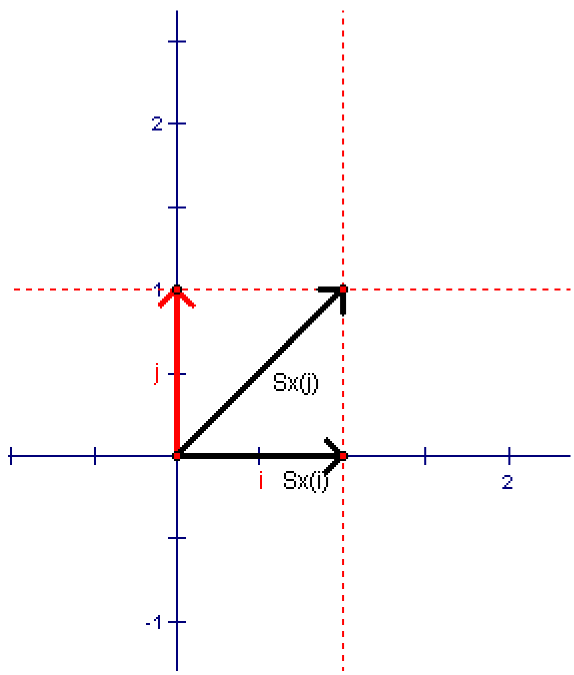
Exercício 55: Mostre que um shear, S_x , se pode escrever como um produto de similitudes e de strains.

Resolução do exercício:

Considere-se o caso particular de um shear de razão 1 aplicado à recta y=0:

$$A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}.$$



Seguindo o raciocínio de Martin, mostrar-se-á que A se pode escrever como produto de similitudes e de strains:

$$A=\begin{bmatrix}\frac{5-\sqrt{5}}{20} & \frac{5-3\sqrt{5}}{20}\\ \frac{-5+3\sqrt{5}}{20} & \frac{5-\sqrt{5}}{20}\end{bmatrix}\begin{bmatrix}\frac{3+\sqrt{5}}{2} & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}2 & 1+\sqrt{5}\\ -1-\sqrt{5} & 2\end{bmatrix}$$

Verificação:

$$\begin{bmatrix} \frac{5-\sqrt{5}}{20} & \frac{5-3\sqrt{5}}{20} \\ \frac{-5+3\sqrt{5}}{20} & \frac{5-\sqrt{5}}{20} \end{bmatrix} \begin{bmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1+\sqrt{5} \\ -1-\sqrt{5} & 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5-\sqrt{5}}{20} \times \frac{3+\sqrt{5}}{2} & \frac{5-3\sqrt{5}}{20} \\ \frac{-5+3\sqrt{5}}{20} \times \frac{3+\sqrt{5}}{2} & \frac{5-\sqrt{5}}{20} \end{bmatrix} \begin{bmatrix} 2 & 1+\sqrt{5} \\ -1-\sqrt{5} & 2 \end{bmatrix}$$

$$= \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

onde:

$$\begin{aligned} A &= \frac{5-\sqrt{5}}{20} \times \frac{3+\sqrt{5}}{2} \times 2 + \frac{5-3\sqrt{5}}{20} \times \left(-1-\sqrt{5}\right) \\ &= \frac{15+5\sqrt{5}-3\sqrt{5}-5}{20} + \frac{-5-5\sqrt{5}+3\sqrt{5}+15}{20} \\ &= \frac{20}{20} = 1 \end{aligned}$$

$$\begin{aligned} B &= \frac{5-\sqrt{5}}{20} \times \frac{3+\sqrt{5}}{2} \times \left(1+\sqrt{5}\right) + \frac{5-3\sqrt{5}}{20} \times 2 \\ &= \frac{15+5\sqrt{5}-3\sqrt{5}-5}{40} \times \left(1+\sqrt{5}\right) + \frac{10-6\sqrt{5}}{20} \\ &= \frac{10+2\sqrt{5}}{40} \times \left(1+\sqrt{5}\right) + \frac{10-6\sqrt{5}}{20} \\ &= \frac{5+\sqrt{5}}{20} \times \left(1+\sqrt{5}\right) + \frac{10-6\sqrt{5}}{20} \\ &= \frac{5+5\sqrt{5}+\sqrt{5}+5+10-6\sqrt{5}}{20} \\ &= \frac{20}{20} = 1 \end{aligned}$$

