Geometry of moduli spaces of Higgs bundles

Peter Gothen

Centro de Matemática da Universidade do Porto

Based on joint work with
S. Bradlow, O. García-Prada, and I. Mundet i Riera

XIX International Fall Workshop on Geometry and Physics
Porto, September 7, 2010

Why Higgs Bundles?

- Hyper-Kähler manifolds.
- Integrable systems: the Hitchin system.
- Mirror symmetry and geometric Langlands.
- Non-abelian Hodge theory.
- Geometric structures and representations of fundamental groups.
- ...

Today:

- Higgs bundles on Riemann surfaces (rather than higher dimensional complex varieties);
- *G-Higgs bundles*, where G is real Lie group.

What is a Higgs bundle?

- X – closed Riemann surface of genus g;
- $K_X = T^{*1,0}X$ – the holomorphic cotangent bundle of X.

Definition

A *Higgs bundle* on X is a pair (E, ϕ), where

- $E \to X$ is holomorphic vector bundle;
- ϕ is a holomorphic 1-form with values in $\mathrm{End}(E)$, i.e., $\phi \in H^0(X, \mathrm{End}(E) \otimes K_X)$.

Discrete invariants: $d = \deg(E)$, $n = \mathrm{rk}(E)$.

Example

A rank one Higgs bundle is a pair (L, α), where $L \to$ is a line bundle and $\alpha \in H^0(X, K_X)$ is a holomorphic 1-form.

Abelian Hodge Theory

Harmonic theory shows that $H^1(X,\mathbb{C}) \cong H^{1,0}(X) \oplus H^{0,1}(X)$.
In fact, this is the infinitesimal version of an isomorphism

$$H^1(X,\mathbb{C}^*) \cong T^* \operatorname{Jac}(X):$$

a flat complex line bundle $\mathcal{L}$ is given by a closed $B \in A^1(X,\mathbb{C})$ which, after a gauge transformation, has harmonic $(1,0)$ and $(0,1)$-parts.
Using $\mathbb{C} = \mathfrak{gl}(1,\mathbb{C}) = \mathfrak{u}(1) \oplus \mathbb{R}$, write

$$B = A + \theta, \qquad A \in A^1(X, i\mathbb{R}) \quad \text{and} \quad \theta \in A^1(X,\mathbb{R}),$$

then $d_A\theta = 0 \iff \bar{\partial}_A\theta^{1,0} = 0$, so we obtain a rank 1 Higgs bundle:

- $\bar{\partial}_A$ defines a *holomorphic line bundle* L_A.
- $\alpha := \theta^{1,0} \in H^{1,0}(X) \cong H^0(X, K_X)$ is the *Higgs field*.

Note:

- $T^* \operatorname{Jac}(X) \ni (L_A, \alpha)$ is the moduli space of rank 1, degree 0 Higgs bundles.
- $H^1(X,\mathbb{C}^*) \cong \operatorname{Hom}(\pi_1 X, \mathbb{C}^*)$.

Non-abelian Hodge Theory — 1

- Let $\mathcal{E} \to X$ be a fixed C^∞ complex vector bundle of rank n and degree d.

Theorem (Donaldson, Corlette)

Let B be a flat reductive connection in $\mathcal{E}$. Then there exists a **harmonic metric** *in $\mathcal{E}$, meaning the following:*
Writing $B = A + \theta$, where A is a unitary connection and θ is a self adjoint 1-form, define $\phi \in H^0(X, \mathrm{End}(\mathcal{E}) \otimes K_X)$ by $\theta = \phi + \phi^*$. Then (A, ϕ) satisfies **Hitchin's equations:**

$$F(A) + [\phi, \phi^*] = 0.$$
$$\bar{\partial}_A \phi = 0.$$

Note: Let $E = E_A \to X$ be the holomorphic bundle given by the operator $\bar{\partial}_A$ on $\mathcal{E}$. Then (E_A, ϕ) is a Higgs bundle which is *polystable* in the following sense:

Non-abelian Hodge Theory — 2

Definition

The *slope* of E is $\mu(E) = \deg(E)/\operatorname{rk}(E) = n/d$. A Higgs bundle (E, ϕ) is

- *semistable* if $\mu(F) \leqslant \mu(E)$ for all $F \subset E$ such that $\phi(F) \subset F \otimes K_X$.
- *stable* if $\mu(F) < \mu(E)$ for all $0 \neq F \subsetneq E$ such that $\phi(F) \subset F \otimes K_X$.
- *polystable* if $(E, \phi) = (E_1, \phi_1) \oplus \cdots \oplus (E_r, \phi_r)$, where each (E_i, ϕ_i) is stable with $\mu(E_i) = \mu(E)$.

