Exploring Born-Infeld electrodynamics using plasmas

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Abraham-Lorentz-Dirac equation

What is the right equation of motion for a classical point charge?

[Dirac, Proc. Roy. Soc. (1938)]

$$m\mathcal{A}_a = qF^{\text{ext}}_{ab}\dot{x}^b + \frac{1}{6\pi}q^2(\eta_{ab} + \dot{x}_a\dot{x}_b)\frac{D\mathcal{A}^b}{d\tau} \qquad (1)$$

N.B. $\varepsilon_0 = \mu_0 = c = 1$

Problems :

- Infinite negative bare mass to compensate for infinite self-energy of the electron
- Abraham-Lorentz-Dirac equation is *third order* in time derivatives of the electron's worldline
  - Acceleration in regions where there is no external field
  - Runaway solutions

[Recent derivation of ALD equation : DAB, J Gratus, RW Tucker, Ann. Phys. 322 3, 599 (2007)]

Constitutive equations

$$\mathbf{D} = \hat{\mathbf{D}}[\mathbf{E}, \mathbf{B}], \quad \mathbf{H} = \hat{\mathbf{H}}[\mathbf{E}, \mathbf{B}]$$

In particular,

- in the vacuum in *classical* Maxwell electrodynamics :

$$\mathbf{D} = \mathbf{E}, \quad \mathbf{H} = \mathbf{B}, \tag{2}$$

$$\text{or} \quad G = F \tag{3}$$

- non-linear when QED vacuum effects are included (Euler-Heisenberg electrodynamics)

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- Could electrodynamics be *fundamentally* non-linear?

Vacuum Born-Infeld equations

$$dF = 0, \quad d \star G = 0, \tag{4}$$

$$G = \frac{1}{\sqrt{1 - \kappa^2 X - \kappa^4 Y^2/4}} \left(F - \frac{\kappa^2 Y}{2} \star F \right) \tag{5}$$

$$X = \star(F \wedge \star F), \qquad Y = \star(F \wedge F) \tag{6}$$

κ is a new constant of nature

[Born et al Proc. Roy. Soc. (1934)]

- The field of a point charge at rest is finite, and the charge's self energy is finite
- Appears in low-energy string field theory [Fradkin et al, PLB (1985)]
  - Considerable interest from a string theory perspective
    [e.g. Gibbons et al]
- Investigations in waveguides and in background magnetic fields [e.g. Ferraro et al, PRL (2007); Tucker et al, EPL (2010)]

High intensity lasers

Two examples :

- VULCAN (UK) : $10^{22}\,\text{W/cm}^2$ (2016 upgrade to 10PW, intensity $10^{23}\,\text{W/cm}^2$)
- ELI (2012, located in Hungary, Czech Republic, Romania, +1 EU) : intensity $\sim 10^{25}\,\text{W/cm}^2$ or field strength $10^{16}\,\text{V/m}$

Motivation :

- Numerous applications : fast ignition, compact electron acceleration, material science,...
- Opportunity to test QED vacuum phenomena in a controllable environment (Schwinger limit $10^{18}\,\text{V/m}$)

Plasma waves

A sufficiently intense and short laser pulse with sufficiently high peak frequency propagating through a plasma creates a wave (wake behind the pulse)

- The wake contains a large longitudinal electric field ($\sim 10^{11}$ V/m) $\implies$ Laser Wakefield Accelerator.

[Tajima et al, PRL (1979);

Mangles et al, Nature (2004)]

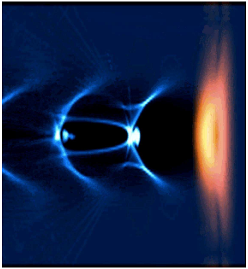


Image courtesy of W. Mori

Electric waves in a cold Born-Infeld plasma

- Maxwell's equations

$$dF = 0, \quad d \star G = -q(\star\widetilde{N} - \star\widetilde{N_{\text{ion}}})$$

where

$$G = \frac{1}{\sqrt{1 - \kappa^2 X - \kappa^4 Y^2/4}}\left(F - \frac{\kappa^2 Y}{2} \star F\right)$$

- Employing an action principle $\Longrightarrow$ Lorentz force equation

$$\nabla_V \widetilde{V} = \frac{q}{m}\iota_V F, \qquad g(V,V) = -1$$

Electric waves in a cold Born-Infeld plasma

- Metric tensor g in ion rest frame (i.e. laboratory frame)

$$g = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3$$

- Seek solutions in $\zeta = x^3 - v\,x^0$ with $F = E(\zeta)\,dx^0 \wedge dx^3$ (numerous simplifying assumptions...)
- Adapted basis in the "wave frame"

