Special Hermitian structures

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Bismut connection

On any Hermitian manifold (M^{2n}, J, g) $\exists!$ connection ∇^B such that

$$\nabla^B g = 0 \quad \text{(metric)}$$
$$\nabla^B J = 0 \quad \text{(Hermitian)}$$
$$c(X,Y,Z) = g(X, T(Y,Z)) \;\; \text{3-form}$$

where T is the torsion of ∇^B

$\nabla^B = \nabla^{LC} + \frac{1}{2} c$ is the Bismut connection and $c = -JdJF$, where $F = g(J\cdot,\cdot)$ is the associated fundamental form.

∇^B is also called a *KT* connection.

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SKT and astheno-Kähler metrics

$c = 0 \iff \nabla^B = \nabla^{LC} \iff (M^{2n}, J, g)$ is Kähler

Definition

(J, g) on M^{2n} is said to be strong Kähler with torsion (SKT) or pluriclosed if $dc = 0$, i.e. if $\partial\overline{\partial}F = 0$.

Definition (Jost, Yau)

(J, g) on M^{2n} is called astheno-Kähler if $\partial\overline{\partial}\, F^{n-2} = 0$.

If $n = 2 \Rightarrow$ any Hermitian metric is astheno-Kähler.

If $n = 3 \Rightarrow$ SKT= astheno-Kähler.

• If $\exists$ a astheno-Kähler metric on a compact (M^{2n}, J), then any holomorphic 1-form must be closed [Jost-Yau].

$\Rightarrow$ a complex parallelizable (M^{2n}, J) cannot admit any astheno-Kähler metric compatible with J.

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Link with standard metrics

Definition

A Hermitian structure (J, g) on M^{2n} is called standard if $\partial\overline{\partial}F^{n-1} = 0$ or equivalently if the Lee form θ is co-closed.

Theorem (Gauduchon)

For a compact (M^{2n}, J) a standard metric can be found in the conformal class of any given J-Hermitian metric.

If $n = 2 \Rightarrow$ standard = SKT

If $n > 2$ a SKT g is standard $\Leftrightarrow |dF|^2 = (n-1)|\theta \wedge F|^2$.

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Link with generalized Kähler Structures

Definition (Hitchin, Gualteri)

A generalized Kähler structure on M^{2n} is a pair $(\mathcal{J}_1, \mathcal{J}_2)$ of generalized complex structures such that
- $\mathcal{J}_1$ and $\mathcal{J}_2$ commute,
- $\mathcal{J}_1$ and $\mathcal{J}_2$ are compatible with the indefinite metric $(\, , \,)$ on $TM \oplus T^*M$,
- $-(\mathcal{J}_1\mathcal{J}_2 \cdot , \cdot)$ is positive definite.

Theorem (Apostolov, Gualtieri)

A GK structure on M^{2n} is equivalent to a triple (g, J_+, J_-) with $(J_\pm, g)$ SKT structures such that $J_+ dF_+ = -J_- dF_-$.

The previous conditions appear on the general target space geometry for a $(2, 2)$ supersymmetric sigma model [Gates, Hall and Roček].

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Hermitian flow

Let (M^{2n}, J, g_0) be an Hermitian manifold. Streets and Tian introduced a flow

$$\frac{\partial F(t)}{\partial t} = \Phi(F), \quad F(0) = F_0,$$

where $\Phi(F) = -\partial\partial^* F - \overline{\partial\partial^*} F - \frac{i}{2}\partial\overline{\partial} \log \det g = -(\rho^B)^{1,1}$.

Proposition (Streets,Tian)

Let (M^{2n}, J, g) be a SKT manifold. Then $F \to \Phi(F)$ is a real quasi-linear second-order elliptic operator when restricted to $\{J - \text{Hermitian SKT metrics}\}$.

If $g(0)$ is SKT (Kähler), then $g(t)$ is SKT (Kähler) for all t.

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Hermitian-symplectic structures

Definition (Streets, Tian)

A SKT metric g on compact (M^{2n}, J) is static if $\Phi(F) = \lambda F$, or equivalently if $(\rho^B)^{1,1} = \lambda F$.

If g is Kähler and static, then it is Kähler-Einstein.

Proposition (Streets,Tian)

Let (M^{2n}, J) be compact with a static SKT metric g. If $\lambda \neq 0$, then $F = \Omega^{1,1}$, where Ω is a symplectic form Ω taming J, i.e. such that $\Omega(X, JX) > 0, \quad \forall X \neq 0$.

Definition

A Hermitian-symplectic structure on (M^{2n}, J) is a symplectic form Ω taming J.

