpha\beta}\boldsymbol{\Theta}^\alpha \wedge \boldsymbol{\Theta}^\beta \\ \mathbf{T}^\alpha_{jk}\boldsymbol{\Theta}^j \wedge \boldsymbol{\Theta}^k + \mathbf{T}^\alpha_{j\beta}\boldsymbol{\Theta}^j \wedge \boldsymbol{\Theta}^\beta \end{bmatrix} \tag{2.40}$$

or (compare with [5], eq. (8')):

$$\left\{ \begin{array}{lllllll} d\boldsymbol{\Theta}^i & = & \boldsymbol{\omega}^i_j \wedge \boldsymbol{\Theta}^j & + & \mathbf{T}^i_{j\alpha}\boldsymbol{\Theta}^j \wedge \boldsymbol{\Theta}^\alpha & + & \mathbf{T}^i_{\alpha\beta}\boldsymbol{\Theta}^\alpha \wedge \boldsymbol{\Theta}^\beta \\ d\boldsymbol{\Theta}^\alpha & = & \boldsymbol{\omega}^\alpha_\beta \wedge \boldsymbol{\Theta}^\beta & + & \mathbf{T}^\alpha_{jk}\boldsymbol{\Theta}^j \wedge \boldsymbol{\Theta}^k & + & \mathbf{T}^\alpha_{j\beta}\boldsymbol{\Theta}^j \wedge \boldsymbol{\Theta}^\beta \end{array} \right. \tag{2.41}$$

with the following symmetries:

$$\boldsymbol{\omega}^i_j = -\boldsymbol{\omega}^j_i, \quad \boldsymbol{\omega}^\alpha_\beta = -\boldsymbol{\omega}^\beta_\alpha, \quad \mathbf{T}^i_{k\alpha} = \mathbf{T}^k_{i\alpha}, \quad \mathbf{T}^\alpha_{j\beta} = \mathbf{T}^\beta_{j\alpha}, \quad \sum_{ij} \mathbf{T}^\alpha_{ij}\mathbf{T}^\beta_{ij} = \delta^{\alpha\beta} \tag{2.42}$$

This finish Cartan's intrinsic geometrization of 2-step non-holonomic systems. In section 4 we examine a detailed example.

# 3 Cartan's affine generalized spaces. Development

Denote by $\mathcal{A}^n$ the space $\mathbb{R}^n$ with its canonical affine structure, and its canonical affine frame, $\{\mathbf{0}; E_a\}$. As usual we identify a point $P \in \mathcal{A}^n$ with its position vector $\mathbf{P} = \overrightarrow{\mathbf{0}P}$. An **affine isomorphism** $A : \mathcal{A}^n \to \mathcal{A}^n$ is a mapping of the form:

$$A : \mathbf{P} \mapsto \mathbf{a} + \mathbf{A}(\mathbf{P}), \qquad \mathbf{P} \in \mathcal{A}^n \tag{3.1}$$

where $\mathbf{A} \in GL(n, \mathbb{R})$ and $\mathbf{a} = A(\mathbf{0}) \in \mathcal{A}^n$ (only depends on $A$). They form a group $GA(n)$, which is the semi-direct product of $\mathbb{R}^n$ by $GL(n)$, and for which we use the following **homogeneous representation** $GA(n) \hookrightarrow GL(n+1, \mathbb{R})$:

$$A \stackrel{\text{def}}{=} (\mathbf{a}, \mathbf{A}) \cong \begin{bmatrix} 1 & 0 \\ \mathbf{a} & \mathbf{A} \end{bmatrix} \qquad \text{with} \quad \mathbf{A} \in GL(n, \mathbb{R}),\ \mathbf{a} \in \mathbb{R}^n \tag{3.2}$$

which corresponds to identifying $\mathcal{A}^n$ with the affine hiperplane $\mathbb{R}^n \times \{1\} \subset \mathbb{R}^n \times \mathbb{R}$, through $P \mapsto \begin{bmatrix} 1 \\ \mathbf{P} \end{bmatrix}$.
The Lie algebra $\mathfrak{ga}(n)$ can be identified with the Lie subalgebra of $\mathfrak{gl}(n+1, \mathbb{R})$ consisting of matrices of the form:

$$\begin{bmatrix} 0 & 0 \\ \xi & \Lambda \end{bmatrix} \stackrel{\text{def}}{=} \xi \oplus \Lambda \qquad \text{with} \quad \xi \in \mathbb{R}^n,\ \Lambda \in \mathfrak{gl}(n) \tag{3.3}$$

The Lie brackett $\mathfrak{ga}(n)$ is given by:

$$[\xi \oplus \Lambda, \eta \oplus \Psi] = (\Lambda\eta - \Psi\xi) \oplus [\Lambda, \Psi] \tag{3.4}$$

and the adjoint representation $GA(n)$ on $\mathfrak{ga}(n)$, by:

$$Ad_{(\mathbf{a},\mathbf{A})}(\xi \oplus \Lambda) = (-\mathbf{A}\Lambda\mathbf{A}^{-1}\mathbf{a} + \mathbf{A}\xi) \oplus (\mathbf{A}\Lambda\mathbf{A}^{-1}) \tag{3.5}$$

So:

$$\mathfrak{ga}(n) = \mathbb{R}^n \oplus \mathfrak{gl}(n) \tag{3.6}$$

and this direct sum is reductive:

$$Ad_{GA(n)}\mathbb{R}^n \subseteq \mathbb{R}^n \tag{3.7}$$

In fact:

$$Ad_{(\mathbf{a},\mathbf{A})}(\xi \oplus 0) = \mathbf{A}\xi \oplus 0, \qquad \forall (\mathbf{a}, \mathbf{A}) \in GA(n), \forall \xi \in \mathbb{R}^n \tag{3.8}$$

Let $Q$ be a $n$-dimensional smooth manifold, and for each point $q \in Q$ let $\mathcal{A}_qQ$ be the affine tangent space, i.e., the tangent space $T_qQ$ with its canonical affine structure. Points in $\mathcal{A}_qQ$ will be denoted by $\mathbf{0}_q, \mathbf{P}_q, \mathbf{Q}_q, ...$, and vectors on $T_qQ$ by $\mathbf{X}_q, \mathbf{Y}_q, ...$ (but we omit the subscrite $q$ when there is no danger of confusion). An affine frame for $\mathcal{A}_qQ$ consists on a point $\mathbf{P} \in \mathcal{A}_qQ$ together with a linear frame $\{\mathbf{X}_a\}_{a=1,\cdots,n}$ for $T_qQ$. We denote such a frame by $\{\mathbf{P};\, \mathbf{X}_a\}$.

Let $\mathcal{A}(Q)$ the affine frame bundle over $Q$ (see [11], section III.3 or [15]), which is a principal fiber bundle with group $GA(n)$, acting on the right of $\mathcal{A}(Q)$ by:

$$\{\mathbf{P};\, \mathbf{X}_a\} \cdot (\mathbf{a} = (\mathbf{a}^a), \mathbf{A} = (\mathbf{A}^a_b)) = \{\mathbf{P} + \mathbf{a}^a\mathbf{X}_a;\, \mathbf{X}_a\mathbf{A}^a_b\}$$

If $\{\mathbf{P};\, \mathbf{X}_a\}$ and $\{\mathbf{Q};\, \mathbf{Y}_a\}$ are two frames for $\mathcal{A}_qQ$, then there is a unique $g = (\mathbf{a}, \mathbf{A}) = (\mathbf{a}^a, \mathbf{A}^a_b) \in GA(n)$ such that $\{\mathbf{P};\, \mathbf{X}_a\} \cdot (\mathbf{a}^a, \mathbf{A}^a_b) = \{\mathbf{Q};\, \mathbf{Y}_a\}$. In fact we determine $\mathbf{a}$ and $\mathbf{A}$ by the equations $\mathbf{Q} = \mathbf{P} + \mathbf{a}^a\mathbf{X}_a$ and $\mathbf{Y}_b = \mathbf{X}_a\mathbf{A}^a_b$, i.e., $(\mathbf{a}, \mathbf{A})$ measure the relative position of the second frame relative to the first one.

