Cohomology and Stiefel-Whitney Classes of Flat Manifolds

joint work with Roberto Miatello and Juan Pablo Rossetti, Cordoba, Argentina

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Sergio Console
Dipartimento di Matematica
Università di Torino

Compact flat manifolds & Bieberbach groups
- Flat manifolds and spectra
- Bieberbach groups
- Bieberbach groups
- Cohomology of B'bach grps

The LHS Spectral Sequence
- The Charlap-Vasquez method
- Use of the LHS Spectral Sequence
- 2. Stiefel-Whitney class

Topology and spectra
- Cohomology and spectral properties
- Stiefel-Whitney classes and spectral properties

1. **Compact flat manifolds and Bieberbach groups**
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   - Bieberbach groups
   - Cohomology of Bieberbach groups

2. **The Lyndon-Hochschild-Serre Spectral Sequence**
   - The Charlap-Vasquez method
   - Use of the LHS Spectral Sequence
   - Second Stiefel-Whitney class

3. **Topology and spectra**
   - Cohomology and spectral properties
   - Stiefel-Whitney classes and spectral properties

Isospectral manifolds

The original examples of isospectral but not isometric manifolds were found by Milnor – these are flat tori.

Since then, effort was:
finding manifolds which are isospectral but not isometric
(`“one cannot hear the shape of a drum”`)
and which even have different topologies.

These kind of problems have been investigated in the context of nilmanifolds, solvmanifolds and compact flat manifolds.

The latter turn out to be a rich family where one can rather explicitly compute the multiplicities of eigenvalues of Laplace type operators, the real cohomology and the lengths of closed geodesics.

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Flat manifolds and spectra

M, M' are *p*-isospectral $\iff$ have the same spectrum with respect to the Hodge Laplacian Δ_p acting on p-forms.

M, M' p-isospectral $\Longrightarrow b_p(M) = b_p(M')$

(the ($\mathbb{Z}$)-Betti number b_p equals the multiplicity of the 0 eigenvalue of Δ_p)

So, the torsionfree part cannot be distinguished $\Longrightarrow$ it is not so easy to exhibit p-isospectral manifolds for all p having different cohomological properties.

We construct for instance M, M', p-isospectral for all p with

- $H^1(M, \mathbb{Z}_2) \cong H^1(M', \mathbb{Z}_2)$ but $H^2(M, \mathbb{Z}_2) \ncong H^2(M', \mathbb{Z}_2)$
- same ($\mathbb{Z}_2$)-cohomology but such that $w_2(M) \neq 0$ and $w_2(M') = 0$

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Bieberbach groups

A *crystallographic group* is a discrete, cocompact subgroup Γ of the isometry group of $\mathbb{R}^n$, $\mathrm{I}(\mathbb{R}^n) \cong \mathrm{O}(n) \ltimes \mathbb{R}^n$.

If Γ is also torsion-free, then Γ is a *Bieberbach group*.

Such Γ acts properly discontinuously and freely on $\mathbb{R}^n$, thus $M_\Gamma = \Gamma \backslash \mathbb{R}^n$ is a compact flat Riemannian manifold with fundamental group Γ.

Any compact flat manifold arises in this way.

> $M_\Gamma = \Gamma \backslash \mathbb{R}^n$ is an Eilenberg-MacLane space $\Longrightarrow$
> the cohomology of M = the group cohomology of Γ

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Bieberbach groups

Any $\gamma \in I(\mathbb{R}^n)$ can be written uniquely

$$\boxed{\gamma = BL_b}$$

where $B \in O(n)$ and L_b is a translation by $b \in \mathbb{R}^n$.

The restriction to Γ of $r : I(\mathbb{R}^n) \to O(n)$, $r(BL_b) = B$, is a homomorphism with kernel Λ=lattice, $r(\Gamma) \cong F$ is a finite subgroup of $O(n)$: the *holonomy group* (or point group) of Γ.

Algebraically, Γ is an extension of F by Λ, i.e., there is an exact sequence $0 \to \Lambda \to \Gamma \xrightarrow{r} F \to 1$

Conjugation by BL_b induces an action of F on Λ which is given by $\lambda \in \Lambda \mapsto B\lambda$ and is called the *holonomy representation*.

If the holonomy representation diagonalizes the corresponding manifolds are called *compact flat manifolds of diagonal type*.

