Exotic automorphisms of the Schouten algebra of polyvector fields

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Poisson brackets

Let f and g be arbitrary differentiable functions on $\mathbb{R}^{2n} = \{q^1, \ldots, q^n, p_1, \ldots, p_n\}$.
In [*J. de l'École* **8** *(1809), p.209*] Poisson introduced the expression,

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i} \right)$$

which is antisymmetric, $\{f, g\} = -\{g, f\}$, obeys the Jacobi identity,

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0, \quad \forall\ f, g, h \in C^\infty(\mathbb{R}^{2n}),$$

and is a derivation on $C^\infty(\mathbb{R}^{2n})$ for one of the inputs (say, g) fixed,

$$\{f_1 f_2, g\} = f_1\{f_2, g\} + f_2\{f_1, g\}, \quad \forall\ f_1, f_2, h \in C^\infty(\mathbb{R}^{2n}).$$

They are called *Poisson brackets*.

Poisson structures on $\mathbb{R}^d$

One can attempt to generalize original Poisson brackets to arbitrary (not necessarily even) dimensional space $\mathbb{R}^d = \{x^1, \dots, x^d\}$ by setting

$$\{f, g\} = \sum_{i,j=1}^{d} \pi^{ij}(x) \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j} \quad \text{for some } \pi^{ij}(x) \in C^\infty(\mathbb{R}^d).$$

Then all the properties of $\{\ ,\ \}$ are preserved provided

$$\pi^{ij}(x) = -\pi^{ji}(x)$$

$$\sum_{l=1}^{d} \left(\pi^{il} \partial_l \pi^{jk} + \pi^{kl} \partial_l \pi^{ij} + \pi^{kl} \partial_l \pi^{ij} \right) = 0, \quad \forall\ i, j, k.$$

Such a collection of functions, $\{\pi^{ij}(x)\}_{1 \le i < j \le d}$, is called a Poisson structure on $\mathbb{R}^d$. Denote the set of such structures by $\mathit{Poisson}(\mathbb{R}^d)$.

Poisson manifolds

The group $Diff(\mathbb{R}^d)$,

$$x^i \longrightarrow \hat{x}^i = f^i(x), \quad \det\left(\frac{\partial f^i(x)}{\partial x^j}\right) \neq 0, \quad f^i(x) \in C^\infty(\mathbb{R}^d)$$

acts on $Poisson(\mathbb{R}^d)$ by the formula,

$$\pi^{ij}(x) \longrightarrow \hat{\pi}^{ij}(\hat{x}) := \frac{\partial f^i(x)}{\partial x^p}\frac{\partial f^j(x)}{\partial x^q}\pi^{pq}(x)|_{x=f^{-1}(\hat{x})}.$$

Hence the tensor $\pi(x) := \pi^{ij}(x)\partial_i \wedge \partial_j$, as well as the basic equation

$$\sum_{l=1}^{d}\left(\pi^{il}\partial_l\pi^{jk} + \pi^{kl}\partial_l\pi^{ij} + \pi^{kl}\partial_l\pi^{ij}\right) = 0, \quad \forall\ i, j, k.$$

have $Diff(\mathbb{R}^n)$-invariant meaning and hence can be defined on arbitrary manifold. This leads to the notion of Poisson manifold.

Poisson structures as Maurer-Cartan elements

Introduce a degree 1 formal variable, $\psi_i := \Pi \frac{\partial}{\partial x^i}$, and consider a graded commutative algebra

$$\mathcal{T}_{poly}(\mathbb{R}^d) = C^\infty(\mathbb{R}^d)[[\psi_1, \ldots, \psi_d]] \subset \mathbb{K}[[x^1, \ldots, x^d, \psi_1, \ldots, \psi_d]]$$

$$x^i x^j = x^j x^i, \qquad \psi_i \psi_j = -\psi_j \psi_i$$

equipped with Poison type degree -1 brackets,

$$[f(x,\psi) \bullet g(x,\psi)] := \sum_{i=1}^{d} \frac{\partial f}{\partial x^i} \frac{\partial g}{\psi_i} + (-1)^{|f|} \frac{\partial f}{\partial x^i} \frac{\partial g}{\psi_i}$$

Then

$$Poisson(\mathbb{R}^d) = \{\pi \in \mathcal{T}_{poly}(\mathbb{R}^d) |\ |\pi| = 2 \ \&\ [\pi \bullet \pi] = 0\},$$

and $Diff(\mathbb{R}^d)$ acts on the Schouten algebra $(\mathcal{T}_{poly}(\mathbb{R}^d), [\ \bullet\ ])$ as automorphisms via $x \to f^i(x)$, $\psi_i \to \sum_{j=1}^{d} (\frac{\partial f(x)}{\partial x}^{-1})^j_i \psi_j$.

