Ricci solitons with large symmetry group

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Joint work with M. Wang (McMaster)

Ricci flow

$$\frac{\partial g_\tau}{\partial \tau} = -2\text{Ric}\,(g_\tau)$$

Einstein metrics g have Ric(g)$= -\frac{\epsilon}{2}g$, so give solution evolving by homotheties

$$g_\tau = (1+\epsilon\tau)g$$

Generalise? Allow flow of diffeomorphisms

$$g_\tau = (1+\epsilon\tau)\psi_\tau^* g$$

These arise from *Ricci solitons*

$$\text{Ric}(g) + \frac{1}{2}L_X g + \frac{\epsilon}{2}g = 0$$

Equation for metric g and vector field X.

$\epsilon = 0$ steady soliton

$\epsilon = 1$ expanding

$\epsilon = -1$ shrinking

Important special case–*gradient solitons*

Now $X = \text{grad}\, u$, so equation is

$$\text{Ric}(g) + \text{Hess}(u) + \frac{\epsilon}{2} g = 0$$

How to find solutions?

Symmetries?

No compact homogeneous examples except Einstein metrics.

Weaken homogeneity assumption?

Cohomogeneity one metrics.

Take g (and u) to depend only on one variable t

eg if G acts isometrically on manifold M (preserving u), with generic hypersurface orbits.

Example

$M = \mathbb{R}^n \quad : \quad G = SO(n) \quad : \quad$ generic orbit S^{n-1}

$$g = dt^2 + f(t) g_{\text{round}} \quad : \quad u = u(t)$$

Nonlinear ODE system.

Special orbit at origin–boundary conditions.

$n = 2$ Hamilton-Witten cigar (asymptotic to cylinder)

$n \geq 3$ Bryant soliton (asymptotic to paraboloid)

Kähler examples

(Cao, Koiso, Guan, Feldman-Ilmanen-Knopf, Pedersen-Tonneson-Friedman-Valent, Apostolov-Calderbank-Gauduchon-Tonneson-Friedman...)

Hypersurface circle bundle over $\mathbb{CP}^n$ or more general Kähler-Einstein base.

Noncompact steady, expanding.

Compact shrinking.

FIK produce some complete noncompact *shrinking* solitons in $\mathbb{CP}^n$ case.

Gradient Ricci solitons

Metric $dt^2 + g_t$: $X = \text{grad } u$.

Cohomogeneity 1 equations:

$$\begin{aligned} \text{tr } \dot{L} + \text{tr } (L^2) - \ddot{u} &= \frac{\epsilon}{2} \\ \dot{L} + (\text{tr } L)L - r - \dot{u}L &= \frac{\epsilon}{2}. \end{aligned}$$

L is shape operator (endomorphism wrt the metric g_t on hypersurface orbit), so

$$\dot{g}_t(Y, Z) = 2g_t(LY, Z)$$

Also r is Ricci endomorphism wrt g_t

Motivated by work in Einstein case (W-W):

Choose hypersurface to be circle bundle over product of Kähler-Einstein factors.

$f(t)$ gives scale in circle; $g_i(t)$ gives scale on base factors.

Constants n_i, p_i, q_i (dims, Einstein constants, bundle)

Equations are:

$$\begin{aligned}
\frac{\ddot{f}}{f} + \sum_{i=1}^{r} 2n_i \frac{\ddot{g}_i}{g_i} - \ddot{u} &= \frac{\epsilon}{2}, \\
\frac{\ddot{f}}{f} + \sum_{i=1}^{r} 2n_i \frac{\dot{f}\dot{g}_i}{fg_i} - \frac{\dot{u}\dot{f}}{f} - \sum_{i=1}^{r} \frac{n_i q_i^2}{2} \frac{f^2}{g_i^4} &= \frac{\epsilon}{2}, \\
\frac{\ddot{g}_i}{g_i} - \left(\frac{\dot{g}_i}{g_i}\right)^2 + \frac{\dot{f}\dot{g}_i}{fg_i} + \sum_{j=1}^{r} 2n_j \frac{\dot{g}_i\dot{g}_j}{g_i g_j} - \frac{\dot{u}\dot{g}_i}{g_i} & \\
- \frac{p_i}{g_i^2} + \frac{q_i^2 f^2}{2g_i^4} &= \frac{\epsilon}{2}.
\end{aligned}$$

