The existence of homogeneous geodesics in homogeneous pseudo-Riemannian manifolds

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Homogeneous geodesics in homogeneous affine manifolds

Definition

Let (M, ∇) be a homogeneous affine manifold.
A geodesic is homogeneous if it is an orbit of an one-parameter group of affine diffeomorphisms. (Here the canonical parameter of the group need not be the affine parameter of the geodesic.)
An affine g.o. manifold is a homogeneous affine manifold (M, ∇) such that each geodesic is homogeneous.

Lemma

Let $M = G/H$ be a homogeneous space with a left-invariant affine connection ∇. Then each regular curve which is an orbit of a 1-parameter subgroup $g_t \subset G$ on M is an integral curve of an affine Killing vector field on M.

Definition

Let (M, ∇) be a manifold with an affine connection.
A vector field X on M is called a Killing vector field if

$$[X, \nabla_Y Z] - \nabla_Y [X, Z] - \nabla_{[X,Y]} Z = 0$$

is satisfied for arbitrary vector fields Y, Z.

Lemma

*Let (M, ∇) be a homogeneous affine manifold and $p \in M$.
There exist $n = \dim(M)$ affine Killing vector fields which are linearly independent at each point of some neighbourhood $\mathcal{U}$ of p.*

Definition

A nonvanishing smooth vector field Z on M is geodesic along its regular integral curve γ if $\gamma(t)$ is geodesic up to a possible reparametrization. If all regular integral curves of Z are geodesics up to a reparametrization, then the vector field Z is called a geodesic vector field.

For example, a round two-sphere with the corresponding Levi-Civita connection does *not* admit any geodesic affine Killing vector field. Still, all geodesics are homogeneous.

Lemma

Let Z be a nonvanishing Killing vector field on $M = (G/H, \nabla)$.
1) *Z is geodesic along its integral curve γ if and only if*

$$\nabla_{Z_{\gamma(t)}} Z = k_\gamma \cdot Z_{\gamma(t)}$$

holds along γ. Here $k_\gamma \in \mathbb{R}$ is a constant.
2) *Z is a geodesic vector field if and only if*

$$\nabla_Z Z = k \cdot Z$$

holds on M. Here k is a smooth function on M which is constant along integral curves of Z.

$\dim(M) = 2$

Theorem (Opozda; Arias-Marco, Kowalski)

Let ∇ be a locally homogeneous affine connection with arbitrary torsion on a 2-dimensional manifold $\mathcal{M}$. Then, either ∇ is locally a Levi-Civita connection of the unit sphere or, in a neighbourhood $\mathcal{U}$ of each point $m \in \mathcal{M}$, there is a system (u, v) of local coordinates and constants A, B, C, D, E, F, G, H such that ∇ is expressed in $\mathcal{U}$ by one of the following formulas:

$$\underline{TypeA}: \quad \begin{aligned} \nabla_{\partial_u}\partial_u &= A\,\partial_u + B\,\partial_v, & \nabla_{\partial_u}\partial_v &= C\,\partial_u + D\,\partial_v, \\ \nabla_{\partial_v}\partial_u &= E\,\partial_u + F\,\partial_v, & \nabla_{\partial_v}\partial_v &= G\,\partial_u + H\,\partial_v, \end{aligned}$$

$$\underline{TypeB}: \quad \begin{aligned} \nabla_{\partial_u}\partial_u &= \frac{A}{u}\,\partial_u + \frac{B}{u}\,\partial_v, & \nabla_{\partial_u}\partial_v &= \frac{C}{u}\,\partial_u + \frac{D}{u}\,\partial_v, \\ \nabla_{\partial_v}\partial_u &= \frac{E}{u}\,\partial_u + \frac{F}{u}\,\partial_v, & \nabla_{\partial_v}\partial_v &= \frac{G}{u}\,\partial_u + \frac{H}{u}\,\partial_v. \end{aligned}$$

