On the Mechanics of the Poincaré Conjecture: an Heuristic Tour.

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Abstract

This paper deals with aspects of the 3-dimensional Poincaré conjecture. Its main purpose is to explain the conjecture to non specialists and provide the readers with some ideas, drawn on physical (mechanical) intuition and based on some arguments of an heuristic nature, why that conjecture ought to be true.

1 Introduction

" Once and for all there is a great deal I do not want to know - Wisdom sets bounds even to knowledge " (F. Nietzsche - Twilight of the Idols)

This is the first of a series of papers on contractible spaces intended to non-specialist readers - in particular I have in mind students with some knowledge of the generals of a first topology course at the graduate level - and the main purpose is to make those readers acquainted with the many exquisite and intriguing features of some contractible spaces. I intend each paper to be kept open to suggestions for improvements, corrections and updates.

This paper deals with aspects of the 3-dimensional Poincaré conjecture. This conjecture can be formulated in terms of contractible spaces: if a compact 3-manifold is contractible then it is homeomorphic to the 3-ball, B^3. In section 2 some basic definitions and equivalent formulations of the conjecture are reviewed. I assume the reader is acquainted with the basics of homology and homotopy theory and covering spaces.

The three dimensional Poincaré conjecture is one of the most famous open problems in mathematics and, over the years, has resisted many attempts to prove it, some by outstanding topologists and others by less talented but

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more stubborn and foolish guys. In this second group, I have already collected my handful of failures, all following the same approach. Most mathematicians believe the conjecture to be true and that reflects in the fact that, to my knowledge, not so much effort has been put in finding counter-examples (however, there are some sketches of programs in that direction; see [6, Chapter 14] where some approaches are described including the search for counter-examples). This is so, even when that major obstacle in checking potential counter-examples, the non existence until recently of any procedure to recognize S^3, has now been removed, [1] - many people contend, rightly so I think, that almost all examples in low-dimensional topology are easy, or at least not too difficult, and the same would be expected of a counter-example to the Poincaré conjecture, was the conjecture to be false. Due to its long resistance to be settled, some believe the conjecture to be in the role of the undecidable problems. Others hold the softer view, which I share, that the conjecture is likely to be unprovable by algorithmic proofs but may perhaps be settled in other ways. By algorithmic proof I mean a proof that would allows us, at least in theory and by some suitable translation, to start with a triangulation of a homotopy 3-sphere or some equivalent combinatorial data like, for instance, Heegaard diagrams or surgery presentation, and generate (in an algorithmic way!) a finite sequence of (combinatorial) configurations ending in some standard one for S^3. All the attempted proofs of the conjecture that I know of - I mean, direct ones and not the possible corollaries of big programs like Thurstons's Geometrization Conjecture, [2] - are of this type.

The main purpose of this paper is a bit ambitious. It is to provide the readers with some ideas, drawn on physical intuition and based on some arguments of an heuristic nature, why the 3-dimensional Poincaré conjecture ought to be true. A question naturally arises: how can such a task be fulfilled without, a least, sketching a proof? For sure, the paper follows the usual pattern of mathematical reasoning, with a sequence of statements, examples and arguments. Most statements are of a pure mathematical nature and can be proved - some are left as exercises. The point is that some key arguments are just heuristic and I can not even think of how they could possibly be formalised. Before proceeding, please read the *disclaimer* at the end of the final section!

2 Homotopy spheres and contractible spaces

In 1900 Poincaré, [3], asserted that a closed 3-manifold with "trivial homology", that is the same homology groups of the 3-sphere, S^3 - what we now call a *homology 3-sphere* - would be simply connected and therefore homeomorphic to the sphere:

> ...est simplement connexe, c'est-a dire homeómorphe a l'hypersphere. [3, p. 308]

He soon corrected the mistake providing an example of a non simply connected homology sphere, the so called Poincaré homology 3-sphere, [4] (this is also a fascinating space: see [5]). Thereafter the conjecture remained, but in terms of the fundamental group:

Conjecture 1 *If a closed and connected 3-manifold, M^3, is simply connected, that is $\pi_1(M^3) = 0$, then M^3 and S^3 are homeomorphic: $M^3 \cong S^3$.*

It is this statement which came to be known as the (3-dimensional) Poincaré conjecture. Due to its origins, it is probably fair to consider 2004 as the year of its (first?) century!

The existence of the unsettled conjecture didn't hamper the development of *3-manifold theory* through out the last century, even if any possible classification of closed 3-manifolds will necessarily involve its solution. References [6]-[16] and [2] cover all the essential parts of 3-manifold theory. Of these, [6], [14] and [13] are very good introductions; [11] and [12] treat the problems related to the existence and equivalence of *triangulations* for 3-manifolds; the others cover more advanced topics, specially [2] and [16]. One also needs some knowledge of *knot theory* to be able to appreciate many aspects of low dimensional topology: [17], [18] and [19] are references for this subject. I will be using some basic language and concepts of *PL (piecewise linear) topology* at a very intuitive and elementary level; but for those who wish to go into the subject, [24] and [25] are standard references.

2.1 Several faces of the Poincaré conjecture

The following results are well known and lead to several equivalent formulations of the Poincaré conjecture. They have easy proofs, making use of homology sequences, Alexander-Poincaré and Poincaré-Lefschetz duality and the homotopy theorems of Hurewicz and Whitehead. These theorems, which are corner stones in geometric topology, can be learned in many algebraic topology books (see, for instance, [20], [21], [22] and [23]).