Hereafter we identify a point $q \in Q$ with the point $\mathbf{0}_q \in \mathcal{A}_qQ$ (the origin of $T_qQ$). Consider an linear moving frame $\{\mathbf{0}_q; \mathbf{X}_a(q)\}$, defined on an open set $U \subseteq Q$. For example, if $(U; q^a)$ is a local coordinate neighbourhood for $Q$, then, for each $q \in U$, $\{\mathbf{0}_q; \partial/\partial q^a\}$ is an affine frame for $\mathcal{A}_qQ$. Each other frame $\{\mathbf{P};\, \mathbf{Y}_a\}$ for $\mathcal{A}_qQ$ determine unique $(y^a, Y^a_b) \in GA(n)$ such that $\mathbf{P} = y^a\mathbf{X}_a$ and $\mathbf{Y}_b = \mathbf{X}_aY^a_b$. Thus we see that $(q^a, y^a, Y^a_b)$ is a trivializing local coordinate system for $\pi^{-1}(U) \cong U \times GA(n) \subset \mathcal{A}(Q)$.

Now let us consider a **generalized affine connection** on $Q$, i.e., a connection on $\mathcal{A}(Q)$. Let $\widetilde{\boldsymbol{\omega}}$ the corresponding connection 1-form, which is a $\mathfrak{ga}(n) = \mathbb{R}^n \oplus \mathfrak{gl}(n)$-valued 1-form on $\mathcal{A}(Q)$:

$$\widetilde{\boldsymbol{\omega}} = \boldsymbol{\varphi}^a \oplus \boldsymbol{\omega}^a_b \tag{3.9}$$

By the general theory (see [11] or [15]), we know that on $\pi^{-1}(U) = U \times GA(n)$, $\widetilde{\boldsymbol{\omega}}$ has the following expression:

$$\begin{aligned}
\widetilde{\boldsymbol{\omega}} &= \boldsymbol{\varphi} \oplus \boldsymbol{\omega} = \boldsymbol{\varphi}^a \oplus \boldsymbol{\omega}^a_b \\
&= Ad_{(\mathbf{y},\mathbf{Y})^{-1}}(\varphi \oplus \omega) + (\mathbf{y}, \mathbf{Y})^{-1}d(\mathbf{y}, \mathbf{Y}) \\
&= (-\mathbf{Y}^{-1}\mathbf{y}, \mathbf{Y}^{-1})\,(\varphi \oplus \omega)\,(\mathbf{y}, \mathbf{Y}) + (-\mathbf{Y}^{-1}\mathbf{y}, \mathbf{Y}^{-1})(d\mathbf{y}, d\mathbf{Y}) \\
&= \left(\mathbf{Y}^{-1}\,(\varphi + \omega\mathbf{y} + d\mathbf{y})\right) \oplus \left(\mathbf{Y}^{-1}\omega\mathbf{Y} + \mathbf{Y}^{-1}d\mathbf{Y}\right) \\
&= (Y^{-1})^a_b\left(dy^b + \varphi^b + \omega^b_c\, y^c\right) \,\oplus\, (Y^{-1})^a_c\,(dY^c_b + \omega^c_e\, Y^e_b)
\end{aligned} \tag{3.10}$$

(where $\mathbf{y} = y^a, \mathbf{Y} = Y^a_b, \varphi = \varphi^a, \omega = \omega^a_b$), for a unique $\mathfrak{ga}(n) = \mathbb{R}^n \oplus \mathfrak{gl}(n)$-valued local "gauge potential" $\omega = \varphi^a \oplus \omega^a_b$ defined on $U$. If we put:

$$\begin{aligned}
\varphi^a &= \Gamma^a_b\, \theta^b \\
\omega^a_b &= \Gamma^a_{bc}\, \theta^c
\end{aligned} \tag{3.11}$$

where $(\mathbf{0}_q; \mathbf{X}_a(q))$ is a linear (affine) moving frame defined on an open set $U \subseteq Q$, $\theta^a(q)$ the corresponding linear dual coframe, and $\Gamma^a_b, \Gamma^a_{bc} \in C^\infty(U)$, then from (3.10):

$$\begin{aligned} \boldsymbol{\varphi}^a &= (Y^{-1})^a_b \left(dy^b + \varphi^b + \omega^b_c\, y^c\right) \\ &= (Y^{-1})^a_b \left(dy^b + \Gamma^b_c\, \theta^c + \Gamma^b_{ce}\, \theta^e\, y^c\right) \end{aligned} \tag{3.12}$$

$$\begin{aligned} \boldsymbol{\omega}^a_b &= (Y^{-1})^a_c \left(dY^c_b + \omega^c_e\, Y^e_b\right) \\ &= (Y^{-1})^a_c \left(dY^c_b + \Gamma^c_{ef}\, \theta^f\, Y^e_b\right) \end{aligned} \tag{3.13}$$

When $\Gamma^a_b = \delta^a_b$, so that $\varphi^a = \theta^a$, then (the pull-back to the linear frame bundle of) $\boldsymbol{\varphi}^a$ is equal to $\boldsymbol{\Theta}^a \stackrel{\text{def}}{=} (Y^{-1})^a_b\, \theta^b$ which is the canonical (tautological or soldering) form on the linear frame bundle of $Q$. In this case, we call $\widetilde{\boldsymbol{\omega}}$ an affine connection on $Q$ (see [11], pag. 129 or [15]). Moreover we see that (the pull-back to the linear frame bundle of) $\boldsymbol{\omega}^a_b$ defines a linear connection on $Q$. Hereafter we only consider affine connections.

Consider now a curve $\tau = q_t$, $0 \le t \le 1$, contained on an open subset $U \subseteq Q$, where is defined a linear moving frame $\{\mathbf{0}_q; \mathbf{X}_a(q)\}_{q \in U}$. Then we can define the horizontal lift $\widetilde{\tau}$ of $\tau$, with respect to the affine connection $\widetilde{\boldsymbol{\omega}}$, as the curve $\widetilde{\tau}$ on $\mathcal{A}(Q)$ (i.e., a curve of affine frames) such that $\pi(\widetilde{\tau}_t) = q_t$ and $\widetilde{\boldsymbol{\omega}}\left(\dot{\widetilde{\tau}}_t\right) = 0$. In local coordinates, if $q_t = q^a(t)$, then $\widetilde{\tau}_t = (q^a(t), y^a(t), Y^a_b(t))$, and:

$$\dot{\widetilde{\tau}}_t = \theta^a(\dot{q}_t)\mathbf{X}_a(q_t) + \dot{y}^a \frac{\partial}{\partial y^a} + \dot{Y}^a_b \frac{\partial}{\partial Y^a_b}$$

and therefore, by (3.12) and (3.13), the condition of horizontality, $\widetilde{\boldsymbol{\omega}}\left(\dot{\widetilde{\tau}}_t\right) = 0$, translates into the following system of ODE's:

$$\left\{ \begin{array}{lcl} \frac{dy^b}{dt} + \frac{d\theta^b}{dt} + \omega^b_c\, y^c &=& 0 \\ \frac{dY^a_b}{dt} + \omega^a_e\, Y^e_b &=& 0 \end{array} \right. \tag{3.14}$$

or more explicitly:

$$\left\{ \begin{array}{lcl} \frac{dy^b}{dt} + \frac{d\theta^b}{dt} + \Gamma^b_{ce} y^c\, \frac{d\theta^e}{dt} &=& 0 \\ \frac{dY^a_b}{dt} + \Gamma^a_{ef} Y^e_b\, \frac{d\theta^f}{dt} &=& 0 \end{array} \right. \tag{3.15}$$

where $\frac{d\theta^b}{dt}$ means of course $\frac{d\theta^b(\dot{q}_t)}{dt}$.

