Exploring Born-Infeld electrodynamics using plasmas

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Abraham-Lorentz-Dirac equation

What is the right equation of motion for a classical point charge?

[Dirac, Proc. Roy. Soc. (1938)]

\[ m\ddot{A}_a = qF_{ab}^{\text{ext}}\dot{x}^b + \frac{1}{6\pi} q^2 (\eta_{ab} + \dot{x}_a \dot{x}_b) \frac{DA^b}{d\tau} \]  

(1)

N.B. \( \varepsilon_0 = \mu_0 = c = 1 \)

Problems:

- Infinite negative bare mass to compensate for infinite self-energy of the electron
- Abraham-Lorentz-Dirac equation is *third order* in time derivatives of the electron’s worldline
  - Acceleration in regions where there is no external field
  - Runaway solutions

Constitutive equations

\[ D = \hat{D}[E, B], \quad H = \hat{H}[E, B] \]

In particular,

- in the vacuum in \textit{classical} Maxwell electrodynamics:
  \[ D = E, \quad H = B, \quad (2) \]
  or \[ G = F \quad (3) \]

- non-linear when QED vacuum effects are included (Euler-Heisenberg electrodynamics)
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- Could electrodynamics be fundamentally non-linear?
Vacuum Born-Infeld equations

\[ dF = 0, \quad d \star G = 0, \]  

\[ G = \frac{1}{\sqrt{1 - \kappa^2 X - \kappa^4 Y^2/4}} \left( F - \frac{\kappa^2 Y}{2} \star F \right) \]  

\[ X = \star (F \wedge \star F), \quad Y = \star (F \wedge F) \]  

\( \kappa \) is a new constant of nature


- The field of a point charge at rest is finite, and the charge’s self energy is finite
- Appears in low-energy string field theory [Fradkin et al, PLB (1985)]
  - Considerable interest from a string theory perspective
    [e.g. Gibbons et al]
- Investigations in waveguides and in background magnetic fields [e.g. Ferraro et al, PRL (2007); Tucker et al, EPL (2010)]
High intensity lasers

Two examples:

- **VULCAN (UK)**: $10^{22} \text{ W/cm}^2$ (2016 upgrade to 10PW, intensity $10^{23} \text{ W/cm}^2$)

- **ELI (2012, located in Hungary, Czech Republic, Romania, +1 EU)**: intensity $\sim 10^{25} \text{ W/cm}^2$ or field strength $10^{16} \text{ V/m}$

Motivation:

- Numerous applications: fast ignition, compact electron acceleration, material science, . . .

- Opportunity to test QED vacuum phenomena in a controllable environment (Schwinger limit $10^{18} \text{ V/m}$)
Plasma waves

A sufficiently intense and short laser pulse with sufficiently high peak frequency propagating through a plasma creates a wave (wake behind the pulse)

- The wake contains a large longitudinal electric field ($\sim 10^{11} \text{ V/m}$) $\Rightarrow$ Laser Wakefield Accelerator.

[Tajima et al, PRL (1979);
Mangles et al, Nature (2004)]

Image courtesy of W. Mori
Electric waves in a cold Born-Infeld plasma

Maxwell’s equations

\[ dF = 0, \quad d \star G = -q(\star \tilde{N} - \star \tilde{N}_{\text{ion}}) \]

where

\[ G = \frac{1}{\sqrt{1 - \kappa^2 X - \kappa^4 Y^2 / 4}} \left( F - \frac{\kappa^2 Y}{2} \star F \right) \]

Employing an action principle \( \implies \) Lorentz force equation

\[ \nabla_V \tilde{V} = \frac{q}{m} \iota_V F, \quad g(V, V) = -1 \]
Electric waves in a cold Born-Infeld plasma

- Metric tensor $g$ in ion rest frame (i.e. laboratory frame)

\[ g = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3 \]

- Seek solutions in $\zeta = x^3 - v x^0$ with $F = E(\zeta) \, dx^0 \wedge dx^3$
  (numerous simplifying assumptions...)

- Adapted basis in the “wave frame”

\[ e^1 = v \, dx^3 - dx^0, \quad e^2 = dx^3 - v \, dx^0 = d\zeta \]

and for electrons moving slower than wave

\[ \tilde{V} = \mu e^1 - (\mu^2 - \gamma^2)^{1/2} e^2 \]

where $\mu = \mu(\zeta) > 0$
Electric waves in a cold Maxwell plasma

For example, when $\kappa = 0$

$$\frac{1}{\gamma^2} \frac{d^2 \mu}{d\zeta^2} = \frac{q^2}{m} n_{i\text{on}} \gamma^2 \left( \frac{v \mu}{\sqrt{\mu^2 - \gamma^2}} - 1 \right)$$

and $E = \frac{m}{(q\gamma^2)} \frac{d\mu}{d\zeta}$, $\gamma = 1/\sqrt{1 - v^2}$, e.g. for $v = 0.8$
Cold Maxwell plasma wave-breaking limit

Maximum amplitude oscillation for $\kappa = 0$:

$$E_0^{\max} = \frac{m\omega_p}{|q|} \sqrt{2(\gamma - 1)}$$

Oscillation frequency for $\gamma \gg 1$

$$\omega^0 \approx \frac{\pi}{2\sqrt{2\gamma}} \omega_p$$

[Akheizer et al, Sov. Phys. JETP 3 (1956)]
Cold Born-Infeld plasma wave-breaking limit

\[ E_{\text{max}}^{\text{BI}} = \frac{1}{\kappa} \sqrt{1 - \left[ \kappa^2 \left( \frac{E_{\text{max}}^0}{2} \right)^2 / 2 + 1 \right]^{-2}} \]