Definition 2 *A homotopy 3-sphere is a 3-manifold, H^3, homotopy equivalent to S^3, in notation $H^3 \simeq S^3$.*

Theorem 3 *A 3-manifold, H^3, is a homotopy 3-sphere iff H^3 is closed, connected and simply connected.*

Proof. Necessity is easy: a homotopy equivalence induces isomorphisms between the homotopy groups and homology groups; therefore $\pi_1(H^3) = 0$, $H_0(H^3) \cong \mathbb{Z}$ so H^3 is connected and $H_3(H^3) \cong \mathbb{Z}$ so H^3 is closed. For sufficiency: assume that H^3 is closed, $\partial(H^3) = \emptyset$ and $\pi_0(H^3) = \pi_1(H^3) = 0$. Then $H_1(H^3) = 0$ and by *duality* $H_2(H^3) = 0$. Let $C = H^3 - \overset{\circ}{B} = \overline{H^3 - B}$ where B is an embedded 3-cell. C is a 3-manifold with boundary $\partial C \cong S^2$. By Van Kampen's theorem $\pi_1(C) = 0$ and so $H_1(C) = 0$. By Mayer-Vietoris $H_2(C) = 0$ and therefore $H_n(C) = 0$ for all $n \geq 1$. By the Hurewicz theorem $\pi_n(C) = 0$ for all n and so by Whitehead theorem C is contractible. Writing the 3-sphere as the union of the upper and lower hemispheres, each one homeomorphic to B^3, $S^3 = B^3_+ \cup_{S^2} B^3_-$, a homeomorphism $f_+ : B^3_+ \longrightarrow B$ extends to a map $f : S^3 \longrightarrow H^3$ with $f(B^3_-) = C$. Similarly $f_+^{-1} : B \longrightarrow B^3_+$ extends to a map $g : H^3 \longrightarrow S^3$ with $g(C) = B^3_-$. It is easy to check that f and g are homotopy inverses. ■

Definition 4 *A homotopy 3-ball (or 3-cell) is a 3-manifold, C^3, which is compact and contractible.*

In general I'll denote an arbitrary homotopy 3-sphere by H or H^3 and an arbitrary homotopy 3-ball by C or C^3.

Theorem 5 *If C is a homotopy 3-ball then $\partial C \cong S^2$.*

If M is a 3-manifold, I'll use the notation $\overset{n\vee}{M}$ for the manifold obtained from M by removing the interiors of n disjoint embedded 3-cells in the interior of M, $\overset{\circ}{M} = M - \partial M$; when $n = 1$ the index is dropped. The boundary of $\overset{n\vee}{M}$ is the disjoint union of n 2-spheres. Reciprocally if N is a 3-manifold whose boundary contains n 2-spheres, $\overset{n\wedge}{N}$ represents the manifold obtained from N by capping off those n 2-spheres with 3-cells, with the same absence of the index in the case $n = 1$. With these notations we have the following result:

Theorem 6 *H is a homotopy 3-sphere iff $\overset{\vee}{H}$ is a homotopy 3-ball and, reciprocally, C is a homotopy 3-ball iff $\hat{C}$ is a homotopy 3-sphere.*

We come to the following equivalent formulation of the Poincaré conjecture:

Conjecture 7 *If C^3 is a homotopy 3-ball, then $C^3 \cong B^3$.*

A contraction of a 3-dimensional object may be tricky to visualize. The next result, which sharpens the second theorem before, tell us that in the case of homotopy 3-balls, C, the essence is in the contraction of the boundary $\partial C \cong S^2$.

Theorem 8 *A compact and connected 3-manifold, M, is a homotopy 3-ball iff $\partial M \neq \emptyset$ and ∂M is contractible in M in which case $\partial M \cong S^2$ and the contraction of ∂M can't miss any point of M.*

Corollary 9 *A 3-manifold C is a homotopy 3- ball iff there is a map $f : B^3 \longrightarrow C$ which restricts to a homeomorphism between $S^2 = \partial B^3$ and ∂C, in which case f is surjective.*

2.1.1 Non compact analogous

The situation is very different in the case of non compact contractible 3-manifolds. The analogous of the previous conjecture for open 3-manifolds - any open contractible 3-manifold is homeomorphic to the interior of the 3-ball (or $\mathbb{R}^3$) is *false*. Whitehead, [26], provided an example of an open contractible subspace of $\mathbb{R}^3$ which is not homeomorphic to the entire space $\mathbb{R}^3$; in fact there is even an *uncountable* collection of pairwise non homeomorphic such examples in $\mathbb{R}^3$: see [27]. This exotic behaviour is due to non compactness since the one-point compactification of these spaces is never a manifold: their *ends* are tricky. The open contractible 3-manifolds are usually called Whitehead-manifolds or W-manifolds. The study of W-manifolds is also important because of the so called Covering Conjecture:

Conjecture 10 (Covering conjecture) *Any open contractible 3-manifold which covers non trivially a compact 3-manifold is homeomorphic to $\mathbb{R}^3$.*

This conjecture is related to the Geometrization Conjecture which implies it. Due to their tricky ends, the known examples of W-manifolds do not cover non trivially any compact 3-manifold. References for this fascinating subject (that I'm planning to dedicate another paper in the series) are [28]-[31].

2.1.2 The higher dimensional Poincaré conjecture

The 3-dimensional conjecture generalized in the obvious way to n-dimensions:

Conjecture 11 (Generalized Poincaré) *Any homotopy n-sphere is homeomorphic to S^n.*

The first mathematician to envisage a proof for the case $n \geq 5$ was S. Smale, using *handle-decompositions* (that correspond to Morse functions in the case of differentiable manifolds). Smale's proof, [35], also applies in the PL case; there were other approaches for the case $n \geq 5$, notably by Zeeman and Stallings, using *engulfing* technics (see [25] and [24]). The topological case for $n = 4$ was solved by M. Freedman, [36] (see also [37]). In this dimension, the PL case (or equivalently the differentiable case) remains undecided, that is, we still don't know if there are any *exotic* differential (or PL) structures in S^4 (they exist in $\mathbb{R}^4$ which is the only euclidian space where this phenomenon occurs: see [38] for a nice introduction).

2.2 2-dimensional contractible polyhedra and the Zeeman conjecture

As we saw in a previous theorem, the contractibility of a homotopy 3-ball is implied by the contractibility of its 2-dimensional sphere boundary. A more direct reduction of the contracting property of a homotopy 3-ball to a 2-dimensional phenomenon is given by the concept of *spine*.

Definition 12 *If M is a compact 3-manifold with non empty boundary, a 2-dimensional complex K embedded in the interior of M (as a subcomplex of some triangulation) is a spine of M if M* collapses *to K, in notation $M \searrow K$; this means we can go from M to K by a succession of elementary collapses, $M \searrow K_1 \searrow K_2 \searrow \cdots \searrow K_n = K$, where an elementary collapse is the removal of the interior of a simplex from a free face, that is a face which is not a face of another simplex, and whose interior is also removed (see [24], [25]).*

Definition 13 *A manifold M is* collapsible *if for some suitable triangulation, $M \searrow v$, where v is some vertex, that is, if M collapses to a point.*

Note that although the 3-ball, B^3, is obviously collapsible, there are triangulations of B^3 which are not collapsible. Nevertheless it can be shown that any triangulation of the 3-ball has some subdivision which collapses (see [32]). At a certain time some hoped to get counter examples to the Poincaré conjecture by constructing a homotopy 3-ball with a triangulation which had no collapsible subdivision. One such approach was by Steve Armentrout, using *decomposition theory* (for decomposition theory see [33], which is a favourite of mine).