Theorem (Hitchin, Simpson)

(E, ϕ) is polystable iff there exists a hermitean (harmonic!) metric in E such that (A, ϕ) satisfies Hitchin's equations (A is the Chern connection).

Non-abelian Hodge Theory — 3

Notation:

- $\mathcal{M}^d(X, \mathrm{GL}(n,\mathbb{C}))$ — moduli space of rank n, degree d Higgs bundles.
- $\mathcal{M}^d_{\mathrm{dR}}(X, \mathrm{GL}(n,\mathbb{C}))$ — moduli space of (projectively) flat *reductive* connections on $\mathcal{E}$ modulo gauge equivalence.

Together, the two preceding theorems imply:

Theorem (Non-abelian Hodge Theorem)

There is a homeomorphism $\mathcal{M}^d_{\mathrm{dR}}(X, \mathrm{GL}(n,\mathbb{C})) \cong \mathcal{M}^d(X, \mathrm{GL}(n,\mathbb{C}))$.

Note:

$\mathcal{M}^0_{\mathrm{dR}}(X, \mathrm{GL}(n,\mathbb{C})) \cong \mathrm{Hom}^{\mathrm{reductive}}(\pi_1(X), \mathrm{GL}(n,\mathbb{C}))/\mathrm{GL}(n,\mathbb{C})$.

(For non-zero values of d, use representations of a central extension of $\pi_1(X)$.)

G-Higgs bundles

G-Higgs bundles are introduced in order to study representations of $\pi_1(X)$ in groups other than $\mathrm{GL}(n, \mathbb{C})$.

- G — real reductive Lie group; maximal compact subgroup $H \subset G$.
- Cartan decomposition: $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$.
- Restriction of adjoint action gives *isotropy representation:* $\iota \colon H \to \mathfrak{m}$.

Definition

A *G-Higgs bundle* is a pair (E, ϕ), where

- $E \to X$ is a holomorphic principal $H^{\mathbb{C}}$-bundle,
- $\phi \in H^0(X, E(\mathfrak{m}^{\mathbb{C}}) \otimes K_X)$. Here $E(\mathfrak{m}^{\mathbb{C}}) = E \times_\iota \mathfrak{m}^{\mathbb{C}}$.

Example

$G = \mathrm{GL}(n, \mathbb{C})$. A G-Higgs bundle is a Higgs bundle as previously defined.

More examples of G-Higgs bundles

- $G = \mathrm{Sp}(2n, \mathbb{R})$. In this case: $H = \mathrm{U}(n)$ and $\mathfrak{m}^{\mathbb{C}} = S^2\mathbb{C}^n \oplus S^2(\mathbb{C}^n)^*$. Hence a $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundle is given by (V, β, γ), where V is a rank n vector bundle and

  $$\beta \in H^0(X, S^2V \otimes K_X), \qquad \gamma \in H^0(X, S^2V^* \otimes K_X).$$

  The usual Higgs bundle given by the inclusion $\mathrm{Sp}(2n, \mathbb{R}) \subset \mathrm{Sp}(2n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{C})$ is:

  $$(E = V \oplus V^*, \phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}).$$

- A $\mathrm{U}(p, q)$-Higgs bundle is given by $(E = V \oplus W, \phi = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix})$.
- A $\mathrm{GL}(n, \mathbb{R})$-Higgs bundle is given by (W, Q, θ), where (W, Q) is an orthogonal bundle and $\theta \colon W \to W \otimes K_X$ is symmetric with respect to Q.

Stability and non-abelian Hodge Theorem for G-Higgs bundles

Stability of G-Higgs bundles is in general a complicated notion.
But when the complexification $G^{\mathbb{C}} \subset \mathrm{GL}(n, \mathbb{C})$ is a linear group, polystability of (E, ϕ) is equivalent to polystability of the induced rank n usual Higgs bundle.