Action of GT on $(\mathcal{T}_{poly}(\mathbb{R}^d)[[\hbar]], [\bullet])$

Let $\mathcal{D}^\bullet_{poly}(\mathbb{R}^d)$ be the Hochschild dg Lie algebra of polydifferential operators on smooth (formal) functions on $\mathbb{R}^d$. Tamarkin proved existence of a family of L_∞-quasi-isomorphisms,

$$\left\{F_a : \mathcal{D}^\bullet_{poly}(\mathbb{R}^d) \longrightarrow \wedge^\bullet \mathcal{T}_{poly}(\mathbb{R}^d)\right\}_{a \in \mathcal{A}},$$

parameterized by the set, $\mathcal{A}$, of all possible Drinfeld's Lie associators. According to Drinfeld, the Grothendieck-Teichmueller group, GT, acts on $\mathcal{A}$ and hence on the above family, $\{F_a\}$, of formality maps. This in turn defines a map,

$$\begin{array}{rccc} \rho : & GT & \longrightarrow & Aut(\mathcal{T}_{poly}(\mathbb{R}^d)[[\hbar]]) \\ & G & \longrightarrow & F_{G(a)} \circ F_a^{-1}, \end{array}$$

where $F_a^{-1} : \mathcal{T}_{poly}(\mathbb{R}^d) \to \mathcal{D}_{poly}(\mathbb{R}^d)$ is a L_∞-morphism which is homotopy inverse to F_a.

Conjecture: ρ is homotopy non-trivial.

A family of exotic automorphisms of $Poisson(\mathbb{R}^d)$

$$\begin{array}{rccc} \mathcal{F}: & Poisson(\mathbb{R}^d) & \longrightarrow & Poisson(\mathbb{R}^d) \\ & \pi & \longrightarrow & \mathcal{F}(\pi) = \pi + \sum_{n\geq 2} \hbar^{n-1}\mathcal{F}_n(\pi) \end{array}$$

where

$$\mathcal{F}_n(\pi) := \sum_{\Gamma \in \mathfrak{G}_{n,2n-2}} \frac{C_\Gamma}{\#Aut(\Gamma)} \Phi_\Gamma(\underbrace{\pi, \ldots, \pi}_{n})$$

- $\mathfrak{G}_{n,2n-2}$ is family of graphs with n vertices and $2n-2$ edges,
- $\#Aut(\Gamma)$ is the cardinality of the group $Aut(\Gamma)$,
- $C_\Gamma \in \mathbb{C}$ depends on Γ and a choice of a propagator ω,
- $\Phi_\Gamma : \otimes^n \mathcal{T}_{poly}(\mathbb{R}^d) \longrightarrow \mathcal{T}_{poly}(\mathbb{R}^d)$ is a polydifferential operator of total order $2n-2$.

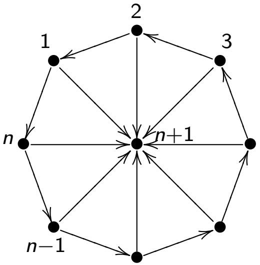
Next we give precise definitions of every item.

$\mathcal{F}(\pi) = \pi + \sum_{n\geq 2} \hbar^{n-1} \sum_{\Gamma \in \mathfrak{G}_{n,2n-2}} \frac{c_\Gamma}{\#Aut(\Gamma)} \Phi_\Gamma(\pi)$

$\mathfrak{G}_{n,l}$ stands for a family of graphs, $\{\Gamma\}$, with n vertices and l edges such that

- the edges of Γ are directed, beginning and ending at *different* vertices;
- the set of vertices, $V(\Gamma)$, is labeled by the set $[n]$;
- the set of edges, $E(\Gamma)$, is totally ordered (up to an even permutation).