Take a linear (affine) frame $\{\mathbf{0}_{q_0}; \mathbf{Y}_a\}$ for $\mathcal{A}_{q_0}Q$. Then the horizontal lift $\widetilde{\tau}_t$, obtained solving the above ODE's (3.15), with initial conditions $y^a(0) = 0$ and $Y^a_b(0) = Y^a_b$, where $\mathbf{Y}_b = Y^a_b \mathbf{X}_a$, defines an affine isomorphism, called the **affine parallel transport** along $\tau$, that we denote by the same symbol:

$$\begin{array}{rccc} \widetilde{\tau}_t : & \mathcal{A}_{q_0}Q & \longrightarrow & \mathcal{A}_{q_t}Q \\ & \{\mathbf{0}_{q_0}; \mathbf{Y}_a\} & \longmapsto & \{\mathbf{P}_t; \mathbf{Y}_b(t)\} \end{array} \tag{3.16}$$

that maps the frame $\{\mathbf{0}_{q_0}; \mathbf{Y}_a\}$ into the frame $\{\mathbf{P}_t; \mathbf{Y}_b(t)\}$, where:

$$\left\{ \begin{array}{lcl} \mathbf{P}_t &=& y^a(t)\, \mathbf{X}_a(q_t) \\ \mathbf{Y}_b(t) &=& Y^a_b(t)\, \mathbf{X}_a(q_t) \end{array} \right. \tag{3.17}$$

and, as above, $y^a(t), Y^a_b(t)$ is the solution to the above ODE's (3.15), with initial conditions $y^a(0) = 0$ and $Y^a_b(0) = Y^a_b$.

Now, from the second equation in (3.17), we see that:

$$\begin{aligned} \mathbf{P}_t &= y^a(t)\, \mathbf{X}_a(q_t) \\ &= y^a(t)(Y^{-1})^b_a(t)\mathbf{Y}_b(t) \end{aligned} \tag{3.18}$$

and so the point $\mathbf{0}_{q_t} \in \mathcal{A}_{q_t}Q$ is the point $-y^a(t)(Y^{-1})^b_a(t)\mathbf{Y}_b(t)$, with respect to the affine frame $(\mathbf{P}_t;\, \mathbf{Y}_a(t))$. Therefore $\widetilde{\tau}_t^{-1}$ maps this position vector onto $-y^a(t)(Y^{-1})^b_a(t)\mathbf{Y}_b$, which, as $t$ varies, describes a curve in $\mathcal{A}_{q_0}Q$, which we denote by:

$$P(t) = -y^a(t)(Y^{-1})^b_a(t)\mathbf{Y}_b \tag{3.19}$$

and is called the **development** of the curve $\tau = q_t$ in $\mathcal{A}_{q_0}Q$. If we differentiate this, taking into account the second equation in (3.14), from which we deduce that $\frac{d(Y^{-1})^b_a}{dt} = (Y^{-1})^b_e\omega^e_a$, we compute that:

$$\begin{aligned} \frac{dP}{dt} &= -\left(\frac{dy^a}{dt}(Y^{-1})^b_a(t) + y^a(t)\frac{d(Y^{-1})^b_a}{dt}\right)\mathbf{Y}_b \\ &= -\left(\left(-\frac{d\theta^a}{dt} - \omega^a_e y^e\right)(Y^{-1})^b_a(t) + y^a(t)(Y^{-1})^b_e(t)\omega^e_a\right)\mathbf{Y}_b \\ &= \frac{d\theta^a}{dt}(Y^{-1})^b_a(t)\,\mathbf{Y}_b \end{aligned} \tag{3.20}$$

In particular, if in (3.20) we take $\mathbf{Y}_b = \mathbf{X}_b(q_0) = \mathbf{X}_b$ as our initial frame and put:

$$\mathbf{e}_a(t) \stackrel{\text{def}}{=} (Y^{-1})^b_a(t)\,\mathbf{X}_b$$

with $(Y^{-1})^b_a(0) = \delta^b_a$, then $\{\mathbf{e}_a(t)\}$ is the image in $\mathcal{A}_{q_0}Q$ of the linear frame $\{\mathbf{X}_a(q_t)\}$ by (the linear part of) $\widetilde{\tau}_t^{-1} : \mathcal{A}_{q_t}Q \to \mathcal{A}_{q_0}Q$, i.e.:

$$\mathbf{e}_a(t) = \widetilde{\tau}_t^{-1}(\mathbf{X}_a(q_t))$$

and $\{\mathbf{e}_a(t)\}$ is a moving frame in $\mathcal{A}_{q_0}Q$. Using the second equation in (3.14), from which we deduce that $\frac{d(Y^{-1})^b_a}{dt} = (Y^{-1})^b_e\omega^e_a$, we compute that:

$$\frac{d\mathbf{e}_a}{dt} = (Y^{-1})^b_e(t)\,\omega^e_a\,\mathbf{X}_b = \Gamma^c_{ae}\frac{d\theta^e}{dt}\,\mathbf{e}_c(t) \tag{3.21}$$

Thus the solution $(P(t);\mathbf{e}_a(t))$ of the system of ODE's:

$$\left\{\begin{array}{lllll} \frac{dP}{dt} &=& \frac{d\theta^a(\dot{q}_t)}{dt}\,\mathbf{e}_a(t) && \\ \frac{d\mathbf{e}_a}{dt} &=& \omega^c_a(q_t)\,\mathbf{e}_c(t) &=& \Gamma^c_{ae}(q_t)\frac{d\theta^e(\dot{q}_t)}{dt}\,\mathbf{e}_c(t) \end{array}\right. \tag{3.22}$$

gives a moving frame in $\mathcal{A}_{q_0}Q$, and $P(t)$ describes a curve starting at the origin $q_0$, which is the development of $\tau = q_t$ in $\mathcal{A}_{q_0}Q$.[4]

The curve $\tau = q_t$ is called a **geodesic** (or auto-parallel curve) of the affine connection $\widetilde{\boldsymbol{\omega}}$ if the development of $\tau$ in $\mathcal{A}_{q_0}Q$ is a straight line. So we must have $P(t) = \mathbf{a}t + \mathbf{b}$, where $\mathbf{a}, \mathbf{b}$ are constant vectors in $T_{q_0}Q$. Differenting $P(t)$ twice we obtain, using (3.22):

$$\frac{d^2\theta^a}{dt^2} + \Gamma^a_{bc}\frac{d\theta^b}{dt}\frac{d\theta^c}{dt} = 0 \tag{3.24}$$

which are the equations of a geodesic.