Definition 14 *A* regular neighbourhood *of a finite complex K in the interior of the manifold M is a neighbourhood, $N(K)$ of K which is a compact manifold with boundary and such that $N(K) \searrow K$.*

Regular neighbourhoods exist (one essentially considers the star of K in the second baricentric subdivision of the triangulation) and are unique up to ambient isotopy. You can think of a *regular neighbourhood* as the result of a small thickening of K in M.

The theory of regular neighbourhoods ([24], [25]) gives the following general result:

Theorem 15 *A 3-manifold M is collapsible iff $M \cong B^3$.*

We can then formulate Poincaré conjecture in terms of collapsibility:

Conjecture 16 *Any homotopy 3-ball is collapsible.*

2.2.1 Delicatessen examples

Now if $M \searrow K$ then K is a (strong deformation) *retract* of $\dot{M}$. Therefore if M is contractible so is K. There are very interesting examples of contractible 2-complexes which are spines of B^3. one such example is the *dunce hat*.

Example 17 (Dunce hat) *The dunce hat, D, is the quotient space obtained from a triangle (2-simplex) by identifying the three oriented edges where the orientations of two of the edges are pointing to the same vertex (Figure 1- left).*

If you identify those two edges first, you get a disc with oriented boundary coming from the third edge and with the two edges identified as an edge in the interior of the disc (Figure 1- right). Then you identify this interior edge to the boundary circle.

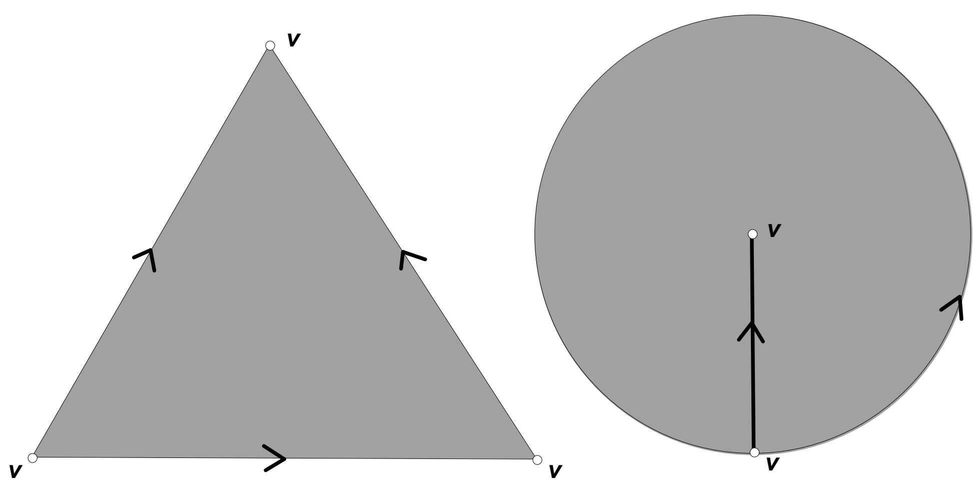

Figure 1

The dunce hat embeds in $\mathbb{R}^3$. Actually it is easy to see that, by stretching the interior of this disc and keeping the boundary circle fixed, you can even perform this identification through a movement in $\mathbb{R}^3$. It is also easy to see that a regular neighbourhood of D in $\mathbb{R}^3$ is homeomorphic to B^3. Therefore the dunce hat is contractible but is not collapsible since there is not even any free edge for a collapsing to start!

Remark 18 *The point I want to stress here is that it is really difficult to visualize a contraction of these spaces even of such a simple one like the dunce hat. I can still do it with the dunce hat, but things became much more difficult with other examples like Bing's house with two rooms (Figure 2).*

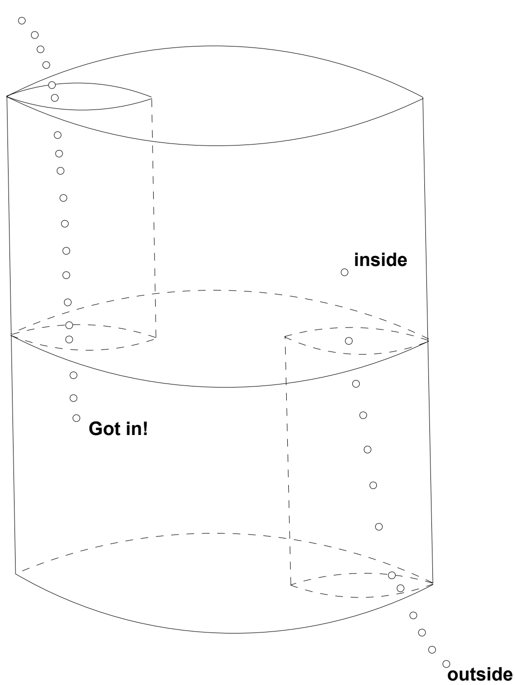

Figure 2

The house is built in the following way. Start with a cylinder, closed at the top and the bottom, and add a slab in the middle; then add two more cylinders, as shown in Figure 2, one from the mid slab to the top and another one from the mid slab to the bottom; remove then the base and top of these two cylinders. You got a house with two separated rooms: you get into the bottom room through the top entrance and into the top room from the bottom entrance (see [11, p.170-174]). You can easily see that a regular neighbourhood of this object in $\mathbb{R}^3$ is a 3-ball. Can you visualize a contraction of this space?

Every compact 3-manifold with non empty boundary has spines; in fact most of the usual representations of 3-manifolds, like *Heegard-diagrams*, *surgery representations* or *branched covers* (see [13] and the references therein) and in particular *handle-decompositions* (that played an essential role in the solutions of the Poincaré-conjecture in higher dimensions), have such spines naturally associated with them.

The dunce hat has many interesting features as you can see in Zeeman's paper [34]. Such a nice property is that although D is not collapsible, $D \times I$ (where I is the unit interval) is collapsible. Zeeman conjectured the general result would be true:

Conjecture 19 (Zeeman) *If K is a compact, contractible 2-dimensional complex, then $K \times I$ is collapsible.*

Theorem 20 *Zeeman's conjecture implies Poincaré's.*

There are some results in the opposite direction, concerning special types of spines, [39], [40].