Homogeneous geodesics in dimension 3

$(\mathbb{R}^3, \nabla)$
connection ∇ with constant Christoffel symbols
group $\mathbb{R}^3$ acting on it by the translations

$$\begin{array}{ccc} \Gamma^i_{11} = A_i, & \Gamma^i_{22} = B_i, & \Gamma^i_{33} = C_i, \\ \Gamma^i_{12} = \Gamma^i_{21} = E_i, & \Gamma^i_{13} = \Gamma^i_{31} = F_i, & \Gamma^i_{23} = \Gamma^i_{32} = G_i. \end{array}$$

The Killing vector field $X = x\,\partial_u + y\,\partial_v + z\,\partial_w$
satisfies the condition $\nabla_X X = kX$ if it holds

$$\begin{aligned} x^2 A_1 + y^2 B_1 + z^2 C_1 + 2\,xy E_1 + 2\,xz F_1 + 2\,yz G_1 &= k\,x, \\ x^2 A_2 + y^2 B_2 + z^2 C_2 + 2\,xy E_2 + 2\,xz F_2 + 2\,yz G_2 &= k\,y, \\ x^2 A_3 + y^2 B_3 + z^2 C_3 + 2\,xy E_3 + 2\,xz F_3 + 2\,yz G_3 &= k\,z. \end{aligned}$$

Families of homogeneous connections on H_3 or on $E(1,1)$
lead to similar equations.

Existence of homogeneous geodesics in dimension 3

Theorem

Let ∇ be a connection with constant Christoffel symbols on $\mathbb{R}^3$. $(\mathbb{R}^3, \nabla)$ admits a geodesic Killing vector field.

Proof. Recall that the Killing vector field $X = x\,\partial_u + y\,\partial_v + z\,\partial_w$ satisfies the condition $\nabla_X X = kX$ if it holds

$$\begin{aligned} x^2 A_1 + y^2 B_1 + z^2 C_1 + 2\,xy\,E_1 + 2\,xz\,F_1 + 2\,yz\,G_1 &= k\,x, \\ x^2 A_2 + y^2 B_2 + z^2 C_2 + 2\,xy\,E_2 + 2\,xz\,F_2 + 2\,yz\,G_2 &= k\,y, \\ x^2 A_3 + y^2 B_3 + z^2 C_3 + 2\,xy\,E_3 + 2\,xz\,F_3 + 2\,yz\,G_3 &= k\,z. \end{aligned}$$

- Sphere S^2 in T_pM, vectors $X = (x, y, z)$ with the norm 1.
- Denote $\underline{v(X) = \nabla_X X}$ and $t(X) = v(X) - \langle v(X), X\rangle X$, then $t(X) \perp X$ and $X \mapsto t(X)$ defines a vector field on S^2.
- According to the Hair-Dressing Theorem for sphere, there is $\bar{X} \in T_pM$ such that $t(\bar{X}) = 0$.
- We see $v(\bar{X}) = k\bar{X}$, hence $\underline{\nabla_{\bar{X}}\bar{X} = k\bar{X}}$.

Existence of homogeneous geodesics in odd dimensions

Theorem

Let $M = (G/H, \nabla)$ be a homogeneous affine manifold of odd dimension n and $p \in M$. There exists a homogeneous geodesic through p.

Proof. Killing vector fields $K_1, \dots, K_n$ independent near p,
$B = \underline{\{K_1(p), \dots, K_n(p)\} \text{ basis of } T_pM}$,
$X \in T_pM$, $X = (x_1, \dots x_n)$ in B,
$\underline{X^* = x_1K_1 + \dots + x_nK_n}$ and an $\underline{\text{integral curve}}$ γ $\underline{\text{of } X^* \text{ through } p}$.
S^{n-1} in T_pM of vectors $X = (x_1, \dots, x_n)$ with the norm 1.
Denote $\underline{v(X) = \nabla_{X^*_{\gamma(t)}} X^*|_{t=0}}$ and $t(X) = v(X) - \langle v(X), X\rangle X$,
then $t(X) \perp X$ and $X \mapsto t(X)$ defines a vector field on S^{n-1}.
Again, there is $\bar{X} \in T_pM$ such that $t(\bar{X}) = 0$.
We obtain $v(\bar{X}) = k_\gamma \bar{X}$, where $k_\gamma = \langle v(\bar{X}), \bar{X}\rangle$ is a constant,
$\underline{\nabla_{\bar{X}^*_\gamma} \bar{X}^* = k_\gamma \bar{X}^*_\gamma}$ and γ is homogeneous geodesic. □