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Theorem (Streets, Tian)

If a compact (M^4, J) admits a Hermitian-symplectic structure, then (M^4, J) has a Kähler metric.

Problem (Streets, Tian)

There exists an example of a compact (M^{2n}, J), with $n > 2$, admitting a Hermitian-symplectic structure, but no Kähler structures?

Proposition (Enrietti, -, Vezzoni)

Giving a Hermitian-symplectic structure Ω on (M^{2n}, J) is equivalent to assign an SKT metric g such that $\partial F = \overline{\partial}\beta$, for some ∂-closed $(2,0)$-form β.

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6-dimensional SKT nilmanifolds

Theorem (-, Parton, Salamon)

$M^6 = G/\Gamma$ nilmanifold, J left-invariant, g any compatible metric. Then

(J, g) SKT $\Leftrightarrow \exists$ a basis (α^i) of $(1,0)$-forms such that

$$\begin{cases} d\alpha^1 = & d\alpha^2 = 0, \\ d\alpha^3 = & A\overline{\alpha}^1 \wedge \alpha^2 + B\overline{\alpha}^2 \wedge \alpha^2 + C\alpha^1 \wedge \overline{\alpha}^1 + \\ & D\alpha^1 \wedge \overline{\alpha}^2 + E\alpha^1 \wedge \alpha^2 \end{cases}$$

with

$$|A|^2 + |D|^2 + |E|^2 + 2Re(\overline{B}C) = 0.$$

G has to be 2-step and the existence of a SKT metric depends only on J.

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Example

Consider the nilpotent Lie group with structure equations

$$\begin{cases} de^i = 0\,, i = 1, \dots, 5, \\ de^6 = e^{12} + e^{34}\,. \end{cases}$$

and J defined by

$$\eta^1 = e^1 + ie^2, \quad \eta^2 = e^3 + ie^4, \quad \eta^3 = e^5 + ie^6.$$

Take $\Gamma \subset G$ such that J is rational on $M = \Gamma/G \Rightarrow$

• any holomorphic 1-form on M is d-closed since $H^{1,0}_{\overline{\partial}}(M, J) = \text{span} < \eta^1, \eta^2 >$.

• (M, J) does not admit any SKT metric compatible with J.

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Twist construction

Let M be a manifold with a T_M-action and a principal torus bundle $\pi : P \to M$ with connection θ.

Definition (Swann)

If the torus action lifts to P commuting with the principal action, then one may construct the twist W of M, as the quotient $W = P/T_M$.

$$M \xleftarrow{\pi} P \xrightarrow{\pi_W} W$$

If the lifted torus action preserves θ, then tensors on M can be transferred to tensors on W if their pullbacks to P coincide on $\mathcal{H} = \mathrm{Ker}\theta$.

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(N^6, J) simply-connected compact with a SKT and standard (J, g).

$M^8 = N^6 \times \mathbb{T}^2$ is astheno-Kähler and SKT with torsion c supported on N^6.

Theorem (Swann)

If there are two l.i. integral closed $(1,1)$-forms Ω_i, $i = 1, 2$, on N^6 with $[\Omega_i] \in H^2(N^6, \mathbb{Z})$ l.i. and such that

$$\sum_{i,j=1}^{2} \gamma_{ij} \Omega_i \wedge \Omega_j = 0$$

for some positive definite $(\gamma_{ij}) \in M_2(\mathbb{R})$, then there is a compact simply connected SKT $\mathbb{T}^2$-bundle $\tilde{W}$ over N^6.

Under the additional condition $c \wedge \Omega_j = 0, j = 1, 2,$ we can prove that $\tilde{W}$ is astheno-Kähler.

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$$\sum_{i,j=1}^{2} \gamma_{ij} \Omega_i \wedge \Omega_j = 0$$

for some positive definite $(\gamma_{ij}) \in M_2(\mathbb{R})$, then there is a compact simply connected SKT $\mathbb{T}^2$-bundle $\tilde{W}$ over N^6.

Under the additional condition $c \wedge \Omega_j = 0, j = 1, 2,$ we can prove that $\tilde{W}$ is astheno-Kähler.

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Twist and SKT nilmanifolds

Example (Swann)

Any 6-dimensional SKT nilmanifold M^6 is the twist of the Kähler product $T^4 \times T^2$ by using the two integral 2-forms Ω_j (supported on T^4) such that $d\alpha^3 = \Omega_1 + i\Omega_2$. The integrability condition for the induced almost complex structure on the twist is

$$(\Omega_1 + i\Omega_2)^{(0,2)} = 0$$

and the SKT condition for the induced Hermitian metric is

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Problem

Study the existence of SKT and Hermitian-symplectic structures on $2n$-dimensional nilmanifolds.