Strange enough Zeeman's conjecture is still open although it is much stronger than the Poincaré conjecture, at least in principle, since it refers to *any* compact,contractible 2-dimensional complex and most of these won't be spines of any 3-manifold C^3 (a homotopy 3-ball). But no counter example was found so far.

2.2.2 Generalized dunce hats

The dunce hat can be generalized to give spines associated with homotopy 3-spheres in the following way. A *generalized dunce hat* (*gdh*), is a 2-complex obtained as an identification space, $\widehat{D} = D/\sim$, where $D = \bigvee_{i=1}^{n} D_i$ is a wedge of n discs with oriented boundaries. Each disc has a connected finite graph with oriented edges in its interior except for one vertex at the wedge point. The space $D/\sim$ is the result of identifying all the vertices of the graphs with the wedge point and each oriented edge with the oriented boundary of one of the discs D_i - the traditional *dunce hat* is the case of having just one disc with its graph consisting of just one edge.

Theorem 21 *Every homotopy 3-sphere, H^3, with the interiors of a finite number of disjoint 3-balls removed, collapses to a* gdh*, $\widehat{D}$: for some n, $\overset{n\vee}{H^3} \searrow \widehat{D}$.*

The proof of this can be seen in [41] (in Portuguese!) where a program to search for counter examples to the conjecture using these objects is also sketched.

It is possible to prove, using a result of Zieschang, [42], that if $n = 1$ in the previous theorem and so $\overset{\vee}{H}^3 = C^3$ is a homotopy 3-ball, which is equivalent to having only *trees* as graphs, then $C^3 \cong B^3$.

3 Heuristic tour

I will now start the heuristic tour. I will be using euclidian cubes instead of the usual 3-ball, B^3. Let J^3 denote the cube $\prod^3 [-1, 1]$. $J^3 = B_m [0; 1]$, the closed ball of centre the origin and radius 1 for the metric of the maximum, $m(x, y) = \max_{i=1,2,3} |\, x_i - y_i|$. I will denote a general homotopy 3-ball by C^3 as before; but now I think of the boundary, ∂C, as a copy of $\partial J^3 = S$, the sphere $S = S_m [0; 1]$. Somehow I find it easier for the purpose of this tour to think of "triangulations" by cubes (*squarings* or *cubings*) instead of 3-simplexes. I will think of cubic lattices dividing J^3 and their iterated subdivisions. A cubing of a 3-manifold, M^3, as a usual triangulation is a subdivision of M^3 by cubes, such that any two cubes intersect, at most, along one common face (meaning square face, edge, or vertex). The usual results about simplicial maps, simplicial approximation, and general position (see [25]) carry on for cubings without to much hassle.

In what follows, the observations, notes or claims of an heuristic nature will be set in several *heuristic criteria*. The mathematical results will be stated as propositions.

3.1 Double immersions

Given a homotopy 3-ball we associate with it an immersion of J^3 in itself, which is the identity in some *collar* of $S = \partial J^3$ - recall that a collar is a neighbourhood homeomorphic to $S \times [0, 1]$. By Corollary 9, there is a (surjective) map $f : J^3 \longrightarrow C$, which restricts to a homeomorphism between S and ∂C - using collars we can further assume it restricts to a homeomorphism between collars at the boundaries. We consider any immersion $\dot{g} : C \longrightarrow J^3$ which, on a collar of ∂C, is the inverse of the previous restriction of f. The double immersion is $h = g \circ f$.

$$J^3 \xrightarrow{f} C \xrightarrow{g} J^3 \quad \underrightarrow{h=g\circ f}$$

I assume C is endowed with a cubical structure and that the three functions are *simplicial* (*cubical*) relative to some cubing of J^3 (it helps to think of cubic lattices and their iterated subdivisions but the arguments are valid in general, for any cubings of J^3): each cube is embedded by h in such a way that all the edges are embedded as line segments; the square faces are embedded linearly with respect to the four 2-simplices formed by the four edges and the centre; in each cube, h is linear with respect to the 3-simplices formed by its centre and the (16) 2-simplices at the boundary. Any such f, g and their composite h can be put in *general position* by an arbitrarily small homotopy (that is, such that for each point the track of the homotopy has diameter less than any prescribed $\varepsilon > 0$): this means that the 1-dimensional complex (vertices and edges) is embedded and the square faces intersect transversely and away of the set of vertices.

In a couple of examples bellow, I will use two dimensional analogues to describe and illustrate a couple of points. Figures 3 and 3-a, next, illustrate a two-dimensional version of a general position map (of J^2 to J^2).

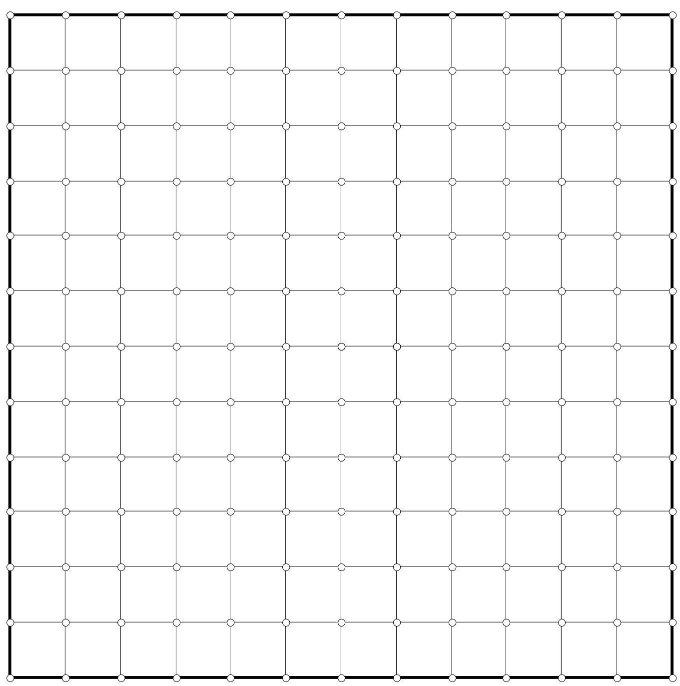

Figure 3

What I just said in the definition of general position applies but decreasing one dimension (only the o-complex is embedded). These pictures were done with the Geometer's Sketchpad; for those familiar with this program here is an alternative description of these mappings $J^{3(2)} \longrightarrow J^{3(2)}$.