Assume now that the linear connection $\boldsymbol{\omega}$ is a $\mathfrak{g}_0$-connection:

$$\boldsymbol{\omega}^a_b = \begin{bmatrix} \boldsymbol{\omega}^i_j & \boldsymbol{\omega}^i_\beta \\ \mathbf{0} & \boldsymbol{\omega}^\gamma_\delta \end{bmatrix}$$

[4]Cartan usually writes the system (3.22) in the simplified form (see, for example, equation (10) in [5],[6],[7] and also reference [9]):

$$\left\{\begin{array}{lll} dP &=& \theta^a\,\mathbf{e}_a \\ d\mathbf{e}_a &=& \omega^c_a\,\mathbf{e}_c \end{array}\right. \tag{3.23}$$

Choose a linear frame $\{\mathbf{0}_q;\, \mathbf{X}_a(q)\}$, so that $\{\mathbf{X}_a\} = \{\mathbf{X}_i; \mathbf{X}_\alpha\}$ and $\mathbf{X}_i$ is a local basis for the distribution $\mathcal{D}$. Then $\mathbf{e}_i(0) = \mathbf{X}_i(q_0)$ is a basis for $\mathcal{D}_0 = \mathcal{D}_{q_0} \cong \mathrm{I\!R}^d$. Let us compute the development in $\mathcal{A}_{q_0}Q$ of a curve $\tau = q_t$ (not necessarilly horizontal). We have:

$$\begin{aligned}
\frac{d\mathbf{e}_i}{dt} &= \Gamma^c_{ie}\frac{d\theta^e}{dt}\,\mathbf{e}_c(t) \\
&= \Gamma^c_{ij}\frac{d\theta^j}{dt}\,\mathbf{e}_c(t) + \Gamma^c_{i\alpha}\frac{d\theta^\alpha}{dt}\,\mathbf{e}_c(t) \\
&= \Gamma^k_{ij}\frac{d\theta^j}{dt}\,\mathbf{e}_k(t) + \Gamma^k_{i\alpha}\frac{d\theta^\alpha}{dt}\,\mathbf{e}_k(t) \\
&= \left(\Gamma^k_{ij}(q_t)\frac{d\theta^j(\dot{q}_t)}{dt} + \Gamma^k_{i\alpha}(q_t)\frac{d\theta^\alpha(\dot{q}_t)}{dt}\right)\mathbf{e}_k(t)
\end{aligned} \tag{3.25}$$

because $\Gamma^\beta_j = 0$, which means that the moving frame $\{\mathbf{e}_i(t)\}$ always evolves within $\mathcal{D}_0$. After solving the ODE's (3.25) for the $\{\mathbf{e}_i(t)\}$, with initial condition $\mathbf{e}_i(0) = \mathbf{X}_i(q_0)$, we substitute in:

$$\frac{dP}{dt} = \frac{d\theta^i(\dot{q}_t)}{dt}\,\mathbf{e}_i(t) \tag{3.26}$$

and we obtain now an ODE for the development $P(t)$ of $\tau = q_t$ in $\mathcal{D}_0 \cong \mathrm{I\!R}^d$. In particular for an horizontal curve, i.e., $\theta^\alpha(\dot{q}_t) = 0$, its development is the curve in $\mathcal{D}_o$ obtained solving the above ODE's. In section 4 we will see an explicit computation of development of a curve.

For an affine connection $\widetilde{\boldsymbol{\omega}} = \boldsymbol{\theta}^a \oplus \boldsymbol{\omega}^a_b$, we can define the corresponding curvature 2-form in the usual way:

$$\widetilde{\boldsymbol{\Omega}} = d\widetilde{\boldsymbol{\omega}} + \widetilde{\boldsymbol{\omega}} \wedge \widetilde{\boldsymbol{\omega}}$$

Then its pull-back to the linear frame bundle is given by:

$$\begin{aligned}
\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{\Omega}^a & \boldsymbol{\Omega}^a_b \end{bmatrix} &= d\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{\theta}^a & \boldsymbol{\omega}^a_b \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{\theta}^a & \boldsymbol{\omega}^a_b \end{bmatrix} \wedge \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \boldsymbol{\theta}^a & \boldsymbol{\omega}^a_b \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ d\boldsymbol{\theta}^a + \boldsymbol{\omega}^a_b \wedge \boldsymbol{\theta}^b & d\boldsymbol{\omega}^a_b + \boldsymbol{\omega}^a_c \wedge \boldsymbol{\omega}^c_b \end{bmatrix}
\end{aligned} \tag{3.27}$$

from which we read the structural equations:

$$\left\{\begin{aligned} \boldsymbol{\Omega}^a &= d\boldsymbol{\theta}^a + \boldsymbol{\omega}^a_b \wedge \boldsymbol{\theta}^b \\ \boldsymbol{\Omega}^a_b &= d\boldsymbol{\omega}^a_b + \boldsymbol{\omega}^a_c \wedge \boldsymbol{\omega}^c_b \end{aligned}\right. \tag{3.28}$$

The first one:

$$d\boldsymbol{\theta}^a = -\boldsymbol{\omega}^a_b \wedge \boldsymbol{\theta}^b + \boldsymbol{\Omega}^a \tag{3.29}$$

defines the torsion $\boldsymbol{\Omega}^a$ of the affine connection - an $\mathrm{I\!R}^n$-valued semi-basic 2-form on te linear frame bundle $\mathcal{L}(Q)$ over $Q$, that can be written in the form:

$$\boldsymbol{\Omega}^a = \boldsymbol{\Omega}^a_{bc}\,\boldsymbol{\theta}^b \wedge \boldsymbol{\theta}^c \tag{3.30}$$

The meaning of this torsion is well known (see for example [9], [6]) - take an ordered pair $(u, v)$ of tangent vectors $u, v \in T_qQ$, and extend them to vector fields $U, V \in \mathfrak{X}(\mathcal{O})$, defined in an open set $\mathcal{O} \subset Q$, containing $q$. We may also assume that $[U, V] = 0$ in $\mathcal{O}$. Consider now a "small" loop $\Lambda^{(U,V)}_\epsilon$, based in $q$, defined by:

$$\Lambda^{(U,V)}_\epsilon = \Phi^V_{-\epsilon}\Phi^U_{-\epsilon}\Phi^V_\epsilon\Phi^U_\epsilon(q) \tag{3.31}$$

where $\Phi^U$ (resp., $\Phi^V$) is the local flow of $U$ (resp., $V$). Then we develop the loop $\Lambda^{(U,V)}_\epsilon$ in $T_qQ$, to obtain a curve $P^\epsilon(t)$, $0 \le t \le 1$ that starts in $q \cong \mathbf{0}_q$. But, in general, this curve $P^\epsilon(t)$ doesn't close, i.e., $P^\epsilon(0) \neq P^\epsilon(1)$. In fact, we can prove that, to second order in $\epsilon$, the failure of $P^\epsilon(t)$ to close is measured by a vector in $T_qQ \cong \mathrm{I\!R}^n$, depending only on $u \wedge v \in \wedge^2 T_qQ$ (and not on the vector fields $U, V$), which is exactly the torsion of the connection at $q$ evaluated in $u \wedge v$: $\Omega^a(u \wedge v)$.