Example : $w_n =$



, $Aut(w_n) = \mathbb{Z}_n$

$\mathcal{F}(\pi) = \pi + \sum_{n\geq 2} \hbar^{n-1} \sum_{\Gamma\in\mathfrak{G}_{n,2n-2}} \frac{C_\Gamma}{\#Aut(\Gamma)} \Phi_\Gamma(\pi,\ldots,\pi)$

With every graph $\Gamma \in \mathfrak{G}_{n,l}$ one can associate a linear map,

$$\begin{array}{cccc} \Phi_\Gamma : & \otimes^n \mathcal{T}_{poly}(\mathbb{R}^d) & \longrightarrow & \mathcal{T}_{poly}(\mathbb{R}^d)[-l] \\ & \gamma_1 \otimes \ldots \otimes \gamma_n & \longrightarrow & \Phi_\Gamma(\gamma_1,\ldots,\gamma_n) \end{array}$$

where

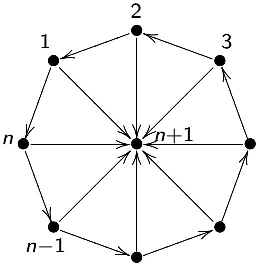
$$\Phi_\Gamma(\gamma_1,...,\gamma_n) = \left[\left(\prod_{e\in E(\Gamma)} \Delta_e\right) \gamma_1(\psi_{(1)}, x_{(1)}) \cdots \gamma_n(\psi_{(n)}, x_{(n)})\right]_{\substack{x_{(1)}=\ldots=x_{(n)} \\ \psi_{(1)}=\ldots=\psi_{(n)}}}$$

Here, for an edge $e = \overset{i}{\bullet}\!\longrightarrow\!\overset{j}{\bullet}$ we set

$$\Delta_e := \sum_{a=1}^{d} \frac{\partial^2}{\psi_{(i)\,a} \partial x^a_{(j)}}.$$

$$\mathcal{F}(\pi) = \pi + \sum_{n\geq 2} \hbar^{n-1} \sum_{\Gamma\in\mathfrak{G}_{n,2n-2}} \frac{c_\Gamma}{\#Aut(\Gamma)} \Phi_\Gamma(\pi, \ldots, \pi)$$

Example: for arbitrary bivector $\pi := \sum_{i,j=1}^{d} \frac{1}{2}\pi^{ij}(x)\psi_i\psi_j$ and the n-wheel (with the total ordering of its edges chosen to be $\{(1,2),(2,3),\ldots,(n-1,n),(1,n+1),\ldots,(n,n+1)\}$),

$w_n =$ 

one easily computes

$$\Phi_{w_n}(\pi,\ldots,\pi) = \frac{(-1)^{1+\frac{n(n-1)}{2}}}{2} \sum \frac{\partial^n \pi^{ij}}{\partial x^{k_1}\ldots\partial x^{k_n}} \frac{\partial \pi^{k_1 l_1}}{\partial x^{l_2}} \frac{\partial \pi^{k_2 l_2}}{\partial x^{l_3}} \cdots \frac{\partial \pi^{k_n l_n}}{\partial x^{l_1}} \psi_i \psi_j .$$

$\mathcal{F}(\pi) = \pi + \sum_{n\geq 2} \hbar^{n-1} \sum_{\Gamma\in\mathfrak{G}_{n,2n-2}} \frac{C_\Gamma}{\#Aut(\Gamma)} \Phi_\Gamma(\pi,\ldots,\pi)$

The numerical coefficient, C_Γ, is given by an integral

$$C_\Gamma = \int_{\widehat{C}_{n,0}} \bigwedge_{e\in Edges(\Gamma)} \frac{\mathfrak{p}_e^*(\omega)}{2\pi},$$

over a compactification, $\widehat{C}_{n,0}$, of the following configuration space of n points in the upper half plane,

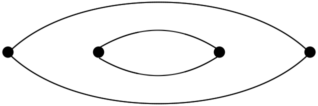
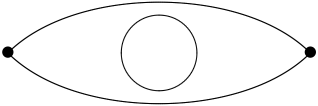
$$C_{n,0} := \frac{\{z_1,\ldots,z_n \in \mathbb{H} \,|\, z_i \neq z_j \text{ for } i\neq j\}}{\{z \to az+b \mid a,b\in\mathbb{R}, a>0\}}.$$

The open space $C_{n,0}$ was introduced by Kontsevich (1997).