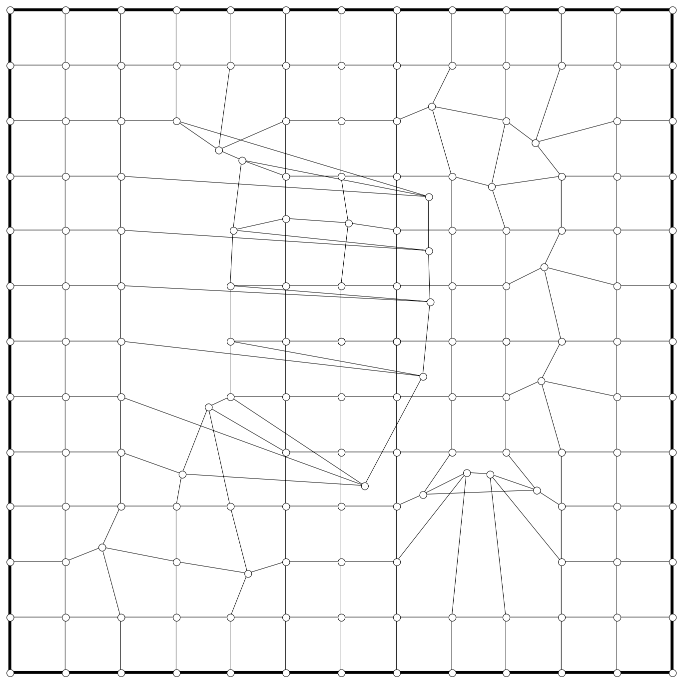

Figure 3-a

Star with a cubical lattice as in Figure 3 (all vertices and edges must be independent) and drag vertices to other positions; but you have to be careful to make sure that, in the end, all the cubes (squares) are embedded. You have to check that no edges attached to the vertex will intersect opposite faces in cubes (squares) where they belong: usually, dragging a vertex will produce intersections of this type that will have to be removed by dragging other vertices; it is not an easy process due to the existence and properties of the folding set that I will examine further down.

Criterion 22 *Think of the standard cube J^3 as an elastic medium, but rigid at the boundary, that reacts to stretching - but not to compression. This can be seen as an approximation process: given a cubing, consider the edges like elastic strings reacting to stretching according to Hook's law (reacting force proportional to displacement by an elasticity coefficient λ). Since we need to consider subdivisions of the given cubing, under such a subdivision we consider the coefficient λ adjusted proportionally to the total length increase so that the total reacting force doesn't change. A continuous elastic medium can be though of as a limit of this iterated process.*

Consider now a double immersion, $h : J^3 \longrightarrow J^3$ which is the identity in a collar at the boundary. This is cubical relative to some cubing of J^3 that will be stretched: think only of the 1-complex of vertices and edges. The following result can be proved; I leave it as an exercise.

Proposition 23 *There are resultant forces, unless $h : J^3 \longrightarrow J^3$ is the identity.*

The idea is then to deform h so that the total force keeps decreasing until we come to an embedding, and in such a away that, at all stages, h is still a double immersion; that is, I want that each deformation of h can factorize through corresponding deformations of f and g. In that situation, the ending f and g will have to be embeddings as well, giving homeomorphisms between C and J^3. As usually we try to built such a global deformation by successive local transformations.

Criterion 24 *Consider a point $b = h(a)$ where there is a resultant force of the stretching - you can think of a as a vertex of the cubing. Let $c = f(a) \in C$ and U, V and W be neighbourhoods of, respectively a, b and c such that $f(U) \subset W$, $g(W) \subset V$ - we can think of these neighbourhoods as the stars of the three vertices relative to a first subdivision of the cubings. We can adjust h near the vertex a, keeping everything fixed outside U, by moving b to a new position - following the natural effect of the resultant -where the stretched incident edges have their lengths decreased (some care must be taken when b is a certain type of singularity as I'll explain further down). Since we are dealing with cubical maps, we can redefine g inside W, accordingly, so as to keep the factorization $h = g \circ \dot{f}$. We managed to decrease the resultant force locally in U. But a problem emerges: when adjusting g in W, there is no guarantee we can decrease the lengths of all the edges incident in c; since $f^{-1}(W)$ may contain open neighbourhoods of vertices other than a, when composing f with the adjusted g we may be increasing the stretching at other points.*

The next example, a 2-dimensional analogue, shows this problem.

Example 25 *Refer to Figure 4. In J^2 consider the two regions (shaded) symmetric in relation to the central vertical. The map $h : J^2 \longrightarrow J^2$ is defined as $h = g \circ f$ where f and g are defined as follows: g is the identity outside the region at the right: the segment $\overline{AB}$ is sent to segment $\overline{ab}$ and we extend linearly to the whole polygon $xyzw$; to help visualize, a couple of segments and their images are shown: $\overline{pA}$ goes to $\overline{pa}$ and $\overline{qB}$ goes to $\overline{qb}$. The first map f is defined symmetrically except that the segment $\overline{cd}$ is now slightly inside the first region. h is the identity outside the two polygons and clearly their interiors are stretched by this map. In Figure 4-a the final representation is shown.*