$$e^1 = v\,dx^3 - dx^0, \quad e^2 = dx^3 - v\,dx^0 = d\zeta$$

and for electrons moving slower than wave

$$\widetilde{V} = \mu e^1 - (\mu^2 - \gamma^2)^{1/2} e^2$$

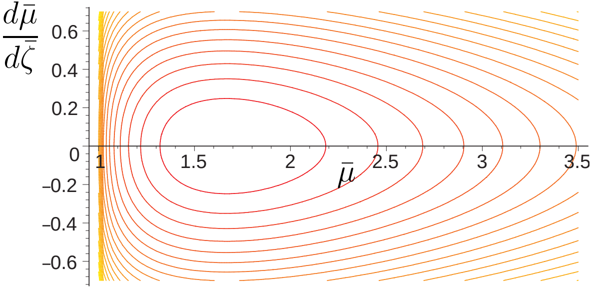
where $\mu = \mu(\zeta) > 0$

Electric waves in a cold Maxwell plasma

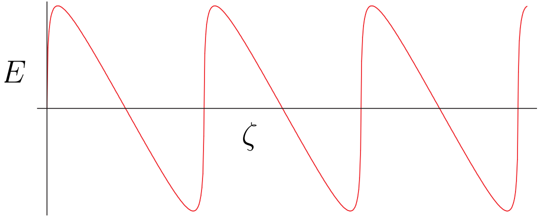
For example, when $\kappa = 0$

$$\frac{1}{\gamma^2}\frac{d^2\mu}{d\zeta^2} = \frac{q^2}{m} n_{\text{ion}} \gamma^2 \left(\frac{v\,\mu}{\sqrt{\mu^2 - \gamma^2}} - 1 \right)$$

and $E = m/(q\gamma^2) d\mu/d\zeta$, $\gamma = 1/\sqrt{1 - v^2}$, e.g. for $v = 0.8$



Cold Maxwell plasma wave-breaking limit



Maximum amplitude oscillation for $\kappa = 0$:

$$E^0_{\mathrm{max}} = \frac{m\omega_p}{|q|}\sqrt{2(\gamma - 1)}$$

Oscillation frequency for $\gamma \gg 1$

$$\omega^0 \approx \frac{\pi}{2\sqrt{2\gamma}}\omega_p$$

[Akheizer et al, Sov. Phys. JETP 3 (1956)]

Cold Born-Infeld plasma wave-breaking limit

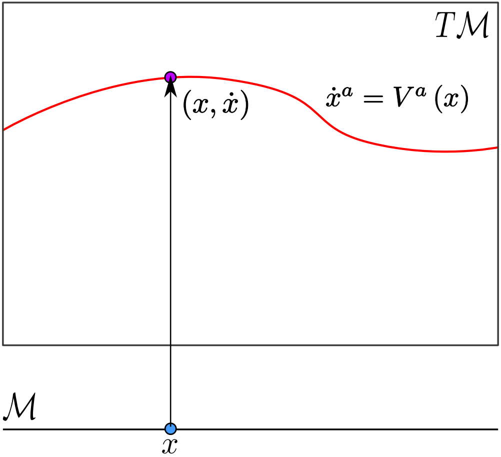
$$\boxed{E^{\mathrm{BI}}_{\mathrm{max}} = \frac{1}{\kappa}\sqrt{1 - [\kappa^2 (E^0_{\mathrm{max}})^2/2 + 1]^{-2}}}$$

Note $\lim_{v\to 1} E^{\mathrm{BI}}_{\mathrm{max}} = 1/\kappa$

$$\boxed{\omega^{\mathrm{BI}} \approx \omega^0 \left[1 - \left(\frac{\kappa m \omega_p c}{2q}\right)^2 \gamma\right]}$$

[DAB, RMGM Trines, TJ Walton, H Wen, arXiv:1006.2246 [physics.plasm-ph] (submitted)]

Cold plasma on $T\mathcal{M}$



Relativistic Vlasov equation

- Not in thermal equilibrium
- Vlasov equation for 1-particle distribution $f(x,\dot{x}) \in \Lambda^0(T\mathcal{M})$

$$Lf = 0$$
$$L = \dot{x}^a\left(\frac{\partial}{\partial x^a} - \Gamma^{b\mathbf{V}}{}_{ac}\dot{x}^c\frac{\partial}{\partial \dot{x}^b} - \frac{q}{m}F^{b\mathbf{V}}{}_a\frac{\partial}{\partial \dot{x}^b}\right)$$

- Electron number 4-current

$$N^a[f](x) = \int_{\mathcal{E}_x} \dot{x}^a f(x,\dot{x})\,\nu_x$$

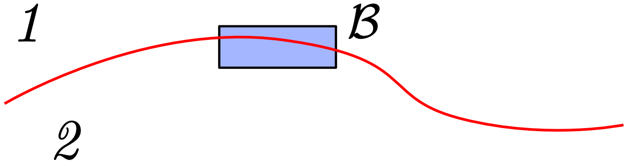
where $\mathcal{E}_x = \{\dot{x}^a \in \mathbb{R}^4 \,|\, g^{\mathbf{V}}_{ab}(x,\dot{x})\dot{x}^a\dot{x}^b = -1,\ \dot{x}^0 > 0\}$ and ν_x is a measure on $\mathcal{E}_x$