Preliminaries on differential topology

Let $f\colon M \to N$ be a smooth map between manifolds of the same dimension.
We say that $x \in M$ is a *regular point* of f if the derivative df_x is nonsingular. In this case, f maps a neighborhood of x diffeomorphically onto an open set in N.
The point $y \in N$ is called a *regular value* if $f^{-1}(y)$ contains only regular points.

- If M is compact and $y \in N$ is a regular value, then $f^{-1}(y)$ is a finite set (possibly empty).

For compact M, smooth map $f\colon M \to N$ and a regular value $y \in N$, we define $\#f^{-1}(y)$ to be the number of points in $f^{-1}(y)$.

- $\#f^{-1}(y)$ is locally constant as a function of y, where y ranges through regular values.

Points, or values, respectively, which are not regular are *critical*.

Theorem (Morse, Sard)

Let $f\colon U \to \mathbb{R}^n$ be a smooth map, defined on an open set $U \subset \mathbb{R}^m$ and let C be the set of critical points; that is the set of all $x \in U$ with $\mathrm{rank}(df_x) < n$. Then the image $f(C) \subset \mathbb{R}^n$ has measure zero.

Corollary (Brown)

The set of regular values of a smooth map $f\colon M \to N$ is everywhere dense in N.

Theorem

Let M and N be manifolds of the same dimension, M compact without boundary, N connected and $f\colon M \to N$ smooth mapping. If y and z are regular values of f, then

$$\# f^{-1}(y) = \# f^{-1}(z) \qquad (\mathrm{mod} 2).$$

This common residue class (called $\mathrm{mod} 2$ degree of f) depends only on the smooth homotopy class of f.

- The antipodal map $x \mapsto -x$ of S^n has mod 2 degree 1.
- Any map $f : S^n \to S^n$ without fixed points has mod 2 degree 1, because it is homotopic to the antipodal map. The homotopy is for example

$$\varphi(x,t) = \frac{(1-t)f(x) - tx}{\|(1-t)f(x) - tx\|}.$$

Clearly, $\varphi(x,0) = f(x)$ and $\varphi(x,1)$ is the antipodal map.

Existence of homogeneous geodesics in any dimension

We refine the proof of previous Theorem to arbitrary dimension.
Recall that $X \mapsto t(X)$ defines a smooth vector field on S^{n-1}.
Assume now that $t(X) \neq 0$ everywhere.
Putting $f(X) = t(X)/\|t(X)\|$, we obtain a smooth map $f\colon S^{n-1} \to S^{n-1}$ without fixed points.
According to a well-known statement from differential topology, the mod 2 degree of f is $\deg(f) = 1$.
On the other hand, we have $v(X) = v(-X)$ and hence $f(X) = f(-X)$ for each X.
If Y is a regular value of f, then the inverse image $f^{-1}(Y)$ consists of even number of elements. Hence $\deg(f) = 0$, which is a contradiction.
This implies that there is $\bar{X} \in T_pM$ such that $t(\bar{X}) = 0$ and again, a homogeneous geodesic exists.

Existence of homogeneous geodesics

Theorem

Let $M = (G/H, \nabla)$ be a homogeneous affine manifold and $p \in M$. Then M admits a homogeneous geodesic through p.

Theorem

Let $M = (G/H, g)$ be a homogeneous pseudo-Riemannian manifold (not necessarily reductive) and $p \in M$. Then M admits a homogeneous geodesic through p.

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