Theorem (Non-abelian Hodge Theorem)

There is a homeomorphism $\mathcal{M}^d_{\mathrm{dR}}(X, G) \cong \mathcal{M}^d(X, G)$, where $\mathcal{M}^d_{\mathrm{dR}}(X, G)$ is the moduli space of gauge equivalence classes of projectively flat G-bundles and $\mathcal{M}^d(X, G)$ is the moduli space of polystable G-Higgs bundles.

Notes:

- d is the topological class of the fixed underlying smooth principal bundle; if G is connected then $d \in \pi_1(G)$.
- $\mathcal{M}^0_{\mathrm{dR}}(X, G) \cong \mathrm{Hom}^{\mathrm{reductive}}(\pi_1(X), G))/G$.

Topological bounds for stable $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles

Let (V,β,γ) be a $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle, i.e.,

- $V \to X$ is a rank n vector bundle;
- $\beta \in H^0(X, S^2V \otimes K_X), \qquad \gamma \in H^0(X, S^2V^* \otimes K_X)$.

Let $N = \ker(\gamma) \subset V$ and let $I \subset V^*$ be the saturation of $\mathrm{im}(\gamma) \otimes K_X^{-1}$. Then

$$N \subset V \oplus V^* \qquad \text{and} \qquad V \oplus I \subset V \oplus V^*$$

are Φ-invariant subbundles of $(E = V \oplus V^*, \Phi = \left(\begin{smallmatrix} \beta & 0 \\ 0 & \gamma \end{smallmatrix}\right))$.

Applying the semistability condition one obtains:

Proposition

Let (V,β,γ) be a semistable $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundle. Then $d \leqslant n(g-1)$.

Moreover, if $d = n(g-1)$ then $\gamma\colon V \to V^* \otimes K_X$ is an isomorphism. □

(Analogously, considering β, one obtains $-d \leqslant n(g-1)$.)

Topological bounds for representations: Milnor–Wood inequality

The degree $d = \deg(V)$ is just the topological degree of the flat $\mathrm{Sp}(2n,\mathbb{R})$-bundle associated via the Non-abelian Hodge Theorem.

In terms of the corresponding representation $\rho\colon \pi_1(X) \to \mathrm{Sp}(2n,\mathbb{R})$, this number is known as the *Toledo invariant* and the inequality

$$|\tau(\rho)| \leqslant n(g-1)$$

is known as the *Milnor–Wood inequality* [Milnor ($n=1$), Dupont, Turaev].

Definition

$\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles for which equality holds – and the corresponding representations of $\pi_1(X)$ – are said to be maximal.

Remark

Generalizations to other isometry groups of hermitean symmetric spaces (of non-compact type) exist.

Relation with Fuchsian representations

Theorem (Goldman 1980)

$\rho\colon \pi_1 X \to \mathrm{Sp}(2,\mathbb{R})$ is maximal if and only if it is Fuchsian, i.e., discrete and faithful

Recall: a Fuchsian representation induces a hyperbolic structure on X:

$$\mathbb{H}^2 \to X = \mathbb{H}^2/\pi_1 X,$$

And *Fricke space* is the space of all such representations:

$$\mathfrak{F} = \{[\rho] \in \mathcal{R}(\pi_1 X, \mathrm{PSp}(2,\mathbb{R})) \mid \rho \text{ is Fuchsian}\}.$$

Remark

(1) $\mathrm{Isom}(\mathbb{H}^2) \cong \mathrm{PSp}(2,\mathbb{R}) = \mathrm{Sp}(2,\mathbb{R})/\{\pm 1\}$;
any Fuchsian $\rho\colon \pi_1 X \to \mathrm{PSp}(2,\mathbb{R})$ lifts to $\mathrm{Sp}(2,\mathbb{R})$ and there are 2^{2g} such lifts, each corresponding to the choice of a spin structure s on X.
(2) By the Uniformization Theorem, $\mathfrak{F}$ can be identified with the Teichmüller space of the Riemann surface X.

Maximal $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles

Assume $d = n(g-1)$ so $\gamma\colon V \xrightarrow{\cong} V^* \otimes K_X$.
Define

- $W = V \otimes K_X^{-1/2}$,
- $Q = \gamma \otimes 1_{K^{-1/2}}\colon W \xrightarrow{\cong} W^*$, and
- $\theta = (\beta \otimes 1) \circ (\gamma \otimes 1)\colon W \to W \otimes K_X^2$.