If we look again to equations (2.41), of section 2:

$$\begin{cases} d\mathbf{\Theta}^i & = & \boldsymbol{\omega}^i_j \wedge \mathbf{\Theta}^j & + & \mathbf{T}^i_{j\alpha}\mathbf{\Theta}^j \wedge \mathbf{\Theta}^\alpha & + & \mathbf{T}^i_{\alpha\beta}\mathbf{\Theta}^\alpha \wedge \mathbf{\Theta}^\beta \\ d\mathbf{\Theta}^\alpha & = & \boldsymbol{\omega}^\alpha_\beta \wedge \mathbf{\Theta}^\beta & + & \mathbf{T}^\alpha_{jk}\mathbf{\Theta}^j \wedge \mathbf{\Theta}^k & + & \mathbf{T}^\alpha_{j\beta}\mathbf{\Theta}^j \wedge \mathbf{\Theta}^\beta \end{cases}$$

that gives the Cartan's intrinsic geometrization of 2-step non-holonomic systems, we see that we have two Euclidean connections $\boldsymbol{\omega}^i_j$ and $\boldsymbol{\omega}^\alpha_\beta$, that conduces to two developments, respectively in $\mathcal{D}_0$ and $\mathcal{D}_0^\perp$ (we choose a complementary subspace $\mathcal{D}_0^\perp$ to $\mathcal{D}_0$ so that $\mathcal{D}_0^\perp \cong V/S$). The first development has torsion along "infinitesimal loops" $u \wedge v \in \wedge^2 T_{q_0}Q$, with $u \in \mathcal{D}_0, v \in \mathcal{D}_0^\perp$ or $u, v \in \mathcal{D}_0^\perp$, and the torsion vanishes if $u, v \in \mathcal{D}_0$. The second development has torsion along "infinitesimal loops" $u \wedge v \in \wedge^2 T_{q_0}Q$, with $u \in \mathcal{D}_0, v \in \mathcal{D}_0^\perp$ or $u, v \in \mathcal{D}_0$ (in this last case the torsion relates to the integrability tensor of the distribution $\mathcal{D}$), and the torsion vanishes if $u, v \in \mathcal{D}_0^\perp$. Moreover we have the symmetries given by (2.42).

# 4 Example. The constrained particle.

Here we apply the above methods to the so called **constrained particle** in $Q = \mathbb{R}^3_{xyz}$ (see [17], pag. 256, [4], pag. 2035 or [3], pag. 53), with kinetic energy:

$$g = 2T = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$$

and constraint:

$$\theta^3 = dz - y\,dx$$

As we have already remarked in the introduction, in these papers, the connection found is neither metric nor unique. On the contrary, and this one the main differences of the approach we develop here, the connection found below is intrinsically associated to the non-holonomic system, and moreover it is a metric connection, though in general with torsion. The difference is therefore very explicit (compared with [2], example 2 and [4], example 6.2). In both these works the connection is not metric. Another subject that is treated here and not elsewhere (to our knowledge), is related to the development of $Q = \mathbb{R}^3$ into (affine) $\mathbb{R}^2 \cong \mathcal{D}_\mathbf{0} \subset \mathcal{A}_\mathbf{0}\mathbb{R}^3$, along any curve starting at $\mathbf{0}$, associated to the intrinsic affine Euclidean connection that is determined below.

We have that $\mathcal{D} = \ker\theta^3 = \mathrm{span}\{Y_1, Y_2\}$, where $Y_1 = \partial_y$ and $Y_2 = \partial_x + y\,\partial_z$. Note that $\{Y_1, Y_2\}$ is a $T$-orthogonal basis for $\mathcal{D}$. Moreover $Y_{12} = [Y_1, Y_2] = \partial_z$, and so the nonholonomy degree is 2. We call $\mathcal{D}$ a 2-step distribution with grow vector $(2, 3)$.

We start with the following 0-adapted orthonormal basis:

$$X_1 = \partial_y, \qquad X_2 = \frac{\partial_x + y\,\partial_z}{\sqrt{1+y^2}}, \qquad X_3 = \frac{-y\,\partial_x + \partial_z}{\sqrt{1+y^2}} \tag{4.1}$$

so that $\mathcal{D} = \mathrm{span}\{X_1, X_2\}$, with corresponding dual basis:

$$\theta^1 = dy, \qquad \theta^2 = \frac{dx + y\,dz}{\sqrt{1+y^2}}, \qquad \theta^3 = \frac{dz - y\,dx}{\sqrt{1+y^2}} \tag{4.2}$$

so that $\mathcal{D} = \ker\theta^3$. By construction, $(\theta^1)^2 + (\theta^2)^2\big|_\mathcal{D} = g|_\mathcal{D}$, and so this is in fact a 0-adapted coframe to $\mathcal{D}$. We consider now the $G_1$-structure $\mathcal{B}_1 \cong \mathbb{R}^3 \times G_1$, over $Q = \mathbb{R}^3$, trivialized with respect to our choice of the initial 0-adapted coframe (4.2), where $G_1$ is the group:

$$G_1 = \left\{ \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & a \end{bmatrix}, \quad \varphi \in \mathbb{R},\, a \in \mathbb{R} - \{0\} \right\}$$

The corresponding tautological form on $\mathcal{B}_1$ is:

$$\mathbf{\Theta} = \begin{bmatrix} \mathbf{\Theta}^1 \\ \mathbf{\Theta}^2 \\ \mathbf{\Theta}^3 \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1/a \end{bmatrix} \begin{bmatrix} \theta^1 \\ \theta^2 \\ \theta^3 \end{bmatrix}$$

So:

$$\mathbf{\Theta}^1 = \cos\varphi\,\theta^1 + \sin\varphi\,\theta^2, \qquad \mathbf{\Theta}^2 = -\sin\varphi\,\theta^1 + \cos\varphi\,\theta^2, \qquad \mathbf{\Theta}^3 = \frac{1}{a}\theta^3 \tag{4.3}$$

where $\theta^a$ are given by (4.2), and we compute that:

$$\begin{aligned} \theta^1\wedge\theta^2 &= \mathbf{\Theta}^1\wedge\mathbf{\Theta}^2 \\ \theta^1\wedge\theta^3 &= a\,\cos\varphi\,\mathbf{\Theta}^1\wedge\mathbf{\Theta}^3 - a\,\sin\varphi\,\mathbf{\Theta}^2\wedge\mathbf{\Theta}^3 \\ \theta^2\wedge\theta^3 &= a\,\sin\varphi\,\mathbf{\Theta}^1\wedge\mathbf{\Theta}^3 + a\,\cos\varphi\,\mathbf{\Theta}^2\wedge\mathbf{\Theta}^3 \end{aligned} \tag{4.4}$$

We also need the following computations:

$$d\theta^1 = 0, \qquad d\theta^2 = \frac{1}{1+y^2}\,\theta^1\wedge\theta^3, \qquad d\theta^3 = -\frac{1}{1+y^2}\,\theta^1\wedge\theta^2 \tag{4.5}$$

Therefore the first derived system $\mathfrak{I}^{(1)}$ is generated by $\theta^1\wedge\theta^2$. So, from (4.3), (4.4) and (4.5), we deduce that:

$$\begin{aligned} d\mathbf{\Theta}^1 &= d\varphi\wedge\mathbf{\Theta}^2 + \frac{\sin\varphi}{1+y^2}\left(a\,\cos\varphi\,\mathbf{\Theta}^1\wedge\mathbf{\Theta}^3 - a\,\sin\varphi\,\mathbf{\Theta}^2\wedge\mathbf{\Theta}^3\right) \\ d\mathbf{\Theta}^2 &= -d\varphi\wedge\mathbf{\Theta}^1 + \frac{\cos\varphi}{1+y^2}\left(a\,\cos\varphi\,\mathbf{\Theta}^1\wedge\mathbf{\Theta}^3 - a\,\sin\varphi\,\mathbf{\Theta}^2\wedge\mathbf{\Theta}^3\right) \\ d\mathbf{\Theta}^3 &= -\frac{da}{a}\wedge\mathbf{\Theta}^3 - \frac{1}{a}\frac{1}{1+y^2}\,\mathbf{\Theta}^1\wedge\mathbf{\Theta}^2 \end{aligned} \tag{4.6}$$