Vlasov equation for a jump

$$\mathcal{E} = \{(x,\dot{x}) \in T\mathcal{M} \,|\, g^{\mathbf{V}}_{ab}(x,\dot{x})\dot{x}^a\dot{x}^b = -1,\ \dot{x}^0 > 0\}$$

$$Lf = 0 \implies d(f\omega) \simeq 0 \text{ (pulled back to unit hyperboloid } \mathcal{E})$$

where $\omega \in \Lambda^6(T\mathcal{M})$, $d\omega \simeq 0$ and $\iota_L\omega = 0$

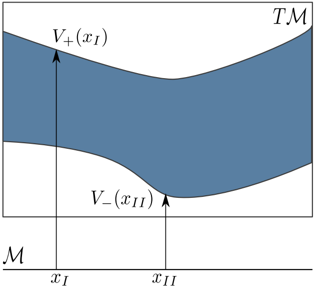


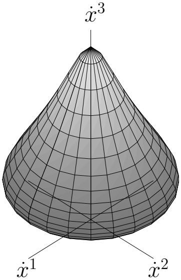
$$d(f\omega) \simeq 0 \implies \int_{\partial B} f\omega = 0$$

$$\implies d\lambda \wedge [f\omega] = 0$$

where $\lambda = 0$ is the interface between 1 and 2

Waterbag distributions





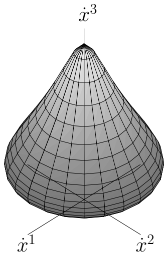
Waterbag distributions

1-particle distribution f described by

$$\Sigma : \mathcal{M} \times S^2 \to T\mathcal{M},$$
$$(x, \xi) \mapsto (x, \dot{x} = V_\xi(x))$$

where $d\lambda \wedge \omega = 0$ is satisfied by

$$\nabla_{V_\xi} \widetilde{V_\xi} = \frac{q}{m} \iota_{V_\xi} F, \qquad g(V_\xi, V_\xi) = -1$$



Axisymmetric waterbags

- For electrons moving slower than the wave

$$\widetilde{V_\xi} = [\mu(\zeta) + A(\xi^1)]\, e^1 + \psi(\xi^1,\zeta)\, e^2 + R\sin(\xi^1)\cos(\xi^2) dx^1 + R\sin(\xi^1)\sin(\xi^2) dx^2$$

with

$$\psi = -\sqrt{[\mu + A(\xi^1)]^2 - \gamma^2[1 + R^2\sin^2(\xi^1)]}$$

for $0 < \xi^1 < \pi$, $0 \le \xi^2 < 2\pi$ and constant R

Electric waves in a warm Maxwell plasma

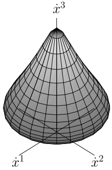
$$\frac{1}{\gamma^2}\frac{d^2\mu}{d\zeta^2} = -\frac{q^2}{m}n_{\mathrm{ion}}\gamma^2 - \frac{q^2}{m}2\pi R^2\alpha\int_0^\pi \Big([\mu + A(\xi^1)]^2 - \gamma^2[1 + R^2\sin^2(\xi^1)]\Big)^{1/2}\sin(\xi^1)\cos(\xi^1)\,d\xi^1$$

with

$$2\pi R^2\int_0^\pi A(\xi^1)\sin(\xi^1)\cos(\xi^1)\,d\xi^1 = -\frac{n_{\mathrm{ion}}\gamma^2 v}{\alpha}$$

and $E = m/(q\gamma^2)d\mu/d\zeta$

Maximum electric field

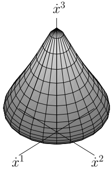


For width $\ll 1 \ll$ height, and $v \to 1$

$$\boxed{E_{\max}^2 \approx \frac{m^2 \omega_p^2}{q^2} \sqrt{\frac{9mc^2}{20 k_B T_{\|\mathrm{eq}}}}}$$

where $T_{\|\mathrm{eq}}$ is an effective longitudinal "temperature".

Maximum electric field



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- Does the wave actually break? Only tip reaches v

Summary

- Understanding radiation reaction is important for contemporary and future acceleration concepts.
  - Abraham-Lorentz-Dirac equation is pathological
- Could Born-Infeld electrodynamics be the answer?
- Test using forthcoming laser facilities and astrophysical systems?
  - Explore large-amplitude waves in Born-Infeld plasmas
    - Cold plasmas
      [DAB, RMGM Trines, TJ Walton, H Wen, arXiv:1006.2246 [physics.plasm-ph] (submitted)]
    - Problems remain to be ironed out in warm plasmas *[DAB, A Noble, arXiv:0908.4498 [physics.plasm-ph] J. Phys. A Math. Theor. 43 075502 (2010)]*

We thank Robin Tucker for useful discussions.