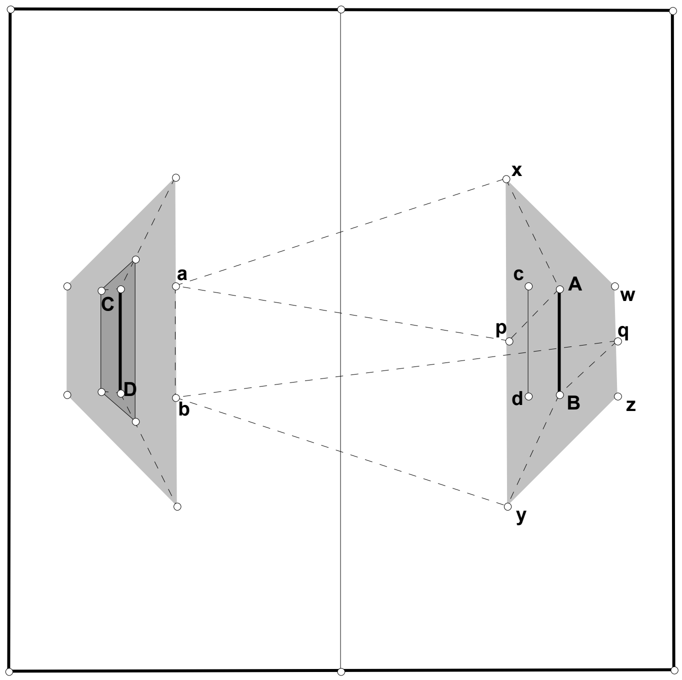

Figure 4

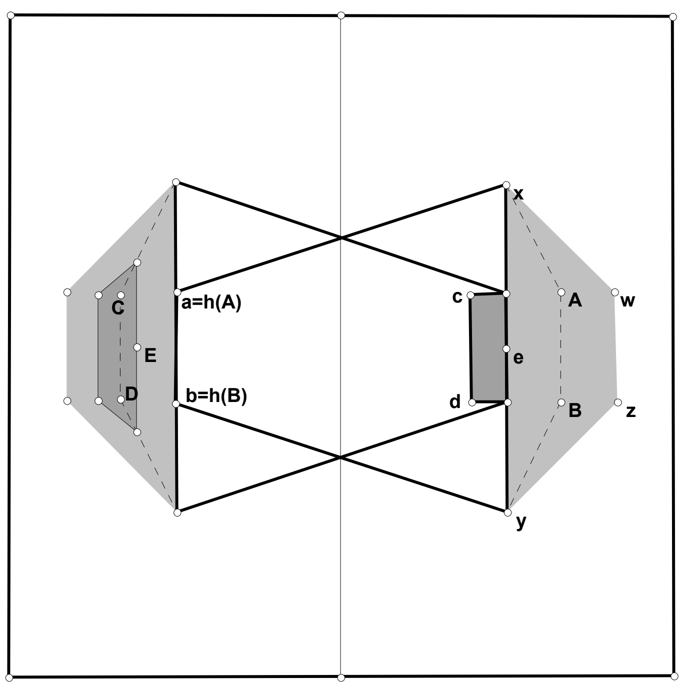

Figure 4-a

Note that in each polygon there are fold lines: on the right, xy which remains fixed and not stretched, and still xA , AB and yB (their images are shown thick); on the left there are more folding lines since the darker region inside is folded again to the left.

Now, if we look, for instance, at a neighbourhood of point E and its image at e as shown, where a resultant force exists, and we deform the map to reduce the stretching, moving $h(E)$ back to the left, we would have to drag the fold line xy along, therefore stretching it. If we don't drag this line along we could only factor $h = g \circ f$ making f discontinuous!

3.2 Singular set

I will now describe the types of singularities that a cubical map, h, may exhibit. A point is *regular* if it has a neighbourhood that is embedded by h. The points in the interior of each cube are of course regular. I will start with the most important type of singular points: the folding set. This is the most important type because, as we shall see further down, it is always present (unless the singular set is empty) and may be even be the only one.

3.2.1 The folding set

Let K be a cubing of J^3. A point, p, on the 2-dimensional complex, $K^{(2)}$, is a *fold-point* if it has a neighbourhood homeomorphic to $B^2 \times J$, with $p = (0,0)$ and $B^2 \times \{0\}$ embedded in $K^{(2)}$, such that $h(B^2 \times I)$ is embedded and for each point $(b,t) \in B^2 \times [-1,0]$ $h(b,t) = h(b,-t)$. It is clear from the definition that all points in $K^{(2)}$ interior to $B^2 \times \{0\}$ are also fold-points and therefore, by connectivity, if the interior of a square, or edge, intersects $B^2 \times \{0\}$ then the whole interior of that square, or edge is made up of fold-points. The folding set is a 2-dimensional stratum that is either closed, as a surface, or ends in edges or points where other types of singularities occur.

3.2.2 Cone and cusp lines

A point, q, on the 1-dimensional complex, $K^{(1)}$, is a point of a cone or cusp line if has a neighbourhood homeomorphic to $B^2 \times J$, with $q = (0,0)$ and $0 \times J$ embedded in $K^{(1)}$ and such that the restriction of h to that cylinder has the form $h = k \circ r$ with k an embedding and r given in cylindrical coordinates by $r(\rho, \theta, t) = (\rho, s(\theta), t)$, $s : S^1 \longrightarrow S^1$ with a finite number of singular points (points where the wrapping direction changes from clockwise to anticlockwise and vice-versa) If the degree of s is zero or one, then q is a cusp point; otherwise it is called a cone point; like the fold-set all the points in the interior of $0 \times J$ and in the interior of edges that intersect it have the same type as q. We call the lines of this 1-dimensional stratum *c-lines.* **Note:** *note that to each singular point of* s *there corresponds a sheet of the folding-set ending at* $0 \times J$.

3.2.3 Singular points

Note that in the 2-dimensional analogues we have, correspondingly, *fold-lines* and *cusp-points* and *cone-points*. In 3-dimensions we have furthermore *singular-points*, points where c-lines and fold set converge - you can visualize those singular points by thinking of their *link* in the first subdivision of K, a 2-sphere, where the transverse intersection with the fold-set and the c-lines gives a corresponding pattern of fold-lines and c-points; you then take the cone on that pattern with vertex the singular point.

The next example gives you an idea of how cone-lines can arise.

Example 26 *Refer to Figure 5 where the spheres for the metric* $m(x, y) = \max_{i=1,2,3} | x_i - y_i|$, *of radius* $1/4$, $1/2$ *and* $3/4$ *are represented in the interior of* J^2. *We represent, by shades, three regions: the outer region (light) the middle region (white) and the inner region (darker). The map* $h : J^2 \longrightarrow J^2$ *is defined on each region in the following way. Identify the upper part of the sphere of radius* $1/2$ *(that is* $\{x = (x_1, x_2) \in J^2 : m(0, x) = 1/2\ ,\ x_2 \geq 0\}$*) to the segment* $\overline{AB}$, *keeping* A *and* B *fixed, sending the two upper vertices to* G *and* H *and extending linearly to the three segments. Do the same for the lower part of that sphere. Finally extend to the outer region linearly: this sends this region onto* J^2. *The inner square,* $CDEF$ *is sent to the dashed square* $cdef$ *by a similarity composed with reflection on the* x_1 *axis. We finally extend linearly to the middle region which is thus mapped onto the square* $cdef$.