$$C = \frac{-5+3\sqrt{5}}{20} \times \frac{3+\sqrt{5}}{2} \times 2 + \frac{5-\sqrt{5}}{20} \times \left(-1-\sqrt{5}\right)$$
$$= \frac{-15-5\sqrt{5}+9\sqrt{5}+15}{20} + \frac{-5-5\sqrt{5}+\sqrt{5}+5}{20}$$
$$= \frac{4\sqrt{5}}{20} - \frac{4\sqrt{5}}{20} = 0$$

$$D = \frac{-5+3\sqrt{5}}{20} \times \frac{3+\sqrt{5}}{2} \times \left(1+\sqrt{5}\right) + \frac{5-\sqrt{5}}{20} \times 2$$
$$= \frac{-15-5\sqrt{5}+9\sqrt{5}+15}{40} \times \left(1+\sqrt{5}\right) + \frac{10-2\sqrt{5}}{20}$$
$$= \frac{2\sqrt{5}}{20} \times \left(1+\sqrt{5}\right) + \frac{10-2\sqrt{5}}{20}$$
$$= \frac{2\sqrt{5}+10+10-2\sqrt{5}}{20}$$
$$= \frac{20}{20} = 1$$

Logo,

$$\begin{bmatrix} \frac{5-\sqrt{5}}{20} & \frac{5-3\sqrt{5}}{20} \\ \frac{-5+3\sqrt{5}}{20} & \frac{5-\sqrt{5}}{20} \end{bmatrix} \begin{bmatrix} \frac{3+\sqrt{5}}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1+\sqrt{5} \\ -1-\sqrt{5} & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = A\,.$$

A similitude acima descrita pela matriz:

$$Sim = \begin{bmatrix} 2 & 1+\sqrt{5} \\ -1-\sqrt{5} & 2 \end{bmatrix}$$

$$= R(0,\beta)D(0,l)\,,$$

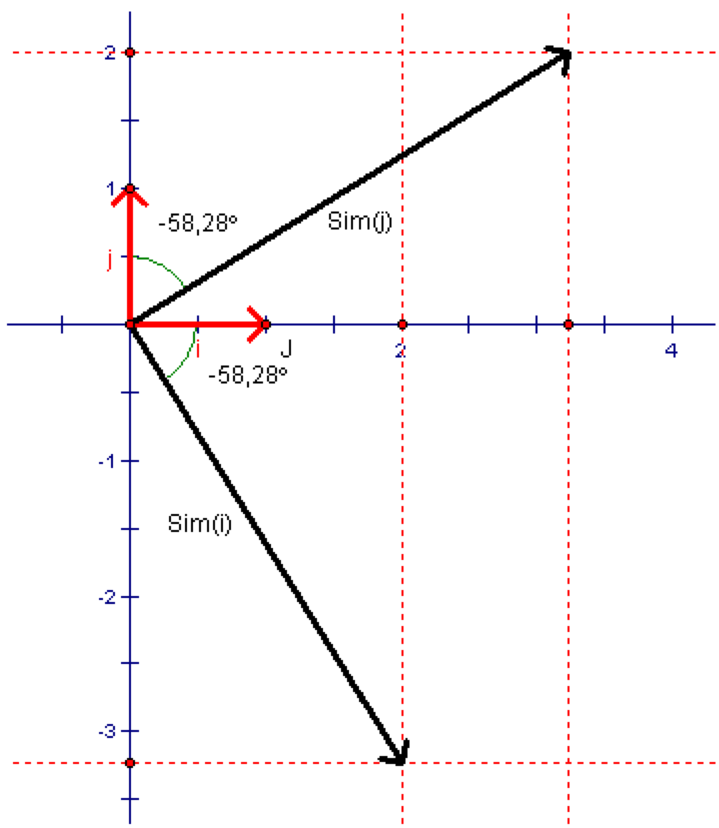
é uma rotação esticada, logo pode ser escrita da seguinte forma:

$$Sim' = R(0,\beta)D(0,l) = \begin{bmatrix} l\cos\beta & -l\sin\beta \\ l\sin\beta & l\cos\beta \end{bmatrix},$$

onde $\beta = \cos^{-1}\left(\dfrac{2}{\sqrt{10+2\sqrt{5}}}\right) \approx -58{,}28^0$ e $l = \sqrt{10+2\sqrt{5}}$.