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Examples in column notation

Hantzsche-Wendt 3-manifold or didicosm

M_Γ: flat manifold of dimension $n = 3$,

Holonomy group of $\Gamma = \mathbb{Z}_2^2$,

generated by $B_1 = \mathrm{diag}(1, -1, -1)$, $B_2 = \mathrm{diag}(-1, 1, -1)$,

$b_1 = \dfrac{e_1 + e_3}{2}$, $b_2 = \dfrac{e_1 + e_2}{2}$; i.e., $\Gamma = \langle B_1 L_{b_1},\ B_2 L_{b_2};\ L_{\mathbb{Z}^3} \rangle$.

M_Γ is orientable since $\det B = 1$ for every $BL_b \in \Gamma$.

In column notation

B_1	B_2
$1\ \frac{1}{2}$	-1
$-1\ \frac{1}{2}$	$1\ \frac{1}{2}$
-1	$-1\ \frac{1}{2}$

or

B_1	B_2	$B_1 B_2$
$1\ \frac{1}{2}$	-1	$-1\ \frac{1}{2}$
$-1\ \frac{1}{2}$	$1\ \frac{1}{2}$	-1
-1	$-1\ \frac{1}{2}$	$1\ \frac{1}{2}$

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$\mathbb{Z}_2$-class polynomial

Fact: $H^*(\mathbb{Z}_2^k, \mathbb{Z}_2) \cong \mathbb{Z}_2[x_1, \ldots, x_k]$, with gen. $x_1, \ldots, x_k$ in dim. 1

Γ of diagonal type with holonomy group $\mathbb{Z}_2^k = \langle B_1, \ldots, B_k \rangle$.

$$\bar{\beta} \in H^2(\mathbb{Z}_2^k, \Lambda^* \otimes \mathbb{Z}_2) \cong \left(H^2(\mathbb{Z}_2^k, \mathbb{Z}_2)\right)^n$$

The components $\bar{\beta}_\ell$ of $\bar{\beta}$ are homogeneous polynomials of degree two called the *$\mathbb{Z}_2$-class polynomials* of Γ

Proposition

$$\bar{\beta}_\ell = \sum_{\substack{i \,:\, B_i e_\ell = e_\ell \\ b_{i\ell} = \frac{1}{2}}} x_i^2 + \sum_{i \,:\, b_{i\ell} = \frac{1}{2}} \sum_{\substack{j \neq i \\ B_j e_\ell = -e_\ell}} x_i x_j \,,$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$.

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Lyndon-Hochschild-Serre spectral sequence

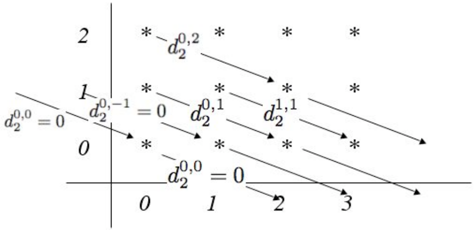
Γ is an extension of F by Λ, i.e., $0 \to \Lambda \to \Gamma \xrightarrow{r} F \to 1$

$$E_r^{p,q} \Longrightarrow H^{p+q}(\Gamma, R)$$

with

$$E_2^{p,q} \cong H^p(F, H^q(\Lambda, R))\,,$$

(the coefficient ring R is regarded as a trivial Γ-module and $p, q \geq 0$)



E_2

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LHS spectral sequence for Bieberbach groups

For Γ Bieberbach group of diag. type with holonomy group $\mathbb{Z}_2^k$

$$E_2^{p,q} = H^p(\mathbb{Z}_2^k, \mathbb{Z}_2) \otimes \bigwedge^q (\mathbb{Z}_2^n)^*$$

and their dimensions are given by

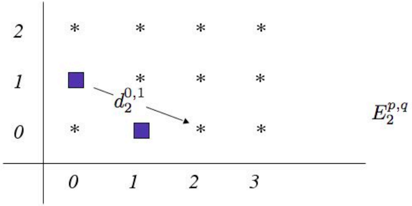
$q \backslash p$	0	1	2	3
2	$\binom{n}{2}$	$k\binom{n}{2}$	$\binom{k+1}{2}\binom{n}{2}$	$\dots$
1	n	$k\,n$	$\binom{k+1}{2}n$	$\dots$
0	1	k	$\binom{k+1}{2}$	$\dots$

$E_2^{p,q}$

Fact: the $\mathbb{Z}_2$-class polynomials determine the differential d_2

Examples of application of the Charlap-Vasquez method

Computation of $H^1(\Gamma, \mathbb{Z}_2)$:



Theorem

Let M be an n-dimensional compact flat manifold with diagonal holonomy $\mathbb{Z}_2^k$. Then

$$\dim H^1(M, \mathbb{Z}_2) = n - \operatorname{rank} d_2^{0,1} + k\,.$$

Note: $\operatorname{rank} d_2^{0,1} = \#$ linearly indep. $\mathbb{Z}_2$-class polynomials $\bar{\beta}_\ell$, $\ell = 1, \ldots, n$.