Exercise 27 *Check that the two points* A *and* B *which remained fixed are cone-points (of order 2).*

Identify the fold-set.

A 3-dimensional analogue can be obtained from this example by spooning around the x_2 axis - and then transposing from the cylinder, so obtained, to J^3 - the points A and B will give rise to a cone-line which is a circle. But we can easily describe 3-dimensional examples, more complicated ones, by using branched covers and *fibered-knots*.

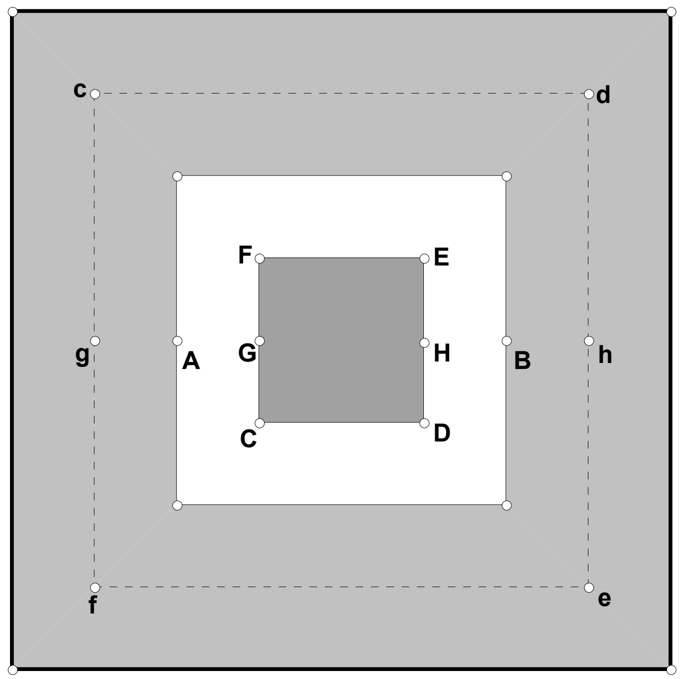

Figure 5

Example 28 *Consider in* $S^3 = R^3 \cup \infty$ *the unknot* K *contained in* J^3*, given by the intersection with the plane* $x_3 = 0$ *of the sphere of radius* $1/2$ *in the metric* m*. A regular neighbourhood of* K *in the interior of* J^3 *is a solid torus with* K *as central curve. The closure of its complement in* S^3 *is also a solid torus, giving the usual genus-1 Heegaard decomposition of . To this decomposition one associates a n-fold branched covering* $k : S^3 \longrightarrow S^3$ *with branching set* K*. Let* D *be the 3-ball which is the complement in* S^3 *of the open ball with centre* 0 *and radius* $3/4$ *in the metric* m*,* $B_m(0, 3/4)$*. You see that* $k^{-1}(D)$ *consists of* n *disjoint 3-balls in each one of which* k *is a homeomorphism to* D*: you can assume one of them is* D *itself and therefore the other ones are* $n-1$ *disjoint 3-balls contained in the interior of* J^3 *(in fact in* $B_m(0, 3/4)$*) and disjoint also from* K*. We redefine* k *in* J^3 *by sending each of those 3-balls to* $B_m(0, 3/4)$ *by composing with the inversion of* S^3 *on the 2-sphere* $\partial D = S_m(0, 3/4)$*.*

You can construct more complicated examples substituting K by other *fibered-knots* or links in the interior of J^3 (see [43]).

3.3 On the folding set

As I mention before the folding-set is the most important part of the singular set. Here are the reasons. I will denote by S the closure of the folding-set and refer to it simply by folding-set

Proposition 29 *Let $h : J^3 \longrightarrow J^3$ be a cubical map (relative to some cubing K) that is the identity on a collar at the boundary and is in general position. Then, either h is an embedding or the folding-set, S, is non empty.*
Proof. *If $S = \emptyset$ and h is the identity on a collar at the boundary, then in the complement of all the edges contained in the interior of J^3, h is a covering...*
■

Exercise 30 *Fulfil the details in the previous proof.*

Lets call a 2-dimensional subcomplex S of the cubing K, *non-collapsible*, if S doesn't have any free edges, that is, any edge of S is an edge of at least two squares of S.

Proposition 31 *In the conditions of the previous proposition, the folding-set S (if non empty) verifies:*
S is non-collapsible.
S separates J^3 and the outer component (the component that contains the boundary) doesn't have any singular points - and therefore each of its points have a singular pre-image.
Proof. *Recall the note at the end of 3.2.2 (with that you can even show that S is a 2-cycle in homology). For the final part adapt the argument with coverings, in the proof of the previous proposition.* ■

Exercise 32 *Fulfil the detail in the previous proof.*

Exercise 33 *Identify the folding-set in Figure 4-a.*

3.4 Corners

Let S be the folding-set of a cubical map $\dot{h} : J^3 \longrightarrow J^3$ in the conditions of the two previous propositions. Let T be the closure of the outer component of $J^3 - S$ and $R = \partial T$ its frontier. It is easy to see that this 2-complex, $R \subset S$, also separates (and is in fact, like S, a cycle in homology). Clearly R consists of all the outermost squares of S: those for which there is a path from the boundary of J^3 to one of their interior points that intersects S only at that end point. I will be interested in the *corners* of R. Let $U = \overline{J^3 - T}$ be the closure of all the internal components of $J^3 - S$; of course $\partial U = R$.