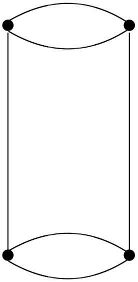
Our compactification, $\widehat{C}_{n,0}$, of $C_{n,0}$ is different from Kontsevich's one, $\overline{C}_{n,0}$ except the cases $n=1$ and $n=2$.

Remark: The product $C_\Gamma\Phi_\Gamma$ is independent of the choice of ordering of edges of Γ.

$C_\Gamma = \int_{\widehat{C}_{n,0}} \bigwedge_{e \in Edges(\Gamma)} \frac{\mathfrak{p}_e^*(\omega)}{2\pi}$

$\widehat{C}_{2,0} =$  $\simeq$  $= \overline{C}_{2,0}.$

Remark: A geometrically more correct picture of $\widehat{C}_{2,0}$ is as of a closed infinitely long cylinder,



rather than as of a topological disk (Kontsevich's eye).

$C_\Gamma = \int_{\widehat{C}_{n,0}} \wedge_{e \in Edges(\Gamma)} \frac{\mathfrak{p}_e^*(\omega)}{2\pi}$

The symbol ω stands any closed semialgebraic 1-form on $\widehat{C}_{2,0}$ whose restriction to the boundary, $\partial \widehat{C}_{2,0} \simeq S^1_{in} \sqcup S^1_{out}$, coincides with the standard homogeneous volume form on each of the two boundary topological circles.

Remark: If we drop the requirement of homogeneity, then our formula describes a L_∞ quasi-isomorphism between certain L_∞-extensions, μ_{in} and μ_{out}, of the Schouten bracket canonically associated with restrictions $\omega|_{S^1_{in}}$ and, respectively, $\omega|_{S^1_{out}}$.

Example: $\omega(z_1, z_2) = dArg \frac{z_1 - z_2}{\overline{z_1} - z_2}$ is *not* homogeneous on S^1_{outer}.

Example: $\omega(z_1, z_2) = \frac{1}{2i} d \log \frac{(z_1 - z_2)^2}{(\overline{z_1} - z_2)(\overline{z_2} - z_1)}$ is homogenous on both boundary circle S^1_{in} and S^1_{outer}.

$C_\Gamma = \int_{\widehat{C}_{n,0}} \bigwedge_{e \in Edges(\Gamma)} \frac{\mathfrak{p}_e^*(\omega)}{2\pi}$

Forgetful map. For any edge $e = \overset{i}{\bullet}\!\longrightarrow\!\overset{j}{\bullet}$ of the graph Γ we set

$$\begin{array}{rccc} p_e : & C_{n,0} & \longrightarrow & C_{2,0} \\ & p = (z_1, \ldots, z_i, \ldots, z_j, \ldots, z_n) & \longrightarrow & (z_i, z_j). \end{array}$$

Renormalized forgetful map is given by

$$\begin{array}{rccc} \mathfrak{p}_e : & C_{n,0} & \longrightarrow & C_{2,0} \\ & p & \longrightarrow & \begin{cases} (z_i - z_j + z_{min}(p), z_{min}(p)) & \text{if } y_i \geq y_j \\ (z_{min}(p), z_j - z_i + z_{min}(p)) & \text{if } y_i \leq y_j \end{cases} \end{array}$$

where

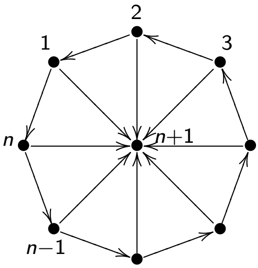
$$z_{min}(p) := \frac{1}{n} \sum_{i \in [n]}^{n} x_i + \mathfrak{i} \inf_{i \in [n]} y_i$$

This is a **semialgebraic** map.

$C_\Gamma = \int_{\widehat{C}_{n,0}} \bigwedge_{e \in Edges(\Gamma)} \frac{\mathfrak{p}_e^*(\omega)}{2\pi}$

Theorem [Me 2008]. The weight, C_{w_n}, of the n-wheel,

$w_n =$



with respect to Kontsevich's singular propagator $\omega = \frac{1}{i} d \log \frac{z_1 - z_2}{\bar{z}_1 - z_2}$ and the ordinary forgetful map p is given by

$$C_{w_n} = (-1)^{n(n-1)/2} \frac{\zeta(n)}{(2\pi\mathrm{i})^n} = (-1)^{n(n-1)/2} \frac{\sum_{p=1}^{\infty} \frac{1}{p^n}}{(2\pi\mathrm{i})^n}.$$

Why my formula works?