and thus the structure equation is:

$$\begin{bmatrix} d\mathbf{\Theta}^1 \\ d\mathbf{\Theta}^2 \\ d\mathbf{\Theta}^3 \end{bmatrix} = \begin{bmatrix} 0 & d\varphi & 0 \\ -d\varphi & 0 & 0 \\ 0 & 0 & -\frac{da}{a} \end{bmatrix} \wedge \begin{bmatrix} \mathbf{\Theta}^1 \\ \mathbf{\Theta}^2 \\ \mathbf{\Theta}^3 \end{bmatrix} + \begin{bmatrix} a\,\frac{\sin\varphi\cos\varphi}{1+y^2}\,\mathbf{\Theta}^1\wedge\mathbf{\Theta}^3 - a\,\frac{\sin^2\varphi}{1+y^2}\,\mathbf{\Theta}^2\wedge\mathbf{\Theta}^3 \\ a\,\frac{\cos^2\varphi}{1+y^2}\,\mathbf{\Theta}^1\wedge\mathbf{\Theta}^3 - a\,\frac{\cos\varphi\sin\varphi}{1+y^2}\,\mathbf{\Theta}^2\wedge\mathbf{\Theta}^3 \\ -\frac{1}{a}\frac{1}{1+y^2}\,\mathbf{\Theta}^1\wedge\mathbf{\Theta}^2 \end{bmatrix} \tag{4.7}$$

Now take a look at the $\mathbf{T}^3_{12}$-torsion term, defined by the last equation $d\mathbf{\Theta}^3 = \mathbf{T}^3_{12}\,\mathbf{\Theta}^1\wedge\mathbf{\Theta}^2 \mod\{\mathbf{\Theta}^3\}$:

$$\mathbf{T}^3_{12}(x,y,z,a,\varphi) = -\frac{1}{a}\frac{1}{1+y^2}$$

Of course we can choose a section of our $\mathcal{B}_1 \cong \mathbb{R}^3\times G_1$ bundle, trivialized with respect to our choice of the initial 0-adapted coframe (4.2), say:

$$\sigma : (x,y,z) \longmapsto (x,y,z,a(x,y,z),\varphi(x,y,z))$$

so that $\mathbf{T}^3_{12}$ becomes constant and equal to 1 (this is called **torsion normalization** or **group parameter normalization**). In fact, take for example:

$$\sigma : (x,y,z) \longmapsto \left(x,y,z,a(x,y,z) = \frac{-1}{1+y^2},\ \varphi(x,y,z) = 0\right)$$

Then, for the corresponding moving coframe, obtained from the initial $O$-adapted coframe (4.2), acting on the right with $g$:

$$\begin{bmatrix} \widehat{\theta}^1 \\ \widehat{\theta}^2 \\ \widehat{\theta}^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -(1+y^2) \end{bmatrix} \begin{bmatrix} \theta^1 = dy \\ \theta^2 = \frac{dx+y\,dz}{\sqrt{1+y^2}} \\ \theta^3 = \frac{dz-y\,dx}{\sqrt{1+y^2}} \end{bmatrix} = \begin{bmatrix} \theta^1 \\ \theta^2 \\ -(1+y^2)\,\theta^3 \end{bmatrix} \tag{4.8}$$

we will have $\mathbf{T}^3_{12}(\sigma(x,y,z,a,\varphi)) \equiv 1$. Now we ask - for which coframes the torsion remains constant and equal to one ? To answer this, let us see how $\mathbf{T}^3_{12}$ changes under the $G_1$-action? We know that $R_g^*\mathbf{\Theta} = g^{-1}\mathbf{\Theta}$, so, with $g^{-1} = \begin{bmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1/a \end{bmatrix}$, with $a \neq 0$, this implies that:

$$R_g^*\mathbf{\Theta}^3 = \frac{1}{a}\,\mathbf{\Theta}^3$$

We take the exterior derivative of both sides mod$\{\mathbf{\Theta}^3\}$:

$$\frac{1}{a}\,d\mathbf{\Theta}^3 = \frac{1}{a}\,\mathbf{T}^3_{12}\mathbf{\Theta}^1\wedge\mathbf{\Theta}^2 = R_g^* d\mathbf{\Theta}^3 = (R_g^*\mathbf{T}^3_{12})R_g^*(\mathbf{\Theta}^1\wedge\mathbf{\Theta}^2) = (R_g^*\mathbf{T}^3_{12})\mathbf{\Theta}^1\wedge\mathbf{\Theta}^2 \mod\{\mathbf{\Theta}^3\}$$

and so the $\mathbf{T}^3_{12}$ changes according to:

$$R_g^*\mathbf{T}^3_{12} = \frac{1}{a}\,\mathbf{T}^3_{12}$$

So we must have $a=1$, and we reduce our $G_1$ group to the $G_2$ group:

$$G_2 = \left\{ \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad \varphi\in\mathbb{R} \right\} \subset G_1$$

and take a $G_2$ bundle $\mathcal{B}_2 \cong \mathbb{R}^3\times G_2$, trivialized with respect to our choice of the 1-adapted coframe given by (4.8) (this is called **group reduction**). We then choose a connection for this $\mathcal{B}_2$ bundle, and compute the corresponding structure equation (we have omited the "hats" over the $\mathbf{\Theta}$'s):

$$\begin{bmatrix} d\mathbf{\Theta}^1 \\ d\mathbf{\Theta}^2 \\ d\mathbf{\Theta}^3 \end{bmatrix} = \begin{bmatrix} 0 & d\varphi & 0 \\ -d\varphi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \wedge \begin{bmatrix} \mathbf{\Theta}^1 \\ \mathbf{\Theta}^2 \\ \mathbf{\Theta}^3 \end{bmatrix} + \begin{bmatrix} -\frac{\sin\varphi\cos\varphi}{(1+y^2)^2}\,\mathbf{\Theta}^1\wedge\mathbf{\Theta}^3 + \frac{\sin^2\varphi}{(1+y^2)^2}\,\mathbf{\Theta}^2\wedge\mathbf{\Theta}^3 \\ -\frac{\cos^2\varphi}{(1+y^2)^2}\,\mathbf{\Theta}^1\wedge\mathbf{\Theta}^3 + \frac{\cos\varphi\sin\varphi}{(1+y^2)^2}\,\mathbf{\Theta}^2\wedge\mathbf{\Theta}^3 \\ \mathbf{\Theta}^1\wedge\mathbf{\Theta}^2 + \frac{2y}{1+y^2}\mathbf{\Theta}^1\wedge\mathbf{\Theta}^3 \end{bmatrix} \tag{4.9}$$