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Hermitian-symplectic structures on nilmanifolds

Theorem (Enrietti, -, Vezzoni)

$(M = G/\Gamma, J)$ with J left-invariant and G any Lie group
If $J\xi \cap [\mathfrak{g}, \mathfrak{g}] \neq \{0\}$, then (M, J) does not admit any compatible Hermitian-symplectic structure.

Theorem (Enrietti, -, Vezzoni)

G/Γ nilmanifold (not a torus), J left-invariant.
1) If $(G/\Gamma, J)$ has a J-Hermitian SKT metric, then G has to be 2-step and the SKT nilmanifold is a twist of a torus.
2) $(G/\Gamma, J)$ does not admit any compatible Hermitian-symplectic structure.

To prove 1) we show that J has to preserve the center ξ of $\mathfrak{g}$ and that a SKT structure on $\mathfrak{g}$ induces a SKT structure on $\mathfrak{g}/\xi$.

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S^1-bundles

Proposition (Kobayashi)

Let $(N^{2n+1}, I, \xi, \eta, h)$ be an almost contact metric manifold and let $[\Omega] \in H^2(N^{2n+1}, \mathbb{Z})$. Then $\exists\, S^1 \hookrightarrow P \xrightarrow{\pi} N^{2n+1}$ with connection 1-form θ on P whose curvature form is $d\theta = \pi^*(\Omega)$.

Proposition (Ogawa)

P has an almost Hermitian structure (J, g) with J defined by

$$\theta(JX) = -\pi^*(\eta(\pi_* X)), \quad \pi_*(JX) = I(\pi_* X) + \tilde{\theta}(X)\xi$$

and

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Theorem (Fernandez, -, Ugarte, Villacampa)

(J, g) on P is SKT if and only if (I, ξ, η, h) is normal, $d\theta$ is J-invariant and

$$d(\pi^*(I(i_\xi d\omega))) = 0,$$
$$d(\pi^*(I(d\omega) - d\eta \wedge \eta)) = (-\pi^*(I(i_\xi d\omega)) + \pi^*\Omega) \wedge \pi^*\Omega.$$

Definition

$(N^{2n+1}, I, \xi, \eta, h)$ is quasi-Sasakian if it is normal and $d\omega = 0$. If $d\eta = -2\omega$, then it is Sasakian.

Corollary

For a quasi-Sasakian $(N^{2n+1}, I, \xi, \eta, h)$, (J, g) on P is SKT if and only if Ω is I-invariant, $i_\xi\Omega = 0$ and

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$(N^{2n+1}, I, \xi, \eta, h)$ *admits* ∇^c *preserving* (I, ξ, η, h) *and with totally skew-symmetric torsion if and only if* $[I, I]$ *is skew-symmetric and* ξ *is a Killing vector field. Moreover,* ∇^c *is unique.*

For a quasi-Sasakian $(N^{2n+1}, I, \xi, \eta, h)$

$$h(\nabla^c_X Y, Z) = h(\nabla^g_X Y, Z) + \frac{1}{2}(d\eta \wedge \eta)(X, Y, Z).$$

Then ∇^B of (J, h) on P and ∇^c on N^{2n+1} are related by

$$g(\nabla^B_X Y, Z) = \pi^* h(\nabla^c_{\pi_* X} \pi_* Y, \pi_* Z),$$

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Riemannian cones

Let $(N^{2n+1}, I, \xi, \eta, h)$ be an an almost contact metric manifold.

Definition

The Riemannian cone of (N^{2n+1}, h) is $N^{2n+1} \times \mathbb{R}^+$ with $g = t^2 h + (dt)^2$.

$N^{2n+1} \times \mathbb{R}^+$ has an almost Hermitian (J, g) with $JX = IX, \; X \in \operatorname{Ker} \eta, \; J\xi = -t\frac{d}{dt}$.

Theorem (Boyer, Galicki)

$(N^{2n+1}, I, \xi, \eta, h)$ is Sasakian if and only if $(N^{2n+1} \times \mathbb{R}^+, J, g)$ is Kähler.

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$(N^{2n+1} \times \mathbb{R}^+, J, g)$ is SKT if and only if (I, ξ, η, h) is normal and $-4\eta \wedge \omega + 2I(d\omega) - 2d\eta \wedge \eta = d(I(i_\xi d\omega))$.

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Blow-ups

Theorem (Blanchard)

The blow-up of a Kähler manifold (M, J, g) at a point p or along a compact complex submanifold Y is still Kähler.