It works because the cell complex of the new compactification $\widehat{C}_{n,0}$ has a natural structure of a **2-coloured operad** describing L_∞-morphisms between an arbitrary pair of L_∞-algebras.

To understand this claim let us study a simpler configuration space

$$C_n := \frac{\{z_1, \ldots, z_n \in \mathbb{C} \mid z_i \neq z_j \text{ for } i \neq j\}}{\{z \to az + b \mid a > 0, b \in \mathbb{C}\}}.$$

It is well known (**Getzler and Jonez 1993**) that the cell complex of its Fulton-McPherson compactification, $\overline{C}_n := \overline{\mathrm{Im}\, f}$,

$$\begin{array}{ccccc} C_n & \longrightarrow & (S^1)^{n(n-1)} & \times & (\mathbb{RP}^2)^{n(n-1)(n-2)} \\ (z_1, \ldots, z_n) & \longrightarrow & \prod_{i \neq j} \frac{z_i - z_j}{|z_i - z_j|} & \times & \prod_{i \neq j \neq k \neq i} [|z_i - z_j| : |z_i - z_k| : |z_k - z_l|] \end{array}.$$

which has a natural structure of an operad of L_∞-algebras.

From $\Omega^1_{PA}(S^1)$ to L_∞ extensions of the Schouten bracket

$\overline{C}_2 = C_2 \simeq S^1$. For $n \geq 3$, $\partial \overline{C}_n = \coprod_{\substack{A \subsetneq [n] \\ \# A \geq 2}} \overline{C}_{\# A} \times \overline{C}_{n - \# A + 1}$

Definition. A Rham field theory on $\{\overline{C}_n\}_{n \geq 2}$ is, by definition, a collections of maps,

$$\left\{ \begin{array}{cccc} \Omega : & \mathcal{G}_{n,l} & \longrightarrow & \Omega^l(\overline{C}_n) \\ & \Gamma & \longrightarrow & \Omega_\Gamma \end{array} \right\}_{n \geq 2, l \geq 1},$$

such that $d\Omega_\Gamma = 0$, $\Omega_{\Gamma_{opp}} = -\Omega_\Gamma$, and, for any $\Gamma \in \mathcal{G}_{n,2n-4}$ and any proper subset $A \subset Vert(\Gamma)$ with $\# A \geq 2$,

$$i_A : \overline{C}_{n - \# A + 1} \times \overline{C}_{\# A} \longrightarrow \overline{C}_n,$$

one has

$$i_A^*(\Omega_\Gamma) \simeq (-1)^{\sigma_A} \Omega_{\Gamma / \Gamma_A} \wedge \Omega_{\Gamma_A}.$$

From $\Omega^1_{PA}(S^1)$ to L_∞ extensions of the Schouten bracket

Theorem [Me 2008]. Given a de Rham field theory on $\overline{C}_\bullet$, then, for any $d \in \mathbb{N}$, there is an associated L_∞-algebra structure,

$$\begin{array}{rccc} \mu_n : & \otimes^n \mathcal{T}_{poly}(\mathbb{R}^d) & \longrightarrow & \mathcal{T}_{poly}(\mathbb{R}^d)[3-2n] \\ & \gamma_1 \otimes \ldots \otimes \gamma_n & \longrightarrow & \mu_n(\gamma_1, \ldots, \gamma_n) \end{array},$$

on $\mathcal{T}_{poly}(\mathbb{R}^d)$ given by

$$\mu_n(\gamma_1, \ldots, \gamma_n) := \begin{cases} 0 & \text{for } n = 1, \\ \sum_{\Gamma \in \mathfrak{G}_{n,2n-3}} c_\Gamma \Phi_\Gamma(\gamma_1, \ldots, \gamma_n) & \text{for } n \geq 2 \end{cases}$$

with

$$c_\Gamma := \int_{\overline{C}_n} \Omega_\Gamma.$$

$\mu_n = \sum_{\Gamma \in \mathfrak{G}_{n,2n-3}} c_\Gamma \Phi_\Gamma$, where $c_\Gamma = \int_{\overline{C}_n} \Omega_\Gamma$