Now changing:

$$d\varphi \longmapsto d\varphi + C_1\mathbf{\Theta}^1 + C_2\mathbf{\Theta}^2 + C_3\mathbf{\Theta}^3$$

we get:

$$\begin{aligned} \begin{bmatrix} d\mathbf{\Theta}^1 \\ d\mathbf{\Theta}^2 \\ d\mathbf{\Theta}^3 \end{bmatrix} &= \begin{bmatrix} 0 & d\varphi & 0 \\ -d\varphi & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{\Theta}^1 \\ \mathbf{\Theta}^2 \\ \mathbf{\Theta}^3 \end{bmatrix} + \begin{bmatrix} C_1\mathbf{\Theta}^1\wedge\mathbf{\Theta}^2 + C_3\mathbf{\Theta}^3\wedge\mathbf{\Theta}^2 \\ -C_2\mathbf{\Theta}^2\wedge\mathbf{\Theta}^1 - C_3\mathbf{\Theta}^3\wedge\mathbf{\Theta}^1 \\ 0 \end{bmatrix} + \\ & \begin{bmatrix} -\frac{\sin\varphi\cos\varphi}{(1+y^2)^2}\,\mathbf{\Theta}^1\wedge\mathbf{\Theta}^3 + \frac{\sin^2\varphi}{(1+y^2)^2}\,\mathbf{\Theta}^2\wedge\mathbf{\Theta}^3 \\ -\frac{\cos^2\varphi}{(1+y^2)^2}\,\mathbf{\Theta}^1\wedge\mathbf{\Theta}^3 + \frac{\cos\varphi\sin\varphi}{(1+y^2)^2}\,\mathbf{\Theta}^2\wedge\mathbf{\Theta}^3 \\ \mathbf{\Theta}^1\wedge\mathbf{\Theta}^2 + \frac{2y}{1+y^2}\mathbf{\Theta}^1\wedge\mathbf{\Theta}^3 \end{bmatrix} \end{aligned} \tag{4.10}$$

and choosing $C_1 = 0 = C_2$ and $C_3 = \frac{1}{2(1+y^2)^2}$, we get the structure equation:

$$\begin{bmatrix} d\mathbf{\Theta}^1 \\ d\mathbf{\Theta}^2 \\ d\mathbf{\Theta}^3 \end{bmatrix} = \begin{bmatrix} 0 & \boldsymbol{\omega} & 0 \\ -\boldsymbol{\omega} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{\Theta}^1 \\ \mathbf{\Theta}^2 \\ \mathbf{\Theta}^3 \end{bmatrix} + \begin{bmatrix} A\,\mathbf{\Theta}^1\wedge\mathbf{\Theta}^3 + B\,\mathbf{\Theta}^2\wedge\mathbf{\Theta}^3 \\ B\,\mathbf{\Theta}^1\wedge\mathbf{\Theta}^3 - A\,\mathbf{\Theta}^2\wedge\mathbf{\Theta}^3 \\ \mathbf{\Theta}^1\wedge\mathbf{\Theta}^2 + \frac{2y}{1+y^2}\mathbf{\Theta}^1\wedge\mathbf{\Theta}^3 \end{bmatrix} \tag{4.11}$$

where:

$$\boldsymbol{\omega} = d\varphi + \frac{1}{2(1+y^2)^2}\,\mathbf{\Theta}^3, \qquad A = -\frac{\sin\varphi\cos\varphi}{(1+y^2)^2}, \qquad B = \frac{1-2\cos^2\varphi}{2(1+y^2)^2} \tag{4.12}$$

The development in $\mathcal{D}_0$, with respect to the $\boldsymbol{\omega}$-connection, has symmetric (according to (2.42)) torsion along "infinitesimal loops" $u\wedge v\in\wedge^2 T_{q_0}Q$, with $u\in\mathcal{D}_0, v\in\mathcal{D}_0^\perp$, and the torsion vanishes if $u, v\in\mathcal{D}_0$.

If we choose a gauge corresponding to $\varphi\equiv 0$, i.e., our 1-adapted coframe given by (4.8), then the base structural equations become:

$$\begin{cases} d\theta^1 = \omega\wedge\theta^2 + B\,\theta^2\wedge\theta^3 \\ d\theta^2 = -\omega\wedge\theta^1 + B\,\theta^1\wedge\theta^3 \\ d\theta^3 = \theta^1\wedge\theta^2 + \frac{2y}{1+y^2}\theta^1\wedge\theta^3 \end{cases} \tag{4.13}$$

where:

$$\omega = \frac{1}{2(1+y^2)^2}\,\theta^3, \qquad B = \frac{-1}{2(1+y^2)^2} \tag{4.14}$$

and:

$$\theta^1 = dy, \qquad \theta^2 = \frac{dx + y\,dz}{\sqrt{1+y^2}}, \qquad \theta^3 = -\sqrt{1+y^2}\,(dz - y\,dx) \tag{4.15}$$

Take a parametrized curve in $\mathbb{R}^3$, $\gamma(t) = (x(t), y(t), z(t))$, so that $\dot\gamma = \dot x\partial_x + \dot y\partial_y + \dot z\partial_z = \theta^a(\dot\gamma)\,X_a(\gamma(t))$, where the $\theta's$ are given by (4.15), and the $X's$ are the corresponding dual basis. We develop this curve into $\mathbb{R}^2_{xy} \cong \mathcal{D}_0$, with respect to the basis $\{\mathbf{e}_1 = X_1(\mathbf{0}) = \partial_y,\ \mathbf{e}_2 = X_2(\mathbf{0}) = \partial_x\}$ for $\mathcal{D}_\mathbf{0}$. The equations for this development are (see (3.22) and (3.25)):

$$\left\{\begin{array}{rcl} \frac{dP}{dt} & = & \frac{d\theta^i(\dot\gamma(t))}{dt}\,\mathbf{e}_i(t) \\ \frac{d\mathbf{e}_i}{dt} & = & \left(\Gamma^k_{ij}(\gamma(t))\frac{d\theta^j(\dot\gamma(t))}{dt} + \Gamma^k_{i\alpha}(\gamma(t))\frac{d\theta^\alpha(\dot\gamma(t))}{dt}\right)\mathbf{e}_k(t) \end{array}\right. \tag{4.16}$$

With $\omega = \omega^1_2 = -\omega^2_1 = \frac{1}{2(1+y^2)^2}\,\theta^3$, we have that the only non trivial $\Gamma's$ are $\Gamma^1_{23} = -\Gamma^2_{13} = \frac{1}{2(1+y^2)^2}$, and so:

$$\left\{\begin{array}{rcccl} \frac{d\mathbf{e}_1}{dt} & = & \Gamma^2_{13}(\gamma(t))\frac{d\theta^3(\dot\gamma(t))}{dt}\,\mathbf{e}_2(t) & = & \frac{1}{2(1+y(t)^2)^2}\frac{d\theta^3(\dot\gamma(t))}{dt}\,\mathbf{e}_2(t) \\ \frac{d\mathbf{e}_2}{dt} & = & \Gamma^1_{23}(\gamma(t))\frac{d\theta^3(\dot\gamma(t))}{dt}\,\mathbf{e}_1(t) & = & -\frac{1}{2(1+y(t)^2)^2}\frac{d\theta^3(\dot\gamma(t))}{dt}\,\mathbf{e}_1(t) \end{array}\right. \tag{4.17}$$

which are the differential equations for the moving frame $\{\mathbf{e}_1(t), \mathbf{e}_2(t)\}$, evolving within $\mathcal{D}_\mathbf{0}$, starting for $t = 0$ with $\{\mathbf{e}_1 = \partial_y,\ \mathbf{e}_2 = \partial_x\}$. After integrating these equations we substitute the $\mathbf{e}_i(t)$ in the first equation (4.16), to obtain the differential equation for the development of $\gamma$ in $\mathcal{D}_\mathbf{0}$:

$$\frac{dP}{dt} = \frac{d\theta^i(\dot\gamma(t))}{dt}\,\mathbf{e}_i(t) \tag{4.18}$$

In particular, if $\gamma$ is an horizontal curve, which implies that $\theta^3(\dot\gamma) \equiv 0$, we obtain $\frac{d\mathbf{e}_1}{dt} = 0$ and $\frac{d\mathbf{e}_2}{dt} = 0$, i.e., $\mathbf{e}_1(t) \equiv \partial_y$ and $\mathbf{e}_2(t) \equiv \partial_x$, and so:

$$\frac{dP}{dt} = \frac{d\theta^1(\dot\gamma(t))}{dt}\,\partial_y + \frac{d\theta^2(\dot\gamma(t))}{dt}\,\partial_x \tag{4.19}$$