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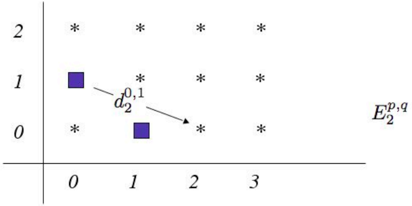

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Second Stiefel-Whitney class

The characteristic classes of a compact Riemannian manifold are closely related to its geometry.

Fact: Pontrjagin classes for any compact flat manifold vanish, since they can be expressed in terms of the curvature tensor (Chern-Weyl theorem),

Surprising: the Stiefel-Whitney classes need not vanish (Auslander and Szczarba)

Recall: $\mathrm{w}_2(M) \in H^2(M, \mathbb{Z}_2)$ and an oriented Riemannian manifold M admits a spin structure if and only if $\mathrm{w}_2(M) = 0$.

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Then,

$$\mathrm{w}_2 \neq 0 \iff \sigma_2(\omega_1, \ldots, \omega_n) \text{ is not a sum of } \mathbb{Z}_2\text{-class polynomials}$$

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Cohomology and spectral properties

- We consider all 4-dimensional flat manifolds of diagonal type with $F \equiv \mathbb{Z}_2^2$ or $F \equiv \mathbb{Z}_2^3$ and we show several isospectral or p-isospectral pairs, with $1 \leq p \leq 3$, having different $\mathbb{Z}_2$-cohomology groups and where some of them have different lengths of closed geodesics.
- We find, for $n = 5$, many isospectral pairs with $F \equiv \mathbb{Z}_2^4$ having different $H^2(M_\Gamma, \mathbb{Z}_2)$ and having the same $H^1(M_\Gamma, \mathbb{Z}_2)$ and the same holonomy representations. Such examples are not possible to obtain in dimension 4.

Example ($\#g1$,$\#g4$ in the CARAT (Aachen) list)

$\#g1$:

B_1	B_2	B_3	B_4
1	1	-1	$1_{\frac{1}{2}}$
$1_{\frac{1}{2}}$	1	1	-1
$1_{\frac{1}{2}}$	1	-1	-1
1	-1	$1_{\frac{1}{2}}$	1
-1	$1_{\frac{1}{2}}$	1	1

$\#g4$:

B_1	B_2	B_3	B_4
1	1	-1	$1_{\frac{1}{2}}$
1	$1_{\frac{1}{2}}$	1	-1
$1_{\frac{1}{2}}$	1	-1	-1
1	-1	$1_{\frac{1}{2}}$	1
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- We consider all 4-dimensional flat manifolds of diagonal type with $F \equiv \mathbb{Z}_2^2$ or $F \equiv \mathbb{Z}_2^3$ and we show several isospectral or p-isospectral pairs, with $1 \leq p \leq 3$, having different $\mathbb{Z}_2$-cohomology groups and where some of them have different lengths of closed geodesics.
- We find, for $n = 5$, many isospectral pairs with $F \equiv \mathbb{Z}_2^4$ having different $H^2(M_\Gamma, \mathbb{Z}_2)$ and having the same $H^1(M_\Gamma, \mathbb{Z}_2)$ and the same holonomy representations. Such examples are not possible to obtain in dimension 4.

Example ($\#g1$,$\#g4$ in the CARAT (Aachen) list)

$\#g1$:

B_1	B_2	B_3	B_4
1	1	-1	$1_{\frac{1}{2}}$
$1_{\frac{1}{2}}$	1	1	-1
$1_{\frac{1}{2}}$	1	-1	-1
1	-1	$1_{\frac{1}{2}}$	1
-1	$1_{\frac{1}{2}}$	1	1

$\#g4$:

B_1	B_2	B_3	B_4
1	1	-1	$1_{\frac{1}{2}}$
1	$1_{\frac{1}{2}}$	1	-1
$1_{\frac{1}{2}}$	1	-1	-1
1	-1	$1_{\frac{1}{2}}$	1
-1	$1_{\frac{1}{2}}$	1	1

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Cohomology and spectral properties

- We determine the $\mathbb{Z}_2$-cohomology of all GHW manifolds in dimensions 3, 4 and 5, listing all isospectral classes.