Proposition. For any cohomologically non-trivial PA 1-form, ω_0, on t $C_2 \simeq S^1$ the associated map,

$$\begin{array}{cccc} \Omega : & \mathcal{G}_{n,l} & \longrightarrow & \Omega^l(\overline{C}_n) \\ & \Gamma & \longrightarrow & \Omega_\Gamma := \bigwedge_{e \in E(\Gamma)} \frac{\pi_e^*(\omega_0)}{\lambda} \end{array}$$

where $\lambda := \int_{C_2} \omega_0$ and, for an edge for an edge $e = \overset{i}{\bullet} \!\longrightarrow\! \overset{j}{\bullet}$,

$$\begin{array}{cccc} \pi_e := \pi_{ij} : & C_n & \longrightarrow & C_2 \\ & (z_1, \ldots, z_i, \ldots, z_j, \ldots, z_n) & \longrightarrow & (z_i, z_j), \end{array}$$

defines a non-trivial de Rham field theory on $\overline{C}_\bullet$ *(and hence an associated non-trivial L_∞-structure on $\mathcal{T}_{poly}(\mathbb{R}^d)$)*.

$\mu_n = \sum_{\Gamma \in \mathfrak{G}_{n,2n-3}} c_\Gamma \Phi_\Gamma$, where $c_\Gamma = \int_{\overline{C}_n} \Omega_\Gamma$

Example. Consider $\omega_0(z_1, z_2) = dArg(z_1 - z_2)$.
By Kontsevich's "vanishing" Lemma,

$$c_\Gamma = 0$$

for all graphs $\Gamma \in \sqcup_{n \geq 2} \mathcal{G}_{n,2n-3}$ except

$\Gamma_1 =$ 

and $\Gamma_2 =$ 

Thus $\mu_{n \geq 3} = 0$ and

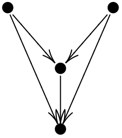
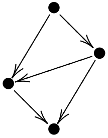
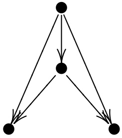
$$\mu_2(\gamma_1, \gamma_2) = c_{\Gamma_1} \Phi_{\Gamma_1}(\gamma_1, \gamma_2) + c_{\Gamma_2} \Phi_{\Gamma_2}(\gamma_1, \gamma_2) = (-1)^{|\gamma_1|}[\gamma_1 \bullet \gamma_2].$$

Thus the homogeneous volume form on S^1 gives us the Schouten bracket on polyvector fields.

$\mu_n = \sum_{\Gamma \in \mathfrak{G}_{n,2n-3}} c_\Gamma \Phi_\Gamma$, where $c_\Gamma = \int_{\overline{C}_n} \Omega_\Gamma$

Example. Consider $\omega_0(z_1, z_2) = dArg \frac{z_1 - z_2}{\overline{z_1} - z_2}\Big|_{Outer\ \ circle\ \ in\ \ \overline{C}_{2,0}}$.
It gives a non-trivial extension of the Schouten Lie algebra structure on $\mathcal{T}_{poly}(\mathbb{R}^d)$,

$$\mu_2 = \text{Schouten bracket}$$

$\mu_4 = \frac{1}{12}$  $+ \frac{1}{12}$  $+ \frac{1}{12}$ 

All $\mu_{2n+1} = 0$. In general, $\mu_{2n} \neq 0$.

This L_∞-structure was studied by **Shoikhet 2008**.

A new semialgebraic structure on $\overline{C}_n$

For any finite set A consider

$$Conf_A(\mathbb{C}) := \{A \hookrightarrow \mathbb{C}\}, \ \ \widetilde{Conf}_A := \{A \to \mathbb{C}\}, \ \ C_A := \frac{Conf_A(\mathbb{C})}{z \to \mathbb{R}^+ z + \mathbb{C}}.$$

Define a section of the projection $Conf_A(\mathbb{C}) \to C_A$,

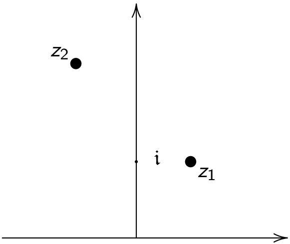
$$\begin{array}{cccc} s: & C_A & \longrightarrow & Conf_A(\mathbb{C}) \\ & p = \{z_i = x_i + \mathrm{i} y_i\}_{i \in A} & \longrightarrow & p^h := \frac{p - z_{min}(p)}{|p - z_{min}(p)|} + \mathrm{i} \end{array}$$

where

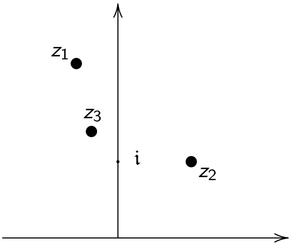
$$z_{min}(p) := \frac{1}{\#A} \sum_{i \in A} x_i + \mathrm{i} \inf_{i \in A} y_i .$$

$$C_n^h := s(C_n) = \{p \in Conf_n(\mathbb{C}) \mid z_{min}(p) = \mathrm{i}, |p - \mathrm{i}| = 1\}.$$