As a concrete example, take $\gamma(t) = (t^2/2, 0, -t^2/2)$. Then $\theta^3(\dot\gamma) = t$ and

$$\left\{\begin{array}{rcl} \frac{d\mathbf{e}_1}{dt} & = & \frac12\,\mathbf{e}_2(t) \\ \frac{d\mathbf{e}_2}{dt} & = & -\frac12\,\mathbf{e}_1(t) \end{array}\right. \quad\Rightarrow\quad \left\{\begin{array}{rcl} \mathbf{e}_1(t) & = & \left(\cos\frac12 t\right)\mathbf{e}_1 + \left(\sin\frac12 t\right)\mathbf{e}_2 \\ \mathbf{e}_2(t) & = & -\left(\sin\frac12 t\right)\mathbf{e}_1 + \left(\cos\frac12 t\right)\mathbf{e}_2 \end{array}\right. \tag{4.20}$$

where $\mathbf{e}_1 = \partial_y$ and $\mathbf{e}_2 = \partial_x$. Thus, since $\theta^1(\dot\gamma) = 0$ and $\theta^2(\dot\gamma) = t$, we have:

$$\frac{dP}{dt} = -\left(\sin\frac12 t\right)\mathbf{e}_1 + \left(\cos\frac12 t\right)\mathbf{e}_2 \quad\Rightarrow\quad P(t) = 2\left(\sin\frac12 t, -1 + \cos\frac12 t\right) \tag{4.21}$$

since $P(t)$ must verify the initial condition $P(0) = (0, 0)$.

Now, with $X_1, X_2$ given by (4.1), we have:

$$\begin{array}{rcl} \dot\gamma & = & \theta^1(\dot\gamma)\,X_1(\gamma(t)) + \theta^2(\dot\gamma)\,X_2(\gamma(t)) \overset{\text{def}}{=} v^1\,X_1 + v^2\,X_2 \\ & = & v^1\partial_y + v^2\frac{\partial_x + y\,\partial_z}{\sqrt{1+y^2}} = \frac{v^2}{\sqrt{1+y^2}}\partial_x + v^1\partial_y + \frac{yv^2}{\sqrt{1+y^2}}\partial_z = \dot x\partial_x + \dot y\partial_y + \dot z\partial_z \\ & \Rightarrow & \left\{\begin{array}{rcl} \dot x & = & \frac{v^2}{\sqrt{1+y^2}} \\ \dot y & = & v^1 \\ \dot z & = & \frac{yv^2}{\sqrt{1+y^2}} \end{array}\right. \quad\Rightarrow\quad \left\{\begin{array}{rcl} v^1 & = & \dot y \\ v^2 & = & \sqrt{1+y^2}\,\dot x \end{array}\right. \end{array} \tag{4.22}$$

and the equations for a geodesic $\gamma$ are:

$$\frac{d}{dt}\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\begin{bmatrix} v^1 \\ v^2 \end{bmatrix}, \qquad \text{with} \qquad \omega = \frac{1}{2(1+y^2)^2}\,\theta^3$$

So they are (compare with [2]):

$$\left\{\begin{array}{lll} \frac{dv^1}{dt} & = \ddot{y} & = 0 \\ \frac{dv^2}{dt} & = \frac{y\dot{y}}{\sqrt{1+y^2}}\dot{x} + \sqrt{1+y^2}\,\ddot{x} & = 0 \end{array}\right. \quad \Rightarrow \quad \left\{\begin{array}{ll} \ddot{y} & = 0 \\ \ddot{x} + \frac{y}{1+y^2}\dot{x}\dot{y} & = 0 \end{array}\right. \tag{4.23}$$

$\blacksquare$.

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# References

[1] Arnold V.I. (Ed.), *"Dynamical Systems III"*. Encyclopaedia of Mathematical Sciences, volume 3, Springer-Verlag, 1988.

[2] Bates L and Sniatycki J., "Nonholonomic Reduction". *Reports on Math. Physics*, vol.32, pag. 99-115, 1993.

[3] Bloch A.M., Krishnaprasad P.S., Marsden J.E., and Murray R. , "Nonholonomic Mechanical Systems with Symmetry". *Arch. Rational Mech. Analysis*, vol.136, pag. 21-99, 1996.

[4] Bloch, A.M.; Crouch, P.E. , "Newton's law and integrability of nonholonomic systems". *SIAM J. Control Optimization* 36, No.6, pag. 2020-2039, 1998.

[5] Cartan E., "Représentation géométrique des systèmes non holonomes". *Oeuvres complètes, vol III*, pag. 253-261, 1928.

[6] Cartan E., "Les variétés a connexion affine et la théorie de la relativité généralisée (première partie)". *Oeuvres complètes, vol III*, pag. 325-412, 1925.

[7] Cartan E., *"La méthode du repère mobile, la théorie des groupes continus et les espaces généralisés"*. Esposés de géometrie, vol V, Hermann, 1935.

[8] Chern S. S., "The geometry of $G$-structures". *Bull. Amer. Math. Soc.* 72, pag. 167-219, 1966.

[9] Favard J., *"Cours de Géométrie Différentielle Locale"*. Gauthiers-Villars, 1957.

[10] Gardner R., *"The Method of Equivalence and its Applications"*. CBMS-NSF Regional Conference Series, vol. 58, SIAM, 1989.

[11] Kobayashi S. and Nomizu K., *"Foundations of Differential Geometry"*, vol. I. Interscience Publishers, John Wiley & Sons, 1963.

[12] Koiller J., Rodrigues P.R., Pitanga P., *"Non-Holonomic Connections following Élie Cartan"*. Preprint, LNCC, Brasil, 2001.

[13] Montgometry R., *"A Tour of Subriemannian Geometries, their Geodesics and Applications"*. Mathematical Surveys and Monographs, vol. 91, AMS, 2002.

[14] Nomizu K., "Kinematics and differential geometry of submanifolds". *Tôhoku Math. Journal* 30, pag. 623-637, 1978.

[15] Okubo T., *" Differential Geometry"*. Monographs and Textbooks in Pure and Applied Mathematics 112, Marcel Dekker, Inc., 1987.

[16] Pars L.A., *"A Treatise on Analytical Dynamics"*. Heinemann Educational Books Ltd., 1968.

[17] Rosenberg R.M., *"Analytical Dynamics of Discrete Systems"*. Plenum Press, 1977.

[18] Sharpe R. W., *"Differential Geometry - Cartan's generalization of Klein's Erlangen programm"*, GTM. 166, Springer-Verlag, 1997.

[19] Spivak M., *"A Comprehensive introduaction to Differential Geometry, vol. 2"*, 2nd edition, Publish or Perish, 1979.

[20] Sternberg S., *"Lectures on Differential Geometry"*. AMS Chelsea Publishing, 1999.

[21] Vershik, A.M.; Gershkovich, V.Ya., "Nonholonomic Dynamical Systems, Geometry of Distributions and Variational Problems", in *Encyclopaedia of Mathematical Sciences, Dynamical Systems VII*, V.I. Arnold and S.P. Novikov eds., Springer-Verlag, 1994.

[22] Vershik, A.M., "Classical and non-classical dynamics with constraints", *Lecture Notes in Mat. 1108*, Springer-Verlag, pag. 278-301, 1984.

João Nuno Tavares

Dept. Matemática Pura
Faculdade de Ciências do Porto
Rua do Campo Alegre, 687
4169-007 Porto, Portugal.

E-mail adress: jntavar@fc.up.pt

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