U= open set centered around p. The blow-up $\tilde{M}_p$ is obtained by adjoining to $M \setminus \{p\}$

$$\tilde{U} = \{(z, l) \in U \times \mathbb{C}P^{n-1} | z \in l\}.$$

by $\tilde{U} \setminus \{z = 0\} \cong U \setminus \{p\}$.

$\pi : \tilde{M}_p \to M$ with $\pi^{-1}(p) \cong \mathbb{C}P^{n-1}$.

$\pi^* F$ is $\partial\overline{\partial}$-closed, but it is not positive definite on $\pi^{-1}(M \setminus \{p\})$.

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The Chern curvature of L is $\hat{\omega} = 0$ on $M \setminus W$ and $\hat{\omega}|_{\mathbb{P}(\mathcal{N}_{Y|M})} = \omega$.

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The same theorem holds for manifolds with $\partial\overline{\partial} F = 0\,, \partial\overline{\partial} F^2 = 0$ ($\Rightarrow$ astheno-Kähler).

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If $M^{2n} \setminus \{p\}$ admits a Kähler metric, then there exists a Kähler metric on the complex manifold M^{2n}.

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Characterization in terms of currents

Let $\mathcal{D}^{p,q}(M)$ be the space of (p,q)-forms with compact support on (M,J).

Definition

The space of currents of bi-dimension (p,q) or of bi-degree $(n-p,n-q)$ is the topological dual $\mathcal{D}'_{p,q}(M)$ of $\mathcal{D}^{p,q}(M)$.

A current of bi-dimension (p,q) on M can be locally identified with a $(n-p,n-q)$-form on M with coefficients distributions.

A current T of bi-dimension (p,p) is real if $T(\varphi) = T(\overline{\varphi})$, for any $\varphi \in \mathcal{D}^{p,p}(M)$.

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Let $\mathcal{D}^{p,q}(M)$ be the space of (p,q)-forms with compact support on (M,J).

Definition

The space of currents of bi-dimension (p,q) or of bi-degree $(n-p,n-q)$ is the topological dual $\mathcal{D}'_{p,q}(M)$ of $\mathcal{D}^{p,q}(M)$.

A current of bi-dimension (p,q) on M can be locally identified with a $(n-p,n-q)$-form on M with coefficients distributions.

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Definition

A real $T \in \mathcal{D}'_{p,p}$ is strictly positive if $T(\frac{i^{p^2}}{2^p}\varphi^1 \wedge \ldots \wedge \varphi^p \wedge \overline{\varphi}^1 \wedge \ldots \wedge \overline{\varphi}^p) \geq 0$, for any $\varphi^j \in \mathcal{D}^{1,0}$ and $T(\frac{i^{p^2}}{2^p}\varphi^1 \wedge \ldots \wedge \varphi^p \wedge \overline{\varphi}^1 \wedge \ldots \wedge \overline{\varphi}^p) > 0$ if $\varphi^1 \wedge \ldots \wedge \varphi^p \neq 0$.

If F is the fundamental form of (J, g) on M, then F corresponds to a real strictly positive current of bi-degree $(1, 1)$.

Theorem (Egidi)

A compact (M, J) has a SKT metric if and only if there is no non-zero positive current of bi-dimension $(1, 1)$ which is $i\partial\overline{\partial}$-exact.

Theorem (–. Tomassini)

Let (M^{2n}, J), $n \geq 2$. If $M^{2n} \setminus \{p\}$ admits a SKT metric, then there exists a SKT metric on M^{2n}.

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Sketch of the proof

It is sufficient to show that if F is the fundamental 2-form of a SKT metric on $\mathbb{B}^n(r) \setminus \{0\}$, $n \geq 2$, then $\exists\, 0 < R < r$ and $\hat{F} \in \Lambda^{1,1}(\mathbb{B}^n(R))$ such that
i) $\hat{F}$ is the fundamental 2-form of a SKT metric on $\mathbb{B}^n(R)$;
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F= fundamental form of a SKT metric on $\mathbb{B}^n(r) \setminus \{0\}$ and set $T = -F$.

Theorem (Alessandrini, Bassanelli)

Y analytic subset in $\Omega \subset \mathbb{C}^n$. If T is a plurisubharmonic, negative current of bi-dimension (p, p) on $\Omega \setminus Y$ and $\dim_{\mathbb{C}} Y < p$, then $\exists$ the simple (or trivial) extension T^0 of T across Y and T^0 is plurisubharmonic.