Definition 34 *A vertex v of R is a corner if there is an euclidian embedded disc with centre v which intersects U only at v (therefore the disc except v is all contained in the outer component of $J^3 - S$).*

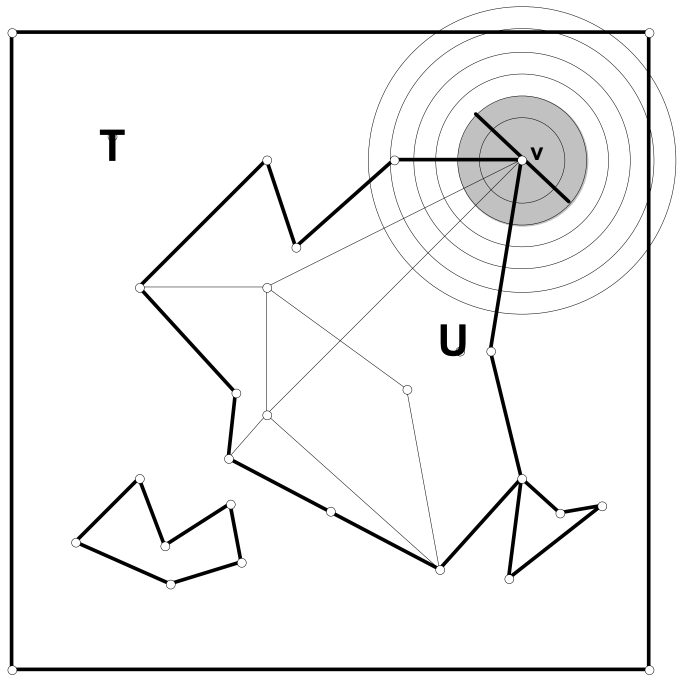

Figure 6

If $v \in R$ is a corner, there is a 3-ball with centre v, I will call a *corner-ball* (represented shaded in Figure 6 above) which contains a 2-disc through v (the thick diameter in Figure 6) dividing it in two halves, an outer one contained in T and the other I will call the "inner" one. We can consider a corner-ball as the star $St(v; K')$ of v in some subdivision of the cubing K; The corresponding $St(v; S)$ is contained in the inner half with the link, $Lk(v; S)$, contained in the corresponding inner hemisphere and away from the equator.

I can now state the next criterion. Recall that we have a double immersion $h = g \circ f$ which is the identity on a collar at the boundary, and S (with the associated T, R and U as before) denoting the folding-set.

Criterion 35 *Consider the situation where we have a resultant at a corner v. Recall the second criterion before, now with $v = b = h(a)$ and the neighbourhood V being a* corner-ball *at v. As before, let W be a neighbourhood of $c = f(a)$. If the resultant points towards the outer half of V, the situation is clear: it is due to stretching within T and therefore v is the end point of some cusp lines (a "cusp point"). In this case we apply first the next criterion to* *create a resultant pointing towards the inner half of V. In this situation, with a resultant pointing inwards, we bring v to a position inside the inner half, along some radius, and adjust all the edges accordingly, keeping $Lk(v;S)$ fixed within the inner hemisphere - if v is a cusp point (which is the situation when $g(W) \cap \overset{\circ}{T} \neq \emptyset$) this will strecth the outside: but you should keep in mind that we are following a resultant and therefore that stretching doesn't prevent the total force to decrease! The problem we faced in the previous criterion when we adjust g in W, accordingly, doesn't occur now: because the map h is one to one to the interior of T, open neighbourhoods of vertices other than a, in $f^{-1}(W)$, are sent to the inner half of V, that is to $V \cap U$, and thus have all their lengths decreasing in the process.*

Before going to the final criterion one needs to show that corners exist. I'll leave it as an exercise (at the level of a first topology course)

Exercise 36 *Show that corners do exist.*

3.5 Waves of (de)compression

Although (by the previous exercise) corners exist, the next simple example shows that not always do we have resultant forces there.

Example 37 *Consider in J^3 the three closed balls , for the metric m, of radius $1/4$, $1/2$ and $3/4$ and centre 0 (refer back to Figure 5 where the 2-dimensional analogue is pictured). The map $h : J^3 \longrightarrow J^3$ is defined as follows: it is the identity in the closed ball of radius $1/2$, $B\,[0, 1/2]$; The sphere of radius $3/4$, $S(0, 3/4)$ is sent, through radial projection from 0, to $S(0, 1/4)$. We extend naturally to J^3: on each radius from 0 to the boundary ∂J^3, the segment from $S(0, 3/4)$ to ∂J^3 is stretched, keeping the end point at ∂J^3 fixed (or even a small segment if we want h to be the identity on a collar at ∂J^3), and the segment between $S(0, 1/2)$ and $S(0, 3/4)$ is reversed into the segment between $S(0, 1/2)$ and $S(0, 1/4)$ (and the segment between 0 and $S(0, 1/2)$ is kept fixed since h is the identity there). It is clear that the folding-set is the union of the two spheres $S(0, 1/4)$ and $S(0, 1/2)$; we have exactly eight corners on this second sphere but no resultant at any of them (nearby there was just a folding with no stretching - even with some transversal shrinking).*

I now come to the last criterion which is the most heuristic one.

Criterion 38 *The general idea is to create a resultant at a corner, by a local stretching, and then compensate somewhere else by some compression. Refer* *back to Figure 6. Assume there is no resultant force at* v*. We can stretch* U *inside the inner half of the corner-ball (represented shaded), keeping* v *fixed. We can do it by composing* h *with a suitable homeomorphism of* J^3*. That homeomorphism would be a similarity, on a cone in the interior of the inner half and with centre* v*, that stretches a portion of a smaller ball (represented in Figure 6 by the circle inside the shaded ball) into the corresponding portion of the corner ball. In Figure 6 several balls, concentric with the corner-ball, are represented: the spacing between consecutive spheres equals the magnitude of the initial stretching. The homeomorphism we want to consider, stretches the inside ball, as explained, while compressing the two shells surrounding the corner-sphere into the outer one. Outside the first ball larger than the corner-ball, the homeomorphism will be the identity. We can now use this first compression to expand to the next shell, moving towards the outside, and then compress the following one, etc. like in a wave of compression-expansion; we do this until we reach some place where there are stretched parts that will have their lengths decreased by a compression crest. Since the changes in the stretching are continuous, we can adjust the process so as to make up for the initial increase, thus keeping the total force unchanged. We can then decrease the total force at the vertex* v *as discussed before...*

This is the end to the heuristic tour.

" ...
C'est même pas toi qui est en avance
C'est déjà moi qui suis en retard
j'arrive, bien sûr j'arrive
Mais ai-je jamais rien fait d'autre qu'arriver. "
(Jacques Brel)

- Disclaimer -

Heuristic Tour is a work of fiction. It was inspired by real mathematics facts - the struggle of an honest conjecture to find its identity - but any resemblance to living or imagined real theorems or proofs is accidental and unintentional.

(As of the 19th March 2003: the conjecture is still alive and on the run; despite continuous and unabated efforts from the *authorities*, the search for its identity continues)

And now a message from our sponsors:

> *Due to the high costs associated with this production, including the long delays to other projects and the heavy burden upon the cast and technical personnel, we are determined not to stage (or sponsor) any sequel in the foreseeable future.*

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