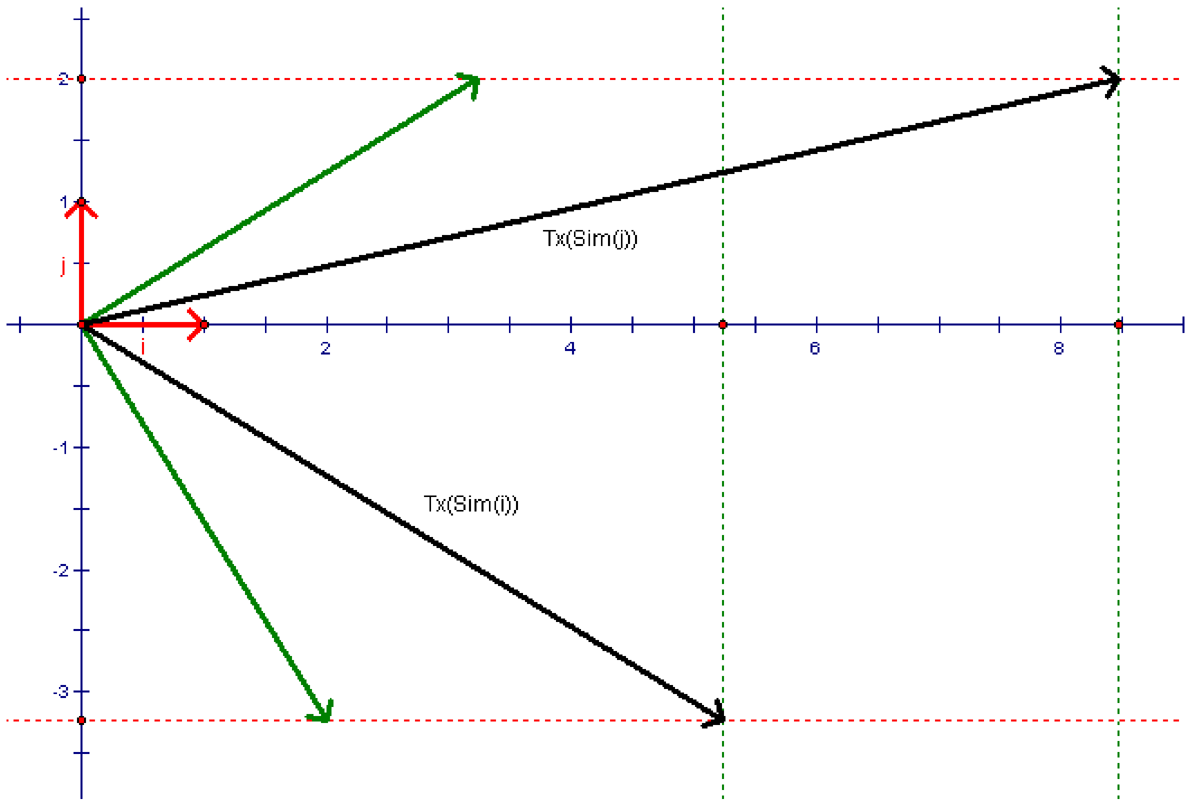
Ou seja,

$$Sim = \begin{bmatrix} \dfrac{2}{\sqrt{10+2\sqrt{5}}} & \dfrac{1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}} \\ \dfrac{-1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}} & \dfrac{2}{\sqrt{10+2\sqrt{5}}} \end{bmatrix} \begin{bmatrix} \sqrt{10+2\sqrt{5}} & 0 \\ 0 & \sqrt{10+2\sqrt{5}} \end{bmatrix}.$$