GHW=generalized Hantzsche-Wendt manifolds: dimension n flat manifolds having holonomy group $\mathbb{Z}_2^{n-1}$

HW = orientable GHW

they are all rational homology spheres

Table: Cohomology classes of GHW manifolds in dimension 5.

$\text{betti}_1^{\mathbb{Z}_2}$	$\text{betti}_2^{\mathbb{Z}_2}$	List of manifolds
4	5	7, 14, 16, 21, 58, 61, 67, 69, 74, 77, 84, 85, 104, 105, 106, 107, 112, 115, 117,118, 121, 122
4	6	2, 3, 4, 6, 8, 9, 10, 11, 13, 17, 18, 20, 22, 23, 36, 37, 38, 39 40, 41, 44, 45, 50, 51, 52, 53, 57, 59, 60, 62, 65, 66, 68 71, 73, 75, 76, 78, 80, 81, 82, 83, 90, 91, 92, 93, 96, 97, 98, 100, 101, 102, 109, 111, 113, 114, 116, 119
4	7	1, 5, 12, 15, 19, 28, 29, 30, 31, 42, 43, 46, 47, 48, 49, 54, 55, 56, 63, 64, 70, 72, 79, 86, 87, 88, 89, 94, 95, 99, 103, 108, 110, 120, 123
5	10	24, 25, 26, 27, 32, 33, 34, 35

some isospectral manifolds have the same colors

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Stiefel-Whitney classes and spectral properties

- For $n = 4$, we exhibit p-isospectral pairs for all p, M, M', that have the same $(\mathbb{Z}_2)$-cohomology but such that $\mathrm{w}_2(M) \neq 0$ and $\mathrm{w}_2(M') = 0$

  See manifolds labelled $(1, 1, 0)$ and $(1, 0, 1)$ in the family $\mathcal{K}_4$:

B_1	B_2	B_3
-1	$1_{\frac{x}{2}}$	$1_{\frac{y}{2}}$
$1_{\frac{1}{2}}$	-1	$1_{\frac{z}{2}}$
1	$1_{\frac{1}{2}}$	-1
1	1	$1_{\frac{1}{2}}$

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Table: Family $\mathcal{K}_4$

(x,y,z)	$\mathrm{betti}_1^{\mathbb{Z}_2} = \mathrm{betti}_3^{\mathbb{Z}_2}$	$\mathrm{betti}_2^{\mathbb{Z}_2}$	w_2	Sunada n.	isospectral pairs
$(0,0,0)$	4	6	$\neq 0$	$\begin{pmatrix}1&0&0\\2&1&0\\3&0&0\end{pmatrix}$	
$(1,0,0)$	3	4	$\neq 0$	$\begin{pmatrix}1&0&0\\1&2&0\\2&1&0\end{pmatrix}$	♣
$(1,1,0)$	3	4	0	$\begin{pmatrix}1&0&0\\2&1&0\\1&2&0\end{pmatrix}$	♡
$(0,1,0)$	3	4	$\neq 0$	$\begin{pmatrix}1&0&0\\1&2&0\\2&1&0\end{pmatrix}$	♣
$(0,0,1)$	4	6	$\neq 0$	$\begin{pmatrix}1&0&0\\3&0&0\\2&1&0\end{pmatrix}$	
$(1,0,1)$	3	4	$\neq 0$	$\begin{pmatrix}1&0&0\\2&1&0\\1&2&0\end{pmatrix}$	♡
$(0,1,1)$	3	4	$\neq 0$	$\begin{pmatrix}1&0&0\\2&1&0\\2&0&1\end{pmatrix}$	
$(1,1,1)$	3	4	0	$\begin{pmatrix}1&0&0\\3&0&0\\1&1&1\end{pmatrix}$	

Sunada numbers: $c_{s,t}$= number of elements in the holonomy F of M_Γ, having exactly s 1's in the diagonal (or column) and t $\frac{1}{2}$'s coming with those 1's, $0 \leq t \leq s \leq n$.