Then Q is a non-degenerate quadratic form on W and θ is symmetric with respect to Q. Hence:

(W, Q, θ) is a *twisted* $\mathrm{GL}(n, \mathbb{R})$-Higgs bundle — the difference to a usual $\mathrm{GL}(n, \mathbb{R})$-Higgs bundle being the twisting by K_X^2 rather than K_X.

New invariants

Recall that (W, Q) is an orthogonal bundle.

$\Longrightarrow$ New invariants of representations / $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles:

- $w_1(V, \beta, \gamma) \in H^1(X, \mathbb{Z}/2)$.
- $w_2(V, \beta, \gamma) \in H^2(X, \mathbb{Z}/2)$.

Remark

- In the case $n = 1$: $w_2 = 0$.
- In the case $n = 2$: when $w_1 = 0$, there is a lift of w_2 to $c \in H^2(X, \mathbb{Z})$, satisfying $|c| \leqslant 2g - 2$.

Notation: Let $\mathcal{M}^d := \mathcal{M}^d(X, \mathrm{Sp}(2n, \mathbb{R}))$ be the moduli space of $\mathrm{Sp}(2n, \mathbb{R})$-Higgs bundles with $\deg(V) = d$ and let $\mathcal{M}^{\max} := \mathcal{M}^{n(g-1)}$.

Connected components of $\mathcal{M}^{\max}$ ($n = 1$)

The following was proved in terms of representations of $\pi_1(X)$:

Theorem (Goldman 1988)

In the case $n = 1$, the connected components of $\mathcal{M}^{\max}$ are the subspaces

$$\mathcal{M}_{w_1} \subset \mathcal{M}^{\max}, \qquad w_1 \in H^1(X, \mathbb{Z}/2),$$

of Higgs bundles having invariant w_1. □

Remark

These components become identified in the moduli space of $\mathrm{PSp}(2, \mathbb{R})$-Higgs bundles and are all homeomorphic to the Fricke (Teichmüller) space $\mathfrak{F}$.
In particular each component $\mathcal{M}_{w_1}$ is homeomorphic to $\mathbb{R}^{6g-6}$.

Goldman's theorems following Hitchin

Let (L, β, γ) be a maximal $\mathrm{Sp}(2, \mathbb{R})$-Higgs bundle. Then

- $L \to X$ is a line bundle of degree $d = g - 1$ and
- $\gamma \in H^0(X, L^{-2}K_X)$ is a non-vanishing section.

Hence $L^{-2}K_X \cong \mathcal{O}_X$, i.e., L is a spin structure.

This spin structure is the one given by the invariant $w_1(L, \beta, \gamma)$.

Note that $L^2 K_X = K_X^2$.

Thus, for any $w_1 \in H^1(X, \mathbb{Z}/2)$,

$$\mathcal{M}_{w_1} = \{\beta \in H^0(X, L^2 K_X)\} = H^0(X, K_X^2).$$

Hitchin components

In analogy with the parametrization $\mathcal{M}_{w_1} \cong H^0(X, K_x^2)$, Hitchin showed the existence of special connected Hitchin components

$$\mathcal{M}^H \subset \mathcal{M}(X, G),$$

when G is a split real form of a simple complex group (classical examples: $G = \mathrm{SL}(n, \mathbb{R})$, $\mathrm{Sp}(2n, \mathbb{R})$, $\mathrm{SO}(n, n)$, $\mathrm{SO}(n+1, n)$).

Hitchin components are parametrized by

$$\mathcal{M}^H \cong \bigoplus H^0(X, K_X^{m_i+1})$$

In the case $G = \mathrm{Sp}(2n, \mathbb{R})$:

- Hitchin components are maximal;
- Under the corresondence with representations $\rho\colon \pi_1(X) \to \mathrm{Sp}(2n, \mathbb{R})$ is a *Hitchin representation* if and only if it can be deformed to a representation of the form $r_{\mathrm{irr}} \circ \rho_0$ with ρ_0 Fuchsian. (Here $r_{\mathrm{irr}}\colon \mathrm{Sp}(2, \mathbb{R}) \to \mathrm{Sp}(2n, \mathbb{R})$ is the irreducible representation given by the action on homogeneous polynomials of degree $2n - 1$.)