Note \( \lim_{v \to 1} E_{\text{max}}^{\text{BI}} = 1/\kappa \)

\[ \omega^{\text{BI}} \approx \omega^0 \left[ 1 - \left( \kappa m \omega_p c / 2q \right)^2 \gamma \right] \]

Cold plasma on $\mathcal{T}\mathcal{M}$

$(x, \dot{x})$

$\dot{x}^\alpha = V^\alpha(x)$
Relativistic Vlasov equation

- Not in thermal equilibrium
- Vlasov equation for 1-particle distribution $f(x, \dot{x}) \in \Lambda^0(TM)$

$$Lf = 0$$

$$L = \dot{x}^a \left( \frac{\partial}{\partial x^a} - \Gamma^b_{ac} \dot{x}^c \frac{\partial}{\partial \dot{x}^b} - \frac{q}{m} F^b_\mathbf{V} a \frac{\partial}{\partial \dot{x}^b} \right)$$

- Electron number 4-current

$$N^a[f](x) = \int_{\mathcal{E}_x} \dot{x}^a f(x, \dot{x}) \nu_x$$

where $\mathcal{E}_x = \{ \dot{x}^a \in \mathbb{R}^4 \mid g^{\mathbf{V}}_{ab}(x, \dot{x}) \dot{x}^a \dot{x}^b = -1, \; \dot{x}^0 > 0 \}$ and $\nu_x$ is a measure on $\mathcal{E}_x$
Vlasov equation for a jump

\[ \mathcal{E} = \{(x, \dot{x}) \in T\mathcal{M} \mid g_{ab}^V(x, \dot{x})\dot{x}^a\dot{x}^b = -1, \dot{x}^0 > 0\} \]

\[ Lf = 0 \implies d(f\omega) \simeq 0 \] (pulled back to unit hyperboloid \( \mathcal{E} \))

where \( \omega \in \Lambda^6(T\mathcal{M}) \), \( d\omega \simeq 0 \) and \( \iota_L\omega = 0 \)

\[ d(f\omega) \simeq 0 \implies \int_{\partial B} f\omega = 0 \]

\[ \implies d\lambda \wedge [f\omega] = 0 \]

where \( \lambda = 0 \) is the interface between 1 and 2
Waterbag distributions
Waterbag distributions

1-particle distribution $f$ described by

$$\Sigma : \mathcal{M} \times S^2 \rightarrow T\mathcal{M},$$

$$(x, \xi) \mapsto (x, \dot{x} = V_\xi(x))$$

where $d\lambda \wedge \omega = 0$ is satisfied by

$$\nabla_{V_\xi} \tilde{V}_\xi = \frac{q}{m} \iota_{V_\xi} F, \quad g(V_\xi, V_\xi) = -1$$
For electrons moving slower than the wave

\[ \vec{V}_\xi = [\mu(\zeta) + A(\xi^1)] e^1 + \psi(\xi^1, \zeta) e^2 \]

\[ + R \sin(\xi^1) \cos(\xi^2) \, dx^1 + R \sin(\xi^1) \sin(\xi^2) \, dx^2 \]

with

\[ \psi = -\sqrt{[\mu + A(\xi^1)]^2 - \gamma^2[1 + R^2 \sin^2(\xi^1)]} \]

for \(0 < \xi^1 < \pi\), \(0 \leq \xi^2 < 2\pi\) and constant \(R\)
Electric waves in a warm Maxwell plasma

\[
\frac{1}{\gamma^2} \frac{d^2 \mu}{d\zeta^2} = -\frac{q^2}{m} n_{\text{ion}} \gamma^2 - \frac{q^2}{m} 2\pi R^2 \alpha \int_0^\pi \left( [\mu + A(\xi^1)]^2 
- \gamma^2 [1 + R^2 \sin^2(\xi^1)] \right)^{1/2} \sin(\xi^1) \cos(\xi^1) d\xi^1
\]

with

\[
2\pi R^2 \int_0^\pi A(\xi^1) \sin(\xi^1) \cos(\xi^1) d\xi^1 = -\frac{n_{\text{ion}} \gamma^2 v}{\alpha}
\]

and \( E = m/(q\gamma^2) \frac{d\mu}{d\zeta} \)
Maximum electric field

For width $\ll 1 \ll$ height, and $v \to 1$

$$E_{\text{max}}^2 \approx \frac{m^2 \omega_p^2}{q^2} \sqrt{\frac{9mc^2}{20k_B T_{||\text{eq}}}}$$

where $T_{||\text{eq}}$ is an effective longitudinal “temperature”.
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▶ Does the wave actually break? Only tip reaches $v$
Summary

- Understanding radiation reaction is important for contemporary and future acceleration concepts.
  - Abraham-Lorentz-Dirac equation is pathological
- Could Born-Infeld electrodynamics be the answer?
- Test using forthcoming laser facilities and astrophysical systems?
  - Explore large-amplitude waves in Born-Infeld plasmas
    - Cold plasmas
      
    - Problems remain to be ironed out in warm plasmas
      

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