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By Alessandrini and Bassanelli $T = -F$ can be extended as a current to $\mathbb{B}^n(r)$ by

$$T^0(\varphi) = \int_{\mathbb{B}^n(r)\setminus\{0\}} F \wedge \varphi, \quad \forall \varphi \in \mathcal{D}^{n-1,n-1}(\mathbb{B}^n(r)).$$

Set $F^0 = -T^0$.

Theorem (Siu, Bassanelli)

Let T be a current of bi-degree (h, k) on Ω. If T is of order 0 and $i\,\partial\overline{\partial}T = 0$, then, locally,

$$T = \partial G + \overline{\partial} H\,,$$

with G and H with locally integrable functions as coefficients.

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In fact, G is smooth on $\mathbb{B}^n(R) \setminus \{0\}$.

Finally, we can regularize G, in order that we obtain a $i\partial\overline{\partial}$-closed and positive $(1, 1)$-form on $\mathbb{B}^n(R)$.

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Resolution of orbifolds

Definition

A complex orbifold is a singular complex manifold M such that each singularity p is locally isomorphic to U/G, where $U \subset \mathbb{C}^n$, $G \subset \mathrm{GL}(n, \mathbb{C})$ finite subgroup with the only one fixed point p and real codim of $S \geq 2$.

Definition

A SKT resolution of a SKT orbifold (M, J, g) is a smooth $(\tilde{M}, \tilde{J})$ endowed with a $\tilde{J}$-Hermitian SKT metric $\tilde{g}$ and of a map $\pi : \tilde{M} \to M$, such that

(i) $\pi : \tilde{M} \setminus E \to M \setminus S$ is a biholomorphism, where $E = \pi^{-1}(S)$.

(ii) $\tilde{g} = \pi^* g$ on the complement of a neighborhood of E.

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By using the Hironaka result that the singularities of a complex variety algebraic can be resolved by a finite number of blow-ups.

Theorem (–, Tomassini)

Let (M, J) be a complex orbifold of complex dimension n endowed with a J-Hermitian SKT metric g. Then there exists a SKT resolution.

The same result holds for Hermitian orbifolds satisfying $\partial\overline{\partial}\, F = 0\,, \partial\overline{\partial}\, F^2 = 0$.

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Sketch of the proof:

$p \in S$: singular point of M . Take a chart $U_p = \mathbb{B}^n(r)/G_p$.

$X = \mathbb{C}^n/G_p$ is an affine algebraic variety which has 0 as the only singular point. By Hironaka there exists a resolution $\pi_X : \tilde{X} \to X$ which is a quasi-projective variety.

Let $E = \pi_X^{-1}(0)$ and $\tilde{U} = \pi_X^{-1}(U_p)$. By identifying $\tilde{U} \setminus E$ with $U_p \setminus \{p\}$, define $\tilde{M} = (M \setminus \{p\}) \cup \tilde{U}$.

It is possible then to define a SKT metric $\tilde{g}$ on $\tilde{M}$ such that $\tilde{g} = \pi_X^* g$ on the complement of a neighborhood of E.

$\tilde{F} = \pi_X^* F + \epsilon i \partial\overline{\partial}(h \iota^* \rho)$, where $h = 1$ on $\mathbb{B}^n(\frac{1}{3}r)/G_p$ and $h = 0$ on $\left(\mathbb{B}^n(r) \setminus \mathbb{B}^n(\frac{2}{3}r)\right)/G_p$.

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A simply-connected example

Let $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ and σ the involution on $\mathbb{T}^{2n}$ induced by

$$\sigma\left((x_1, \ldots, x_{2n})\right) = (-x_1, \ldots, -x_{2n}).$$

Consider on $\mathbb{T}^{2n}$ J defined by

$$\begin{cases} \eta^1 = dx_1 + i\,(f\,dx_n + dx_{n+1})\,, \\ \eta^j = dx_j + i\,dx_{n+j}\,, \quad j = 2, \ldots, n, \end{cases}$$

where $f = f(x_n, x_{2n})$ is a $\mathcal{C}^\infty$, $\mathbb{Z}^{2n}$-periodic and even function.

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Then

- $(\mathbb{T}^{2n}/\langle\sigma\rangle, J)$ is a complex orbifold with singular point set

$$S = \left\{ x + \mathbb{Z}^{2n} \mid x \in \frac{1}{2}\mathbb{Z}^{2n} \right\}.$$

- The J-Hermitian metric $g = \frac{1}{2}\sum_{j=1}^{n}\left(\eta^j \otimes \overline{\eta}^j + \overline{\eta}^j \otimes \eta^j\right)$ is SKT and $\partial\overline{\partial}F^2 = 0$ ($\Rightarrow$ astheno-kähler).
- The strong KT resolution is simply-connected.

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