Connected components of $\mathcal{M}^{\max}$ for $n \geqslant 2$

Recall the invariants of maximal $\mathrm{Sp}(2n,\mathbb{R})$-Higgs bundles described earlier:

- $w_1(V,\beta,\gamma) \in H^1(X,\mathbb{Z}/2)$, $w_2(V,\beta,\gamma) \in H^2(X,\mathbb{Z}/2)$.
- In the case $n = 2$: when $w_1 = 0$, there is a lift of w_2 to $c \in H^2(X,\mathbb{Z})$, satisfying $|c| \leqslant 2g-2$.

Moreover, it turns out that there are 2^{2g} Hitchin components $\mathcal{M}^H_s$ of $\mathcal{M}^{\max}$, indexed by spin structures $s \in H^1(X,\mathbb{Z}/2)$.

Theorem (G., García-Prada–G.–Mundet i Riera)

For $n = 2$, the decomposition in connected components of $\mathcal{M}^{\max}$ is

$$\mathcal{M}^{\max} = \bigcup_{w_1 \neq 0, w_2} \mathcal{M}_{w_1,w_2} \cup \bigcup_{0 \leqslant c < 2g-2} \mathcal{M}_{0,c} \cup \bigcup_{s} \mathcal{M}^H_s.$$

For $n \geqslant 3$, the decomposition in connected components of $\mathcal{M}^{\max}$ is

$$\mathcal{M}^{\max} = \bigcup_{w_1, w_2} \mathcal{M}_{w_1,w_2} \cup \bigcup_{s} \mathcal{M}^H_s.$$

Components of $\mathcal{M}^{\max}(X, G)$

A lot of the preceding can be generalized to isometry groups of (classical) hermitean symmetric spaces of non-compact type.
Again, new invariants permit the detection of "extra" connected components of $\mathcal{M}^{\max}(G)$.

Theorem (Bradlow–García-Prada-G., García-Prada-G.–Mundet)

The number of connected components of $\mathcal{M}^{\max}(X, G)$ is given by the following table:

G	$\#\pi_0(\mathcal{R}_{\max}(G))$	Hitchin Components
$\mathrm{SU}(n, n)$	2^{2g}	– $(2^{2g}$ if $n = 1)$
$\mathrm{SU}(p, q) \quad (p \neq q)$	1	–
$\mathrm{Sp}(2n, \mathbb{R}) \quad (n \geqslant 3)$	$3 \cdot 2^{2g}$	2^{2g}
$\mathrm{SO}_0(2, n) \quad (n \geqslant 4)$	2^{2g+1}	–
$\mathrm{SO}^*(2n)$	1	–

Comments on the Theorem

- The Theorem contains important special cases, due to Goldman, Hitchin, Markman–Xia, Xia.
- Special low dimensional cases:

G	$\#\pi_0(\mathcal{R}_{\max}(G))$	Hitchin Components
$\mathrm{Sp}(4,\mathbb{R}) \cong \mathrm{Spin}_0(2,3)$	$3 \cdot 2^{2g} + 2g - 4$	2^{2g}
$\mathrm{SO}_0(2,3)$	$2^{2g+1} + 4g - 5$	1
$\mathrm{Sp}(2,\mathbb{R}) \cong \mathrm{SL}(2,\mathbb{R})$	2^{2g}	2^{2g}
$\mathrm{SO}_0(2,1) \cong \mathrm{PSL}(2,\mathbb{R})$	1	1

- Representations in $\mathcal{M}^{\max}(X,G)$ have special geometric properties. For example:
  - Maximal representations are discrete, faithful and reductive [Burger–Iozzi–Wienhard 2003].
  - The action of the mapping class group on $\mathcal{R}_{\max}(\pi_1 X, \mathrm{Sp}(2n,\mathbb{R}))$ is proper [Labourie 2005, Wienhard 2006].

Deformation of maximal representations in $\mathrm{Sp}(4, \mathbb{R})$

Question [W. Goldman]: When can a maximal $\mathrm{Sp}(2n, \mathbb{R})$-representation of a surface group be deformed to a representation which factors through a proper reductive subgroup G_* of $\mathrm{Sp}(2n, \mathbb{R})$?