New variable $ds = f(t)\, dt$. Let

$$\alpha(s) = f(t)^2 \quad : \quad \beta_i(s) = g_i(t)^2 \quad : \quad \phi(s) = u(t).$$

Equations become:

$$\begin{aligned}
\frac{1}{2}\alpha'' + \frac{1}{2}\alpha'(\log v)' + \alpha \sum_{i=1}^{r} n_i \left(\frac{\beta_i''}{\beta_i} - \frac{1}{2}\left(\frac{\beta_i'}{\beta_i}\right)^2 \right) & \\
-\alpha\varphi'' - \frac{1}{2}\alpha'\varphi' &= \frac{\epsilon}{2}, \\
\frac{1}{2}\alpha'' + \frac{1}{2}\alpha'(\log v)' - \frac{1}{2}\alpha'\varphi' - \frac{\alpha}{2}\sum_{i=1}^{r}\frac{n_i q_i^2}{\beta_i^2} &= \frac{\epsilon}{2}, \\
\frac{\alpha'}{2}\frac{\beta_i'}{\beta_i} + \frac{\alpha}{2}\left(\frac{\beta_i''}{\beta_i} - \left(\frac{\beta_i'}{\beta_i}\right)^2\right) + \frac{\alpha}{2}\frac{\beta_i'}{\beta_i}(\log v)' - \frac{\alpha}{2}\frac{\beta_i'}{\beta_i}\varphi' & \\
-\frac{p_i}{\beta_i} + \frac{q_i^2\alpha}{2\beta_i^2} &= \frac{\epsilon}{2}.
\end{aligned}$$

where $v = \prod_{i=1} \beta_i^{n_i}$ (vol of orbit)

Observe these imply:

$$\phi'' = \sum_{i=1}^{r} n_i \left(\frac{\beta_i''}{\beta_i} - \frac{1}{2} \left(\frac{\beta_i'}{\beta_i} \right)^2 + \frac{1}{2} \frac{q_i^2}{\beta_i^2} \right).$$

Look for solutions with each bracket zero, so ϕ linear in s.

Kähler ansatz:

$$\beta_i(s) = -q_i(s + \sigma_i)$$

Equations now exactly solvable:

Special orbits: collapse circle fibre

or higher-dim sphere (if one base factor is $\mathbb{CP}^n$).

(i) Steady solitons on bundles (rank ≥ 1)

Asymptotic to cigar-paraboloid

(ii) Expanding solitons on bundles of rank ≥ 1

Asymptotically conical

(iii) Shrinking solitons on bundles of rank ≥ 1

Asymptotically conical

(iv) Compact shrinking solitons (blow-downs)

Other examples :

hypersurface circle bundle over generalised flag variety.

Same types of examples as before.

Non-Kahler examples?

Multiple warped products

$$dt^2 + \sum_{i=1}^{r} g_i^2(t)\mathsf{h}_i$$

Open set is

$$(a, b) \times M_1 \times \ldots \times M_r$$

with possible collapse at one or both ends.

(M_i, h_i) are Einstein with positive scalar curvature.

Generalisation of Bryant-Ivey examples.

Consider steady case.

Equations in suitable coordinates

$$X_i' = X_i \left(\sum_{j=1}^{r} X_j^2 - 1 \right) + \frac{Y_i^2}{\sqrt{d_i}},$$

$$Y_i' = Y_i \left(\sum_{j=1}^{r} X_j^2 - \frac{X_i}{\sqrt{d_i}} \right).$$

Have Lyapunov function

$$\mathcal{L} = \sum_{j=1}^{r} \left(X_j^2 + Y_j^2 \right) - 1$$

$$\mathcal{L}' = 2\mathcal{L} \left(\sum_{j=1}^{r} X_j^2 \right)$$

Pick trajectory in unstable manifold of critical point on unit sphere, flowing into unit ball.

Must establish several limits to ensure metric is C^3. Then regularity theory gives smoothness at critical point (sphere M_1 collapses)

Analysis of ω-limit set and Lyapunov shows trajectory converges to origin.

Obtain complete steady soliton on vector bundle over $M_2 \times \ldots \times M_r$.

Asymptotically paraboloid

$$\sim \quad dt^2 + t\mathbf{h}$$

Ricci is nonnegative.