Posteriormente temos o strain:

$$\hat{T}_x = \begin{bmatrix} \frac{3-\sqrt{5}}{2} & 0 \\ 0 & 1 \end{bmatrix}.$$



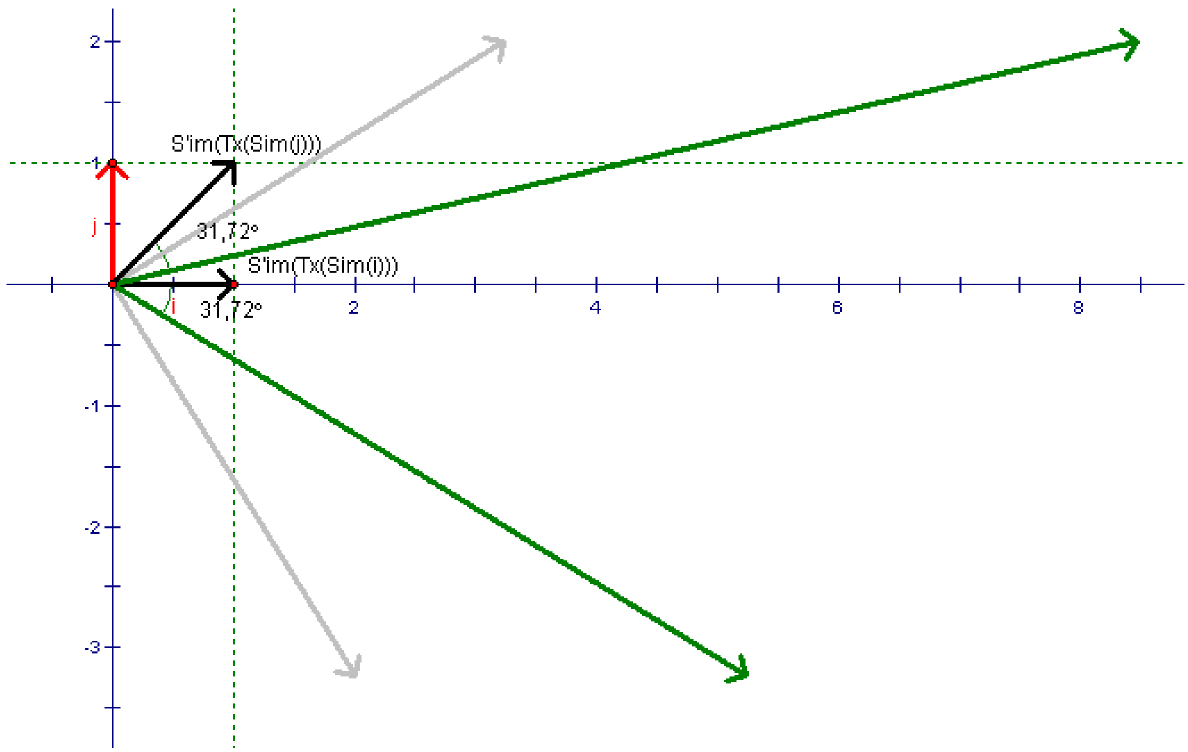
E, por fim, temos a similitude que, analogamente ao que foi feito acima, pode ser escrita como:

$$Sim' = \begin{bmatrix} \frac{5-\sqrt{5}}{20} & \frac{5-3\sqrt{5}}{20} \\ \frac{-5+3\sqrt{5}}{20} & \frac{5-\sqrt{5}}{20} \end{bmatrix} = R(0,\alpha)D(0,k),$$

onde $\alpha = \cos^{-1}\left(\frac{\sqrt{5}-1}{2\sqrt{5-2\sqrt{5}}}\right) \approx 31{,}72^0$ e $k = \sqrt{\frac{5-2\sqrt{5}}{20}}$.

Ou seja,

$$Sim' = \begin{bmatrix} \dfrac{\sqrt{5}-1}{2\sqrt{5-2\sqrt{5}}} & \dfrac{\sqrt{5}-3}{2\sqrt{5-2\sqrt{5}}} \\ \dfrac{-\sqrt{5}+3}{2\sqrt{5-2\sqrt{5}}} & \dfrac{\sqrt{5}-1}{2\sqrt{5-2\sqrt{5}}} \end{bmatrix} \begin{bmatrix} \sqrt{\dfrac{5-2\sqrt{5}}{20}} & 0 \\ 0 & \sqrt{\dfrac{5-2\sqrt{5}}{20}} \end{bmatrix}.$$



Para o caso geral de um shear de razão k aplicado à recta y=0:

$$S_x = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad .$$

$$\begin{aligned} \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{k} & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} k & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dfrac{1}{k} & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad . \end{aligned}$$

E se repararmos:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = Sim^1 \hat{T}_x \, Sim^1 .$$

Logo, temos que um shear pode ser escrito como produto de strains e similitudes.

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TURMA P4