Consider the following representations:

- $r_{\mathrm{irr}} \colon \mathrm{Sp}(2, \mathbb{R}) \to \mathrm{Sp}(4, \mathbb{R})$ — the irreducible representation.
- $r_{\mathrm{p}} \colon \mathrm{Sp}(2, \mathbb{R}) \times \mathrm{Sp}(2, \mathbb{R}) \to \mathrm{Sp}(4, \mathbb{R})$ — the product representation.
- $r_{\Delta} \colon \mathrm{Sp}(2, \mathbb{R}) \to \mathrm{Sp}(4, \mathbb{R})$ — the diagonal representation.

Let $G_* = N_{\mathrm{Sp}(4,\mathbb{R})}(\mathrm{image}(r_*))$ for $* = \mathrm{irr}, \mathrm{p}, \Delta$.

Deformation of maximal representations in $\mathrm{Sp}(4,\mathbb{R})$ – 2

Theorem (Bradlow–García-Prada–G.)

Let $\rho\colon \pi_1 X \to \mathrm{Sp}(4,\mathbb{R})$ be a maximal representation. Then ρ can be deformed to a representation which factors through

- G_{irr} *if and only if* $\rho \in \mathcal{M}_s^H$;
- G_p *if and only if* $\rho \in \mathcal{M}_{w_1,w_2}$ *or* $\mathcal{M}_{0,0}$;
- G_Δ *if and only if* $\rho \in \mathcal{M}_{w_1,w_2}$ *or* $\mathcal{M}_{0,0}$.

Remark

(1) Guichard–Wienhard have recently constructed "model representations" in $\mathcal{M}_{0,c}$ by amalgamating r_{irr} and r_Δ on separate parts of the surface X.

(2) Let $n \geqslant 3$. Then any maximal representation of $\pi_1(S)$ in $\mathrm{Sp}(2n,\mathbb{R})$ can be deformed to one which factors through a proper reductive Zariski closed subgroup of $\mathrm{Sp}(2n,\mathbb{R})$.

Sketch of proof

Idea: Analyse G_*-Higgs bundles and see where they live. Example:

$$G_p = N_{\mathrm{Sp}(4,\mathbb{R})}(\mathrm{Sp}(2,\mathbb{R}) \times \mathrm{Sp}(2,\mathbb{R})) = (\mathrm{Sp}(2,\mathbb{R}) \times \mathrm{Sp}(2,\mathbb{R})) \rtimes \mathbb{Z}/2,$$

where $\mathbb{Z}/2 \cong \{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)\}$ and $H_p = (\mathrm{U}(1) \times \mathrm{U}(1)) \rtimes \mathbb{Z}/2$.

If a maximal G_p-Higgs bundle (V, β, γ) reduces to $\mathrm{Sp}(2,\mathbb{R}) \times \mathrm{Sp}(2,\mathbb{R})$ it is of the form

$$V = L_1 \oplus L_2, \quad L_i^2 = K_X, \quad \beta = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{pmatrix},$$

where $\beta_i \in H^0(X, L_i^2 \otimes K_X)$ and $\gamma_i = 1 \in (X, L_i^{-2} \otimes K_X)$.

It follows that $\mathcal{M}_{w_1,w_2}$ and $\mathcal{M}_{0,0}$ are the components containing $\mathrm{Sp}(2,\mathbb{R}) \times \mathrm{Sp}(2,\mathbb{R})$-Higgs bundles.

If there is no such reduction, the obstruction defines a double cover $X' \to X$ on which (V, β, γ) reduces: this gives the desired conclusion.

Some further reading

[1] S. B. Bradlow, O. García-Prada, and P. B. Gothen, *Surface group representations and* $\mathrm{U}(p,q)$*-Higgs bundles*, J. Differential Geom. **64** (2003), 111–170.

[2] ———, *Maximal surface group representations in isometry groups of classical hermitian symmetric spaces*, Geometriae Dedicata **122** (2006), 185–213.

[3] ———, *Deformations of maximal representations in* $\mathrm{Sp}(4,\mathbb{R})$, preprint, 2009, `arXiv:0903.5496 [math.AG]`.

[4] Oscar García-Prada, Peter B. Gothen, and Ignasi Mundet i Riera, *The Hitchin-Kobayashi correspondence, Higgs pairs and surface group representations*, preprint, 2009, `arXiv:0909.4487 [math.AG]`.

[5] ———, *Representations of surface groups in the real symplectic group*, preprint, 2009, `arXiv:0809.0576 [math.AG]`.