A new semialgebraic structure on $\overline{C}_n$

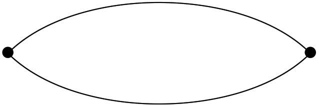


C_2^h



C_3^h

$C_2^h =$



A new semialgebraic structure on $\overline{C}_n$

Define similarly

$$\widetilde{C_n^h} := \{p \in \widetilde{Conf}_n(\mathbb{C}) \mid z_{min}(p) = \mathrm{i}, |p - \mathrm{i}| = 1\}.$$

Contrary to C_n^h, this is a **compact** semialgebraic set for all n.
Define a compactification, $\overline{C}_n$, of C_n as the closure of

$$C_n \xrightarrow{\prod \pi_A} \prod_{\substack{A \subseteq [n] \\ \#A \geq 2}} C_A \xrightarrow{\simeq} \prod_{\substack{A \subseteq [n] \\ \#A \geq 2}} C_A^h \hookrightarrow \prod_{\substack{A \subseteq [n] \\ \#A \geq 2}} \widetilde{C_A^h}.$$

where

$$\begin{array}{rccc} \pi_A : & C_n & \longrightarrow & C_A \\ & p = \{z_i\}_{i \in [n]} & \longrightarrow & p_A := \{z_i\}_{i \in A} \end{array}$$

is the natural forgetful semialgebraic map.

$C_{n,0}$ as a magnified C_n

$$Conf_{A,0}(\mathbb{H}) := \{A \hookrightarrow \mathbb{H}\}, \quad C_{A,0} := \frac{Conf_{A,0}(\mathbb{H})}{\{z \longrightarrow \mathbb{R}^{+}z + \mathbb{R}\}}.$$

There is a semialgebraic homeomorphism,

$$\begin{array}{lcccccc} \Psi_A : & C_{A,0} & \longrightarrow & C_A^h & \times & (0,1) \\ & p & \longrightarrow & \frac{p - z_{min}(p)}{|p - z_{min}(p)|} + \mathfrak{i} & \times & \frac{||p||}{||p||+1}. \end{array}$$

where

$$||p|| := \frac{|p - z_{min}(p)|}{\inf_{i \in A} y_i}.$$

A new compactification $\widehat{C}_{n,0}$

With a subset $A \subset [n]$ one associates a forgetful map,

$$\begin{array}{rccc} \pi_A : & C_{n,0}(\mathbb{H}) & \longrightarrow & C_{A,0}(\mathbb{H}) \\ & p = \{z_i\}_{i\in[n]} & \longrightarrow & p_A = \{z_i\}_{i\in A}. \end{array}$$

A semialgebraic *compactification*, $\widehat{C}_{n,0}$, of $C_{n,0}$ is defined as the closure of the semialgebraic monomorphism

$$C_{n,0}(\mathbb{H}) \xrightarrow{\prod \pi_A} \prod_{\substack{A\subseteq[n]\\A\neq\emptyset}} C_{A,0} \xrightarrow{\prod \Psi_A} \prod_{\substack{A\subseteq[n]\\A\neq\emptyset}} C_A^h \times (0,1) \longrightarrow \prod_{\substack{A\subseteq[n]\\A\neq\emptyset}} \widetilde{C}_A^h \times [0,1].$$

equipped with the induced structure of a semialgebraic set.

Codimension 1 boundary strata in $\widehat{C}_{n,0}$

$$\partial \widehat{C}_{n,0} = \bigsqcup_{\substack{A \subseteq [n] \\ \#A \geq 2}} (C_{n-\#A+1,0} \times C_{\#A}) \quad \bigsqcup_{\substack{[n]=B_1 \sqcup \ldots \sqcup B_k \\ 2 \leq k \leq n \\ \#B_1, \ldots, \#B_k \geq 1}} (C_k \times C_{\#B_1,0} \times \ldots \times C_{